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Ellipsoid Algorithm

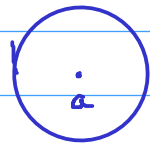
Khachyan 1979
(first polynomial time algorithm)

→ Works for more general convex programming
(minimize a convex function over a convex set)

→ Recall that Optimization $\leq$ Feasibility.

→ Ellipsoid algorithm solves the feasibility problem.

e.g., • is $Ax \leq b$ feasible?



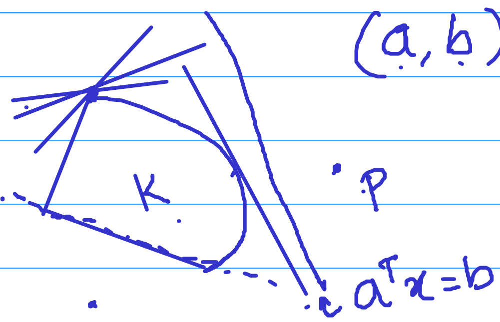
• is $Ax \leq b$, $\|x - a\| \leq r$ feasible?

→ Does not need an explicit description of the constraints
Needs only a separation oracle.

Def: Seperation Oracle

Given a convex set K and a point p
find out whether $p \in K$.

If $p \notin K$ then output a separating hyperplane
(a, b) s.t. $a^T p - b > 0$
and
$a^T x - b \leq 0$ for all $x \in K$.



→ Separation oracle obvious for a given System $Ax \leq b$

→ Ellipsoid algorithm can also work for exponentially many constraints, if efficient separation oracle can be designed. (e.g., Steiner tree LP).

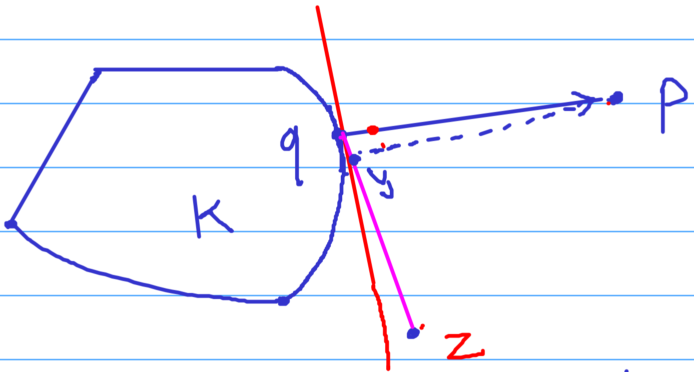
Separating Hyperplane theorem:

Given a convex set $K \subseteq \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$

If $p \notin K$ then there exists a separating hyperplane $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ s.t. $a^T p - b > 0$

and $a^T x - b \leq 0$ for all $x \in K$.

Proof



Take a point $q \in K$ that is closest to p.

That is,
$$\text{dist}(q, p) \leq \text{dist}(x, p) \quad \forall \; x \in K$$

Consider the hyperplane passing through q that is orthogonal to $p - q$.

$$\underbrace{(p-q)^T}_{a} x = \underbrace{(p-q)^T q}_{b}.$$

$$(p-q)^T (x) - (p-q)^T q = 0$$

if $x = p$ $\quad (p-q)^T (p-q) > 0$

$\|p - q\| > 0$ ($\because$ p and q are distinct

To show

$\forall\ x \in K$

$$(p-q)^T x - (p-q)^T q \leq 0$$

Contrapositive

let's take point z

$$(p-q)^T z - (p-q)^T q > 0 \Rightarrow \boxed{(p-q)^T (z-q) > 0}$$

we will show that $z \notin K$

Take a point y on the line segment joining (z, q)

Will show that $y \notin K$.

