Lecture 1 — Jan 11, 2021

Linear Programming

OPTIMISATION

- Variables $\underline{x_1, x_2, x_3, \ldots x_n} \in \mathbb{R}$
- given constraints on these variables.
  - Linear Constraints $\underline{\text{Ex}}$ $x_1 + 2x_2 + x_3 \leq 5$
    $x_1 - x_2 = 7$
- Maximize / Minimize a given function
  $f(x_1, x_2, \ldots, x_n)$ subject to given constraints.

$f : \mathbb{R}^n \to \mathbb{R}$

$f(x_1, x_2, \ldots, x_n) = 3x_5 + 7x_9 - 9x_1 \ldots$

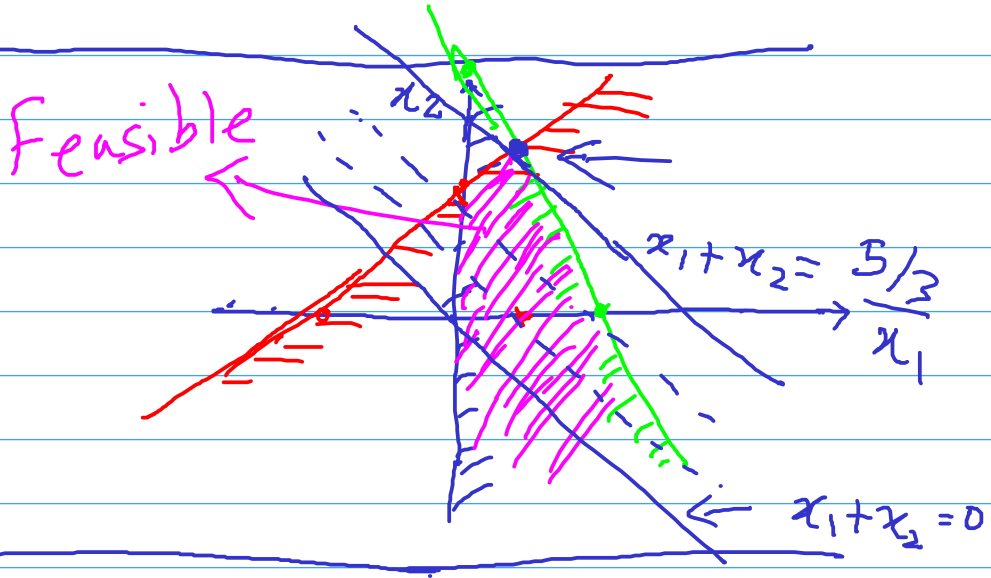
Example 1

$x_1 \geq 0$

$x_2 - x_1 \leq 1$

$2x_1 + x_2 \leq 2$

Max $x_1 + x_2$



Example 2: Diet Plan. Midday Meal.

	Rice	Dal	Egg	…
Calories	100	115	75	
Carb	25	20	1	
Protein	3.5	8	7	
⋮				

$x_R, x_D, x_E \geq 0$

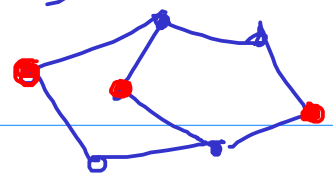
$600 \leq 100x_R + 115x_D + 75x_E \leq 800$

$75 \leq 25x_R + 20x_D + x_E \leq 100$

$10 \leq 3.5x_R + 8x_D + 7x_E \leq 12$

Min $5x_R + 15x_D + 6x_E$

Combinatorial Optimization (discrete)



Maximum Independent Set

Given a graph $G(V, E)$ find a maximum size set of vertices which don't have any edges among them.

Try to express it as an optimization problem with linear constraints.

Int Prog

Variables

$x_v \quad \forall v \in V$

$\longrightarrow x_v \in \{0, 1\}$

$\forall (u, v) \in E \qquad x_u + x_v \leq 1$

$$\text{Max} \sum_{u \in V} x_u$$

Exer verify that any solution satisfying this is an ind. set.

Any Ind set satisfies these const.

