• Prove that any polyhedron (i.e., a set \( \{ x \in \mathbb{R}^n : Ax \leq b \} \) for some matrix \( A \in \mathbb{R}^{m \times n} \) and vector \( b \in \mathbb{R}^m \)) is a convex set.

• Suppose we want to maximize a given function \( w^T x \) over a polyhedron \( P \). If \( z_1, z_2 \) are two maximizing points, then show that their mid-point \( (z_1 + z_2)/2 \) will also be a maximizing point.

• Suppose \( P \) is a polyhedron given by
  \[
  \{ x \in \mathbb{R}^n : a_i^T x \leq b_i \text{ for } 1 \leq i \leq m \}
  \]
  and \( F \) is a face of \( P \) described by the tight constraints
  \[
  a_i^T x = b_i \text{ for } 1 \leq i \leq k,
  \]
  where \( k \leq m \). Prove that there exists a function \( w^T x \) such that \( F \) is the face maximizing \( w^T x \) over \( P \).
  Hint: you can try to express \( w^T x \) in terms of the tight constraints for \( F \).