This will imply that $z \notin K$

Take y to be very close to q and show that

$$\longrightarrow \boxed{\text{dist}(y, p) < \text{dist}(q, p)}$$

$$\Rightarrow y \notin K \Rightarrow z \notin K.$$

$$y = (1-\lambda) q + \lambda z \qquad \lambda \geq 0$$

$$(y-p)^T (y-p)$$

$$= \left[(1-\lambda) q + \lambda z - p\right]^T \left[(1-\lambda) q + \lambda z - p\right]$$

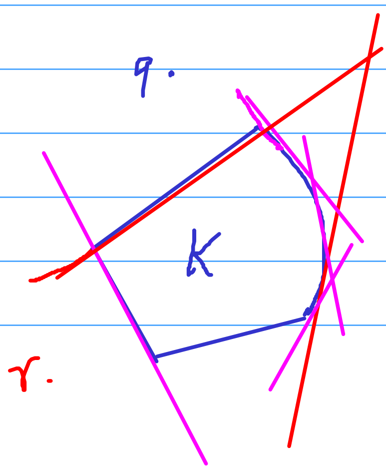
$$[\text{dist}(y,P)]^2 - [\text{dist}(q,P)]^2$$
$$= (y-p)^T(y-p) - (q-p)^T(q-p)$$

$$= [\lambda(z-q)]^T[\lambda(z-q)] + (q-p)^T(\lambda(z-q)) + \lambda(z-q)^T(q-p)$$

$$= \lambda^2 \underbrace{(z-q)^T(z-q)}_{>0} \underbrace{- 2\lambda(p-q)^T(z-q)}_{<0}$$

If you choose small enough λ then $\lambda^2 << \lambda$ and the second term will dominate.

Then the overall sum will be negative.



$\cdot\, p$

Define $H = \{(a,b) :$ for some $p \notin K$
$a^T p > b$
and $a^T x \leq b$
for all $x \in K\}$

H is the set of linear constraints describing K.

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Corollary: Every convex set can be described by a set of linear constraints, (intersection of Halfspaces) possibly infinitely many.

Separating Hyperplane theorem can be used to prove

Farkas' Lemma

Strong LP Duality.

Strong Duality: $\max w^T x$ s.t. $Ax \leq b$

let the opt value be w^*

then there exists

$y \geq 0$ s.t. $A^T y = w$, $b^T y = w^*$

proof outline:

Define $K = \{(A^T y, b^T y) : y \geq 0\}$

Argue that K is a convex set.

Need to show that $(w, w^*) \in K$

For the sake of contradiction assume $(w, w^*) \notin K$

By separating hyperplane theorem

$\exists$ a, α, β s.t.

$a^T w + \alpha w^* > \beta$

but $a^T A^T y + \alpha b^T y \leq \beta$ for every $y \geq 0$

can take $\beta = 0$ (need to argue)

$$\Rightarrow \quad a^T A^T + \alpha b^T \leq 0$$

and $\quad a^T w + \alpha w^* > 0$

HW

Using a, α, one can construct a feasible point for $Ax \leq b$ with $w^T x > w^*$ leading to a contradiction.

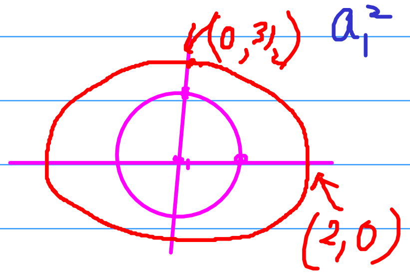
Ellipsoid Algorithm

(Ellipsoid)

Higher dimensional analogue of an ellipse.

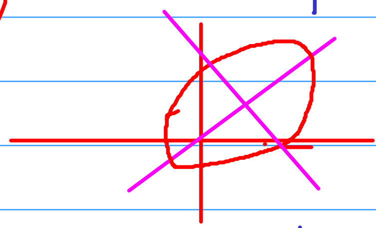
Sphere $\quad x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1 \qquad$ center origin

Ellipsoid $\quad \dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} + \cdots + \dfrac{x_n^2}{a_n^2} \leq 1 \qquad$ Center origin



$$\frac{x_1^2}{4} + \frac{x_2^2}{(3/2)^2} \leq 1$$

axis-parallel



Any ellipsoid can be obtained from a sphere by

(1) stretching the axes

(2) rotation

(3) shifting the center.