Linear Prog Relaxation

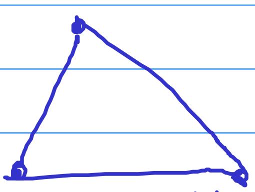
$0 \leq x_v \leq 1$

$\forall (u, v) \in E \qquad x_u + x_v \leq 1$

$$\text{Max} \sum_{u \in V} x_u$$

linear Prog.

1/2



1/2 1/2

Max Ind set size 1

Opt function value $3/2$

Any Clique.

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Linear Program in matrix representation.

$x_1 \geq 0 \quad x_2 \geq 0$
$x_2 - x_1 \leq 1$
$2x_1 + x_2 \leq 2$
Max $x_1 + x_2$

$x_1 \geq 0 \quad x_2 \geq 0$

$A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A \underset{m\times 1}{x} \leq \underset{m\times 1}{b}$

m constraints   n variables.

$A_{m\times n}, \quad x \in \mathbb{R}^n \quad b \in \mathbb{R}^m$

$\geq \;\longrightarrow\; \leq$

$= \;\longrightarrow\; \leq$

$x_3 - x_4 \geq 1 \iff x_4 - x_3 \leq -1$

$x_2 + x_3 = 5 \iff \left.\begin{matrix} x_2 + x_3 \leq 5 \\ -x_2 - x_3 \leq -5 \end{matrix}\right\}$

$w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Max $\underset{1\times n}{w^T} \underset{n\times 1}{x} = w_1x_1 + w_2x_2 - \cdots + w_nx_n$

$\boxed{\text{Min} \iff \text{Max} \\ \text{multiplying } -1}$

Two standard forms

① Minimize c^Tx

s.t. $Ax \geq b$

② Minimize c^Tx

s.t.

$Ax = b$
$x \geq 0$

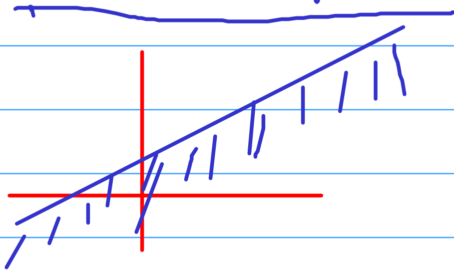
Converting to the second standard form.

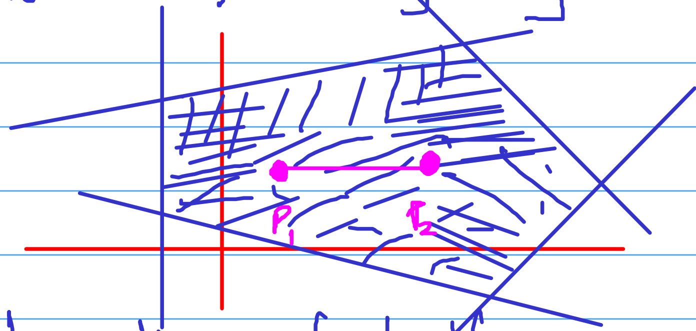
$[x_1 + x_2 \leq 5$

$\left[\begin{matrix} x_1 + x_2 + y = 5 \\ y \geq 0 \end{matrix}\right.$

Geometric Perspective.

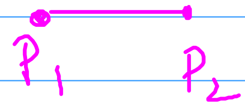
Def (Halfspace): region defined by any linear inequality





Def (Polyhedron): Intersection of halfspaces.
Equivalently, region defined by a set of linear inequalities

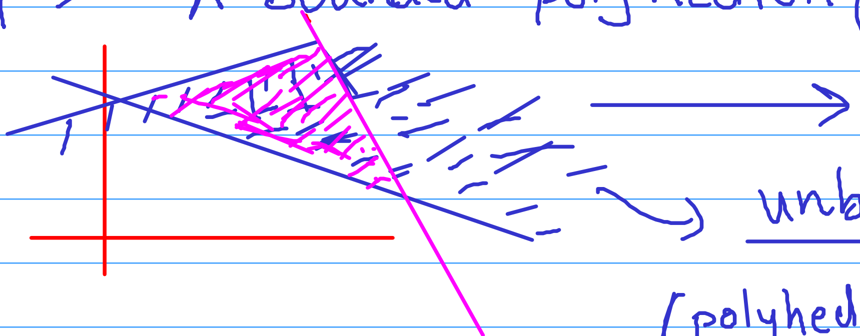
HW (convex) A polyhedron is a convex set



