CS602 Applied Algorithms	**2019-20 Sem II**

Midsem: Feb 27

Total Marks: 80 *Time: 120 minutes*

Note. Write your answers clearly and succinctly. Hints might not be precise. Feel free to ignore the hints. You can use any results proved in the class. For questions with multiple parts, if you cannot solve a particular part, you can still try to answer latter parts by assuming the previous parts as given.

Definition 1 *For a graph $G(V, E)$, a* vertex cover *is a set of vertices $S \subseteq V$ such that every edge has at least one end point in S. When each vertex has a cost, the cost of a vertex set S is the sum of the costs of the vertices in S.*

Definition 2 *For a minimization problem, a 2-approx solution means a solution whose cost is at most twice of the optimal solution.*

Que 1 [10 marks]. Recall the augmenting path algorithm for finding a maximum size matching in a bipartite graph. In every step, it tries to find an augmenting path with respect to the current matching. If it cannot find an augmenting path then the current matching is maximum. Also recall that maximum matching size is same as minimum vertex cover size in bipartite graphs. Explain how to use this algorithm to find a minimum vertex cover in a bipartite graph.

Hint: Consider the situation when you have a matching and you cannot find an augmenting path (from a free vertex to another free vertex). Can you use this situation to generate a vertex cover which has the same size as the current matching. Figure 1 shows such a situation.

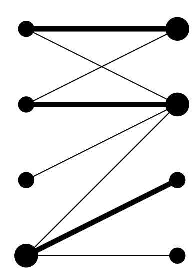

Figure 1: A bipartite graph with a maximum matching (thick edges) and minimum vertex cover (vertices shown by larger circles).

Que 2. Consider a graph $G(V, E)$, with a vertex $v \in V$ having cost $c_v \geq 0$. Following is a natural LP relaxation for the minimum cost vertex cover problem.

Vertex Cover LP:

$$\begin{aligned} \min \quad & \textstyle\sum_{v \in V} c_v y_v \quad \text{subject to} \\ \text{for each } v \in V, \quad & y_v \geq 0, \\ \text{for each } e = (u, v) \in E, \quad & y_u + y_v \geq 1. \end{aligned}$$

(a) [10 marks]. Show that if G is a bipartite graph then the above LP has an integral optimal solution.

Hint: Try to convert a non-integral solution to an integral one while decreasing the function value $\sum_v c_v y_v$ or keeping it the same. Try $+\epsilon$ on the left and $-\epsilon$ on the right or vice versa.

(b) [10 marks]. Show that for a general graph G, the LP always has an optimal solution that is half-integral, that is, $y \in \{0, \frac{1}{2}, 1\}^{|V|}$.

Hint: If you are at some solution, try to modify the coordinates that are not 0,1, or 1/2 so that the function value goes smaller or remains same. Can you think of a natural bipartition of the set of vertices (maybe based on the values of y variables) and try the idea from the previous part?

(c) [3+7 marks]. Suppose one can solve the above LP in polynomial time and get an optimal solution y^* that is half integral. We want to get a vertex cover from this. A natural idea is to take all those vertices v for which y^*_v is nonzero. Show that this set of vertices forms a vertex cover. Show that it is a 2-approx solution for the minimum cost vertex cover problem.

Que 3. Now, let us design a different 2-approximation algorithm for minimum cost vertex cover using Primal-Dual Scheme. Recall that for an LP and a feasible solution, a constraint is said to be tight if it is satisfied by the solution with equality .

(a) [5 marks]. Write the dual LP for the above vertex cover LP.

Hint: Recall that the dual LP will be a maximization LP, and there will be one variable x_e for each edge e and one constraint for each vertex.

(b) [5 marks]. Write the expressions for the following two Complementary slackness (CS) conditions.

- **Primal CS.** $x_e \neq 0$ will imply that the corresponding primal constraint is tight.
- **Dual CS.** $y_v \neq 0$ will imply that the corresponding dual constraint is tight.

Now, we will run the following primal-dual algorithm that maintains a dual feasible solution and a primal solution that satisfies dual CS. We will not care about the primal CS for now. For simplicity, we have assumed that each cost c_v is non-negative.

1. Initialize $y_v = 0$ for each v and $x_e = 0$ for each e.
2. Take any edge e such that all the dual constraints in which x_e participates are non-tight. Keep increasing the dual variable x_e till at least one of the dual constraints becomes tight. As soon as a dual constraint becomes tight we set the corresponding primal variable y_u to 1.
3. Keep repeating the above step until there is no such edge left.
4. Output the set of vertices whose corresponding primal variables are 1.

(c) [5 marks]. Argue that the primal solution obtained after this algorithm is feasible and thus, gives us a vertex cover.

(d) [5 marks]. Show that after the algorithm, the dual CS condition is satisfied exactly and the primal CS condition is satisfied approximately (don't ask me what this means) with an appropriate approximation factor.

(e) [5 marks]. Use this approximate complementary slackness to show that the output vertex cover is a 2-approx solution.

Que 4 Set Cover [5+5+5 marks]. Suppose we are given sets $S_1, S_2, \ldots, S_k \subseteq E$, where the set S_i has cost c_i. Assume that any element $e \in E$ appears in at most γ of these sets. The set cover problems asks for the minimum cost sub-collection of these sets such that their union is equal to the whole set E. Write the natural LP relaxation for this problem and the corresponding dual LP.

If we run the primal-dual algorithm exactly analogous to Que 3, we will get a set cover in the end. What will be the approximation factor of this algorithm : 2, $2k/\gamma$, γ, or 2γ ? Just write the answer directly. No need to describe the algorithm or any proof.