$x^Tx = \|x\|_2^2$

$$S(0,1) \equiv \{ x : x^T x \leq 1 \}$$

Ellipsoid $\rightarrow$ $\{ M^{-1}x + c : \underline{x^T x \leq 1} \}$

for some invertible matrix M and $c \in \mathbb{R}^n$

$y = M^{-1}x + c \Rightarrow x = M(y-c)$

Equivalently $\{ y \in \mathbb{R}^n : \underbrace{(y-c)^T M^T}\ \underbrace{M(y-c)} \leq 1 \}$

Let $A = M^T M$ (a positive definite matrix)

Def (Ellipsoid)

A s.t. for all x $x^T A x \geq 0$

For a PD matrix A, and $c \in \mathbb{R}^n$

$$E(c, A) = \{ y \in \mathbb{R}^n : \underbrace{(y-c)^T A (y-c)} \leq 1 \}.$$

Ellipsoid Algorithm:

(not explicitly)

Input : a convex set K with a separation oracle.

Output : a point in K if K is nonempty or say that K is empty.

Promise :

(1) bounded : a number R s.t. $K \subseteq S(0,R)$

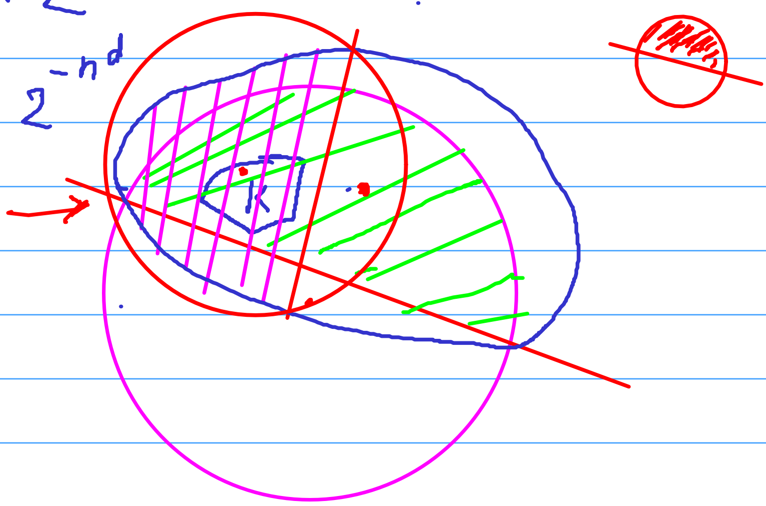
(2) Volume lower bound : number r s.t. $K \supseteq S(r)$ if K is non-empty.

($\Rightarrow$ full dimensional)

Running time: $O\left(n^2 \log \frac{R}{r}\right)$ (no. of iterations).

$\underbrace{n \leftarrow \text{dim}}$

- R should be $\leq 2^{n^d}$ ← const.
- r should be $\geq 2^{-n^d}$



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Algorithm outline:

$E_0 = \underline{S(0,R)}$

Invariant: At round t, we will have an ellipsoid E_t with $K \subseteq E_t$

- Check if center c_t of E_t lies in K
  if yes then done.
- if No then get a hyperplane separating c_t from K.

Say, $a^T x \leq b$ for every $x \in K$
and $a^T c_t > b$.

Find a new Ellipsoid E_{t+1} containing the intersection of E_t and the halfspace.

$$E_{t+1} \supseteq \{ x \in E_t : a^T x \leq b \}$$

Clearly, $K \subseteq E_{t+1}$

Continue while $\mathrm{Vol}(E_t) \geq \mathrm{Vol}(S(0,r))$

If $\mathrm{Vol}(E_t)$ becomes $< \mathrm{Vol}(S(0,r))$

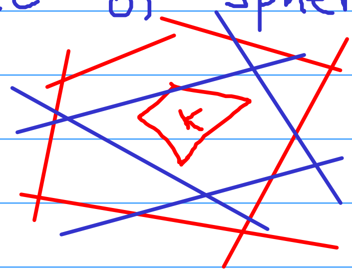
$\Rightarrow$ K is empty.

To bound the running time, we will show that the volume of the ellipsoid decreases by a fraction in every round.

Que: Could we use a sequence of spheres instead of Ellipsoids?

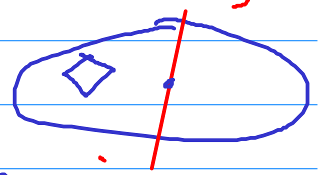
or

sequence of Polytopes?



