Mar 23

So far we have seen some applications of LP techniques but we did not need to solve an LP for those applications.

Next we will see how approximation algorithms can be designed using an LP-solver.

We will see some general purpose LP-solving methods towards the end of this course.

General Strategy:

1) Express your optimization problem as an Integer program

2) Relax the integer constraints to make it a linear program.

3) Find an optimal solution for the LP using an LP-solver.

4) If the solution is non-integral then convert it into an integral solution by a rounding procedure

5) Compare the obtained solution with the optimal solution. Ideally we should compare it with the integral optimal solution. But we have no clue about the integral optimal solution. So, instead compare it with LP optimal solution and get an approximation guarantee.

What is rounding?

$$\underset{\text{LP opt}}{x^*} \longrightarrow \underset{\text{Integral}}{x}$$

Example (1)

$$x_i = \begin{cases} 0 & \text{if } 0 \leq x_i^* < 1/2 \\ 1 & \text{if } \frac{1}{2} \leq x_i^* < 1 \end{cases}$$

(2) Threshold $0 \leq \lambda \leq 1$

$$x_i = \begin{cases} 0 & \text{if } x_i^* < \lambda \\ 1 & \text{if } x_i^* \geq \lambda \end{cases}$$

(3) Randomized rounding

$$x_i = \begin{cases} 0 & \text{with prob } 1 - x_i^* \;/\; 1 - f(x_i^*) \\ 1 & \text{with prob } x_i^* \;/\; f(x_i^*) \end{cases}$$

Reference: Williamson Shmoys Section 5.1 - 5.5

Maximum Satisfiability Problem (NP-hard)

Given a set of clauses we want satisfy maximum number of clauses

Def: clause is an OR of literals

$$x_1 \vee \overline{x}_2$$

Ex

$$(x_1 \vee \overline{x}_2), (x_2 \vee \overline{x}_3), (\overline{x}_3 \vee \overline{x}_1), (x_3 \vee x_4), (\overline{x}_4 \vee x_3)$$

$x_1 = 1$ $x_2 = 1$ $x_3 = 1$ $x_4 = 0$

(4) HW max

Maximum Weight Satisfiability

Each clause comes with a weight.

Maximize total weight of satisfied clauses

Greedy Algorithm?

$x_i \rightarrow$ no. of clauses with x_i

$\quad \rightarrow$ no. of clauses with $\bar{x}_i$

1/2 - approx

A Randomized algorithm:

set each x_i to 1 with prob. 1/2

and to 0 with Prob 1/2

Independently.

What is the expected weight of the satisfied clauses?

How does it compare with the optimal?

$W \leftarrow$ total weight of satisfied clauses.

$$E[W] = \sum_{A \in \{0,1\}^n} \frac{1}{2^n} \left(\text{total weight of satisfied clauses by } A \right)$$

$$\boxed{\Pr(x = A) = 1/2^n}$$

Linearity of expetation:

Let $w_1, w_2, \ldots, w_m$ be the weights of the given clauses.

Let us define a random Variable $W(C_j)$ for each clause C_j.

$$W(C_j) = \begin{cases} w_j & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise.} \end{cases}$$

Claim $\quad W = \sum_{j=1}^{m} W(C_j)$

By linearity of expectation

$$E[W] = \sum_{j=1}^{m} E[W(C_j)]$$

HW

$$= \sum_{j=1}^{m} w_j \times \Pr[C_j \text{ is satisfied}] + 0 \times \Pr[C_j \text{ is not satisfied}]$$

What is the probability that C_j is satisfied when each variable is assigned 0 or 1 with prob $1/2$.

$C_j = x_1 \vee \bar{x}_2 \quad \Pr = 3/4$

$C_j = x_2 \vee x_3 \vee \bar{x}_4$

$\Pr[C_j \text{ is satisfied}] = 7/8$

$$\Pr[C_j \text{ is satisfied}] = 1 - \frac{1}{2^{l_j}}$$

where l_j is the number of literals in clause C_j

$$E[W] = \sum_{j=1}^{m} w_j \left(1 - \frac{1}{2^{l_j}}\right)$$

$$l_j \geq 1 \Rightarrow 1 - \frac{1}{2^{l_j}} \geq 1/2$$

$$E[W] \geq \sum_{j=1}^{m} w_j \times 1/2 = \frac{1}{2}\left(\sum_{j=1}^{m} w_j\right)$$

$$\geq \frac{1}{2} OPT$$

If all clauses are large then better approximation factor.