$\lambda P_1 + (1-\lambda) P_2 \quad 0 \le \lambda \le 1$

Def (Polytope): A bounded polyhedron. ($\exists\, r$, s.t. Polyhedron $\subseteq S_r$)

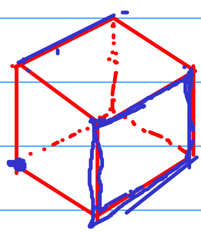
(2 dim, Polygon)



Def (Face): A boundary of the polytope (polyhedron), which itself is a polytope.

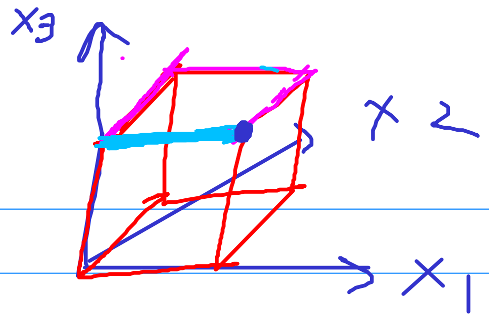
A face can have dimension $0, 1, 2, \cdots, n-1$.

corner edges



$$\boxed{\begin{array}{c} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \\ 0 \le x_3 \le 1 \end{array}}$$

Formal Definition (Face): a face is what you get when some of the inequalities are replaced by equalities and you get a non-empty region.



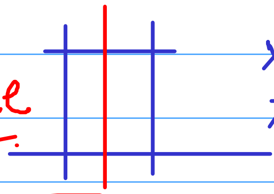
$x_3 = 1$

$x_2 = 0,\ x_3 = 1$

$\boxed{\begin{matrix} x_3 = 1 \\ x_3 = 0 \end{matrix}}$ not defined

$x_2 = 0,\ x_3 = 1,\ x_1 = 1$

Not a face

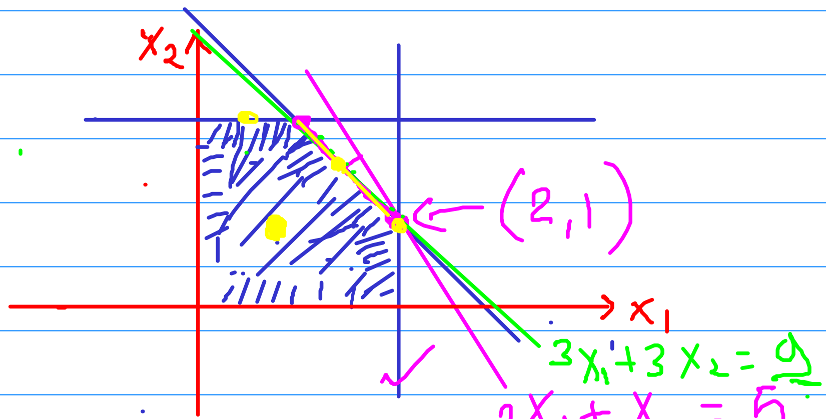


$x_1 \geq 0 \quad x_1 \leq 2$
$x_2 \geq 0 \quad x_2 \leq 2$
$x_1 = 1$

Lecture 3 Jan 14 Faces of a Polyhedron

Lemma 1 For any polyhedron P ($Ax \leq b$) and any optimizing function $w^T x$, the set of points in P that maximize $w^T x$ form a face of P.
(if P is non-empty and the max is not unbounded).

$x_1 \geq 0$
$x_2 \geq 0$
$x_1 \leq 2$
$x_2 \leq 2$
$x_1 + x_2 \leq 3$



Max $2x_1 + x_2$

(2,1) $\quad x_1 = 2,\ x_1 + x_2 = 3$ tight.

Max $3x_1 + 3x_2$

Tight constraints $x_1 + x_2 = 3$

Towards the proof of the lemma 1

Claim 1: Let P be a polyhedron given by $Ax \leq b$

$$a_1^T x = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots + a_{1n}x_n \leq b_1$$
$$a_2^T x = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_m^T x = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$

Consider a point z that maximizes the function $w^T x$. Then every point in the minimal face containing z maximizes $w^T x$.