Finding E_{t+1} containing $E_t \cap \{x : a^T x \leq b\}$

simplest case:



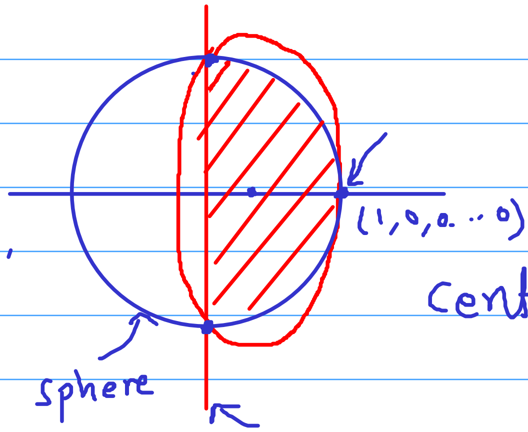
$$E_t = S(0,1) \qquad x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1$$

for simplicity, we will always take the separating hyperplane passing through the center.

Let the hyperplane be

$$\underline{x_1 \geq 0}$$

Find an ellipsoid E_{t+1} containing



center $(\alpha, 0, 0, \cdots 0)$

$\alpha \geq 0.$

Ellipsoid E_{t+1}

$$\frac{(x_1-\alpha)^2}{\beta^2} + \frac{x_2^2}{\gamma^2} + \frac{x_3^2}{\gamma^2} + \cdots + \frac{x_n^2}{\gamma^2} \leq 1$$

boundary should passes through

$\longrightarrow (1, 0, 0, \cdots, 0)$ and

$\longrightarrow$ any $(0, x_2, x_3 \cdots, x_n)$ s.t. $\underline{x_2^2 + x_3^2 + \cdots + x_n^2 = 1}$

$$\underline{\frac{(1-\alpha)^2}{\beta^2} = 1} \qquad \left(\frac{-\alpha}{\beta}\right)^2 + \frac{1}{\gamma^2} = 1$$

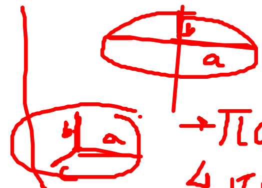
$$\boxed{\beta^2 = (1-\alpha)^2} \qquad \boxed{\gamma^2 = \frac{1}{1-\left(\frac{\alpha}{1-\alpha}\right)^2} = \frac{(1-\alpha)^2}{1-2\alpha}}$$

verify that for any such α, β, γ,

HW E_{t+1} contains the half-sphere

$$\left\{ (x_1, x_2, \cdots x_n) : x_1 \geq 0 \text{ and } \sum_{i=1}^{n} x_i^2 \leq 1 \right\}$$

Volume of E_{t+1} $= c_n \, \beta \cdot \gamma^{n-1}$



$\rightarrow \pi a b \cdot$

$\frac{4}{3}\pi abc$

Minimize $(1-\alpha)^n / (1-2\alpha)^{\frac{n-1}{2}}$

$\alpha \geq 0$

$$\Rightarrow \quad \alpha = \frac{1}{n+1} \quad \Rightarrow \quad \beta = \frac{n}{n+1}, \quad \gamma = \sqrt{\frac{n^2}{n^2-1}}$$

E_{t+1}

$$\frac{\left(x_1 - \frac{1}{n+1}\right)^2}{\left(\frac{n}{n+1}\right)^2} + \frac{x_2^2}{n^2/n^2-1} + \cdots + \frac{x_n^2}{n^2/n^2-1} \leq 1.$$

Bounding $\quad \mathrm{Vol}(E_{t+1}) / \mathrm{Vol}(E_t)$

$$= \frac{c_n \cdot \frac{n}{n+1} \cdot \left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}}}{c_n \cdot 1^n}$$

$$\frac{n}{n+1} \cdot \left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} = \left(1 - \frac{1}{n+1}\right) \cdot \left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}}$$