$\forall j \; l_j \geq 3 \Rightarrow$ Approx factor $= 7/8$.

. Deterministic Implementation of above

there exists at least one assignment A for which $W_A \geq E[W]$

Find A via self reducibility and conditional expectation

$$E[W \mid x_1 = 1] = \sum_{A \in \{0,1\}^{n-1}} \frac{1}{2^{n-1}} \cdot \left(\begin{array}{l}\text{total weight of the} \\ \text{satisfied clauses} \\ \text{under } A \text{ and} \\ x_1 = 1\end{array}\right)$$

Claim

HW

$$E[W] = \frac{E[W \mid x_1=0] + E[W \mid x_1=1]}{2}$$

More generally,

$$E[W \mid x_2=0, x_3=1] = \frac{E[W \mid x_2=0, x_3=1, x_1=0] + E[W \mid x_2=0, x_3=1, x_1=1]}{2}$$

Fact: computing $E[W]$ is easy by linearity of expectation.

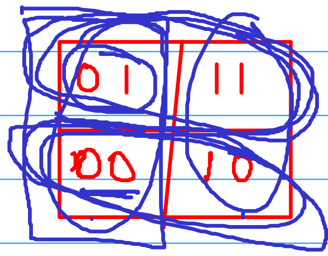
Similarly, computing $E[W \mid x_2=0, x_3=1]$ etc. is easy.

HW

Deterministic Strategy

One of $E[W \mid x_1=0]$ and $E[W \mid x_1=1]$ must be larger than $E[W]$.

Set x_1 based on that.



If $E[W \mid x_1=0] > E[W \mid x_1=1]$ then set $x_1=0$. otherwise $x_1=1$.

Say we have set $x_1=0$.

If $E[W \mid x_1=0, x_2=1] > E[W \mid x_1=0, x_2=0]$ then set $x_2=1$.

continue for each variable

HW Prove that the above deterministic algorithm gives objective value as good as the expected objective value of the randomized algorithm.

Example

$$(\bar{x}_1 \vee x_2),\ (\bar{x}_2 \vee x_3),\ (\bar{x}_3 \vee \bar{x}_1),\ (x_1)$$

weights 16 20 12 18

$$E[W] = \frac{3}{4} \times 16 + \frac{3}{4} \times 20 + \frac{3}{4} \times 12 + \frac{1}{2} \times 18$$

$$= 12 + 15 + 9 + 9 = 45$$

when each of x_1, x_2, x_3 is set with prob $1/2$

The above numbers in red show the probability of a clause being satisfied under a random assignment.

Now we want to find $E[W \mid x_1 = 0]$ & $E[W \mid x_1 = 1]$

we will fix x_1 and let x_2 and x_3 be randomly chosen

$$E[W \mid x_1 = 0] = 1 \times 16 + \frac{3}{4} \times 20 + 1 \times 12 + 0 \times 18$$

$$= 43$$

$$E[W \mid x_1 = 1] = \frac{1}{2} \times 16 + \frac{3}{4} \times 20 + \frac{1}{2} \times 12 + 1 \times 18$$

$$= 47$$

Since $E[W \mid x_1=1] > E[W \mid x_1=0]$

let us set $x_1=1$

Now we want to decide about x_2.

Let x_1 be fixed to 1.

Compute $E[W \mid x_1=1, x_2=0]$ and $E[W \mid x_1=1, x_2=1]$

Here x_2 will be fixed.

and only x_3 will be randomly chosen.

$$E[W \mid x_1=1, x_2=0] = 0 \times 16 + 1 \times 20 + \frac{1}{2} \times 12 + 1 \times 18$$
$$= 44$$

$$E[W \mid x_1=1, x_2=1] = 1 \times 16 + \frac{1}{2} \times 20 + \frac{1}{2} \times 12 + 1 \times 18$$
$$= 50$$

Since $E[W \mid x_1=1, x_2=1] > E[W