Equivalently,
Let the set of all tight constraint for z (i.e., satisfied by equality) be given by.

$$\left.\begin{aligned} a_1^T x &= b_1 \\ a_2^T x &= b_2 \\ &\vdots \\ a_k^T x &= b_k \end{aligned}\right] \leftarrow \text{define the minimal face containing } z.$$

Then every point in P that satisfies the above equalities maximizes $w^T x$.

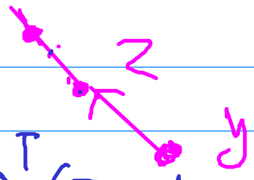
Proof of Claim 1 Let y be such a point

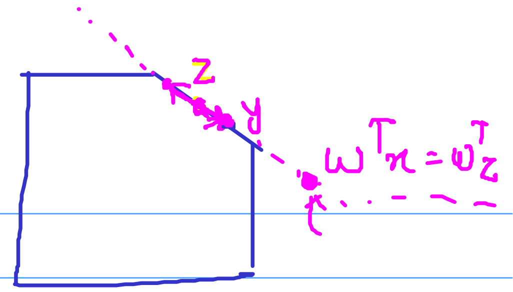
For the sake of contradiction y is not an optimizing point.

$$w^T y < w^T z$$

$$w^T(z-y) > 0$$

$\alpha = z + \varepsilon(z-y)$
small $\varepsilon > 0$





$$w^T \alpha = w^T z + \underbrace{\varepsilon w^T (z-y)}_{>0}$$

$\underline{w^T \alpha > w^T z.}$

This would be a contradiction if $\underline{\alpha \in P}$.

Argue $\underline{\text{Now}}$ $\underline{\alpha \in P}$.

Observe that α satisfies the first k constraints with equality.

$\underline{\text{We know}}$ $a_m^T z < b_m$ $\quad \underbrace{a_m^T \alpha = \underbrace{a_m^T z}_{<b_m} + \underbrace{a_m^T \varepsilon (z-y)}_{\text{very small}}}_{<b_m}$

$\Rightarrow$ $\underline{\alpha \in P}$.

Claim 2: If face F_1 maximizes $w^T x$ and face F_2 maximizes $w^T x$ then the minimal face containing $F_1 \cup F_2$ maximizes $w^T x$.

Equivalently,

Let $S_1 = \{i_1, i_2, i_3 \ldots\}$ be the set of indices for all the tight constraints for F_1.

Let $S_2 = \{j_1, j_2, \ldots\}$ be the same for F_2.

Then, the face defined by the tight constraints $S_1 \cap S_2$ will maximize $w^T x$.

proof of $\underline{\text{Claim 2}}$ consider points $z_1 \in F_1$ and $z_2 \in F_2$ s.t. For z_1, all the constraints in S_1 are tight and all the constraints outside S_1 are non-tight (strict inequality).

Similary for z_2, all the constraints in S_2 are tight and other constraints are non-tight.

Consider $\frac{z_1+z_2}{2}$. The set of tight constraints will exactly $S_1 \cap S_2$.

Argue that $\frac{z_1+z_2}{2}$ is a maximizing point. (Homework)

From claim 1, we will get that the minimal face containing $\frac{z_1+z_2}{2}$ (i.e., face defined by $S_1 \cap S_2$) is maximizing.

Proof of Lemma 1: follows from Claim1 & Claim 2.

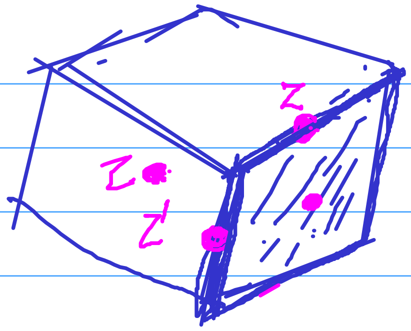
Homework

Lemma 2: For every face F of a polyhedron P, there exists a function $\omega^T x$ such that the face maximizing $\omega^T x$ is F.

Homework

Corollary of Lemma 1: For any optimizing function $\omega^T x$, there is some corner/vertex of the polyhedron that maximizes $\omega^T x$.

How do you define a corner?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