$\boxed{\text{Use } \; 1+y \leq e^y \;\; \forall y \in \mathbb{R}}$

$$\leq e^{\frac{-1}{n+1}} \left(e^{\frac{1}{n^2-1}}\right)^{\frac{n-1}{2}}$$

$$= e^{\frac{-1}{n+1}} \; e^{\frac{1}{2(n+1)}}$$

$$= e^{\frac{-1}{2(n+1)}}$$

If in every round we have volume decrease by a factor of $e^{-1/2n+2}$

then

$$e^{-k/2n+2} \cdot Vol(S(R)) = Vol \cdot LB$$

number of rounds $= (2n+2)\log\left(\frac{Vol(S(R))}{Vol\ LB}\right)$

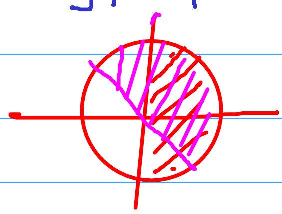
$$= (2n+2)\cdot\log\left(\frac{R^n}{r^n}\right)$$

$$= O(n^2)\cdot\log\frac{R}{r}.$$

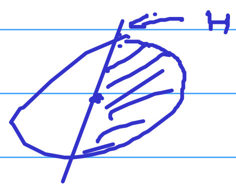
General Case: Arbitrary Ellipsoid and hyperplane.

$(r,0,0,\dots,0)$

(1) $S(0,1)$ and $a^T x \geq 0$



Just need to rotate the above computed ellipsoid appropriately.



(2) Arbitrary ellipsoid

$$E_t,\ H \xrightarrow{T} S(0,1),\ H'$$

$$\downarrow$$

$$E_{t+1} \xleftarrow{T^{-1}} E'_{t+1} \supseteq S(0,1)\cap H'$$

Claim Containment, and ratio of volumes invariant under Linear Transformation

Examples

$$\frac{x_1^2}{4} + \frac{x_2^2}{9} \le 1$$

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \longrightarrow$$

$y_1 = x_1/2$ $\qquad$ $y_1^2 + y_2^2 \le 1$

$y_2 = x_2/3$

$$(x_1+x_2)^2 + (x_1-x_2)^2 \le 1$$

$y_1 = x_1 + x_2$ $\qquad$ $x_1 = \frac{y_1+y_2}{2}$

$y_2 = x_1 - x_2$ $\qquad$ $x_2 = \frac{y_1-y_2}{2}$

$$(x_1-\alpha)^2 + (x_2-\beta)^2 \le 1$$

$\longrightarrow$ $y_1 = x_1 - \alpha$ $\qquad$ $y_2 = x_2 - \beta$

Steiner Tree LP

HW separation oracle in poly time

R $\quad$ $K \subseteq S(0,R)$ $\qquad$ $\text{vol}(K) \ge$ ...

Solving an LP via Ellipsoid.

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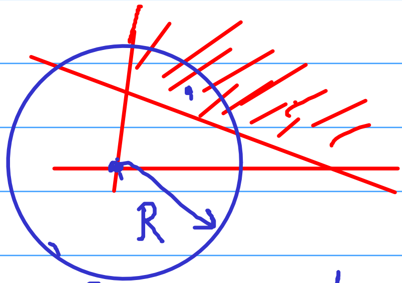
Que: is $Ax \le b$ feasible.

Bounding R and r for the convex set

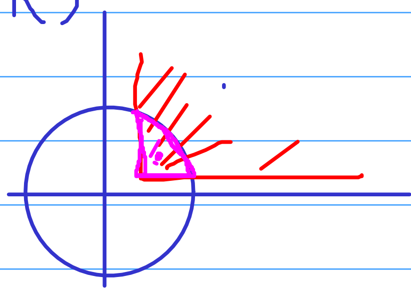
$Ax \le b$



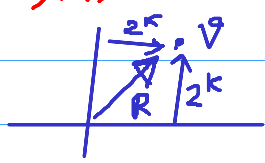
An LP can be unbounded. So cannot bound R.

It is good enough if we can guarantee that if $Ax \le b$ is feasibe then there is at least one point in

$$\{x : Ax \le b\} \cap S(0,R)$$



Just take $K = \{x : Ax \le b\} \cap S(0,R)$



say, A is $m \times n$.

Entries are rational numbers with b bits

To show: $R \le 2^{\text{poly}(m,n,b)}$ $\quad \log R = \text{poly}(m,n,b)$

Claim: Any vertex has coordinates with at most poly(m,n,b) bits. $\quad R \le 2^k \sqrt{n}$

$$\boxed{\begin{array}{l} A'x = b \\ x = A'^{-1}b \end{array}}$$

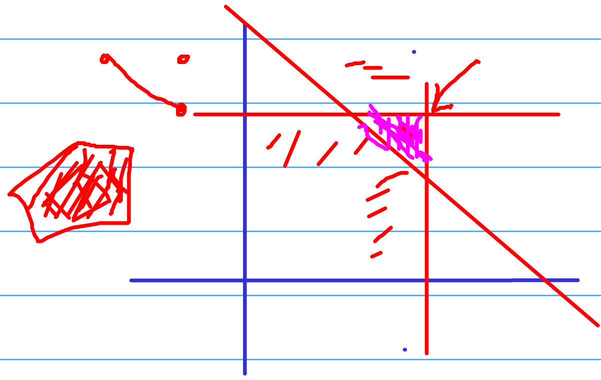
Thus, we can bound $R \le 2^{\text{poly}(m,n,b)}$

Lower bounding the volume.

① If not Full dimensional (nonzero volume)
→ perturb constraints by small amount

$$a_j^T x \le b_j + \varepsilon$$

Example



$x \le 1 + \varepsilon$

$y \le 1 + \varepsilon$

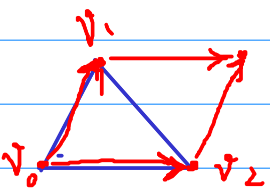
$x + y \ge 2 - \varepsilon$

If $Ax \le b$ non-feasible $\Rightarrow$ $Ax \le b + \varepsilon$ non-feasible.

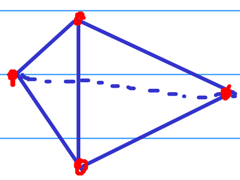
② volume lower bound

co-ordinates of vertices are like P/q
where $\underline{q} \le \underbrace{2^{\text{Poly}(m,n)}}$

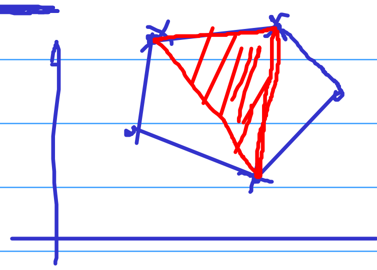
Let us lower bound the volume
when there are exactly $\underline{n+1}$ vertices.



2-simplex



3-simplex

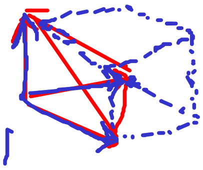


n-dim

n-simplex

$$\frac{1}{2} \det \left[v_1 - v_0 \,,\, v_2 - v_0 \right]$$

Let vertices be $v_0, v_1, v_2, v_3, \cdots, v_n$

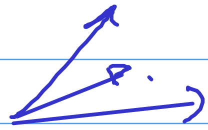


$\frac{1}{6}$

$$\text{volume}\left(\text{parallelopiped}\left(v_1 - v_0,\ v_2 - v_0,\ v_3 - v_0,\ \cdots,\ v_{n+1} - v_0\right)\right)$$

$$= (n!)\ \text{volume}\left(\text{simplex}\left(v_0, v_1, v_2, \cdots, v_{n+1}\right)\right)$$

HW



vol (parllelopiped)

$$= \det \begin{bmatrix} p/q & & & \\ v_1 - v_0 & v_2 - v_0 & \cdots & v_n - v_0 \\ & & & \end{bmatrix} \geq q^{-n}$$

$\geq 2^{-poly}$

If the largest denominator of any entry is q the det $\geq q^{-n}$

$$\text{Volume (simplex)} \geq 2^{-poly(m,n,b)}$$

No. of iterations in Ellipsoid

$$= O\left(n \log \frac{R^n}{\text{Vol} \cdot LB}\right)$$

$$= O\left(\text{poly}(m, n, b)\right)$$

Unfortunately the poly( ) is too large to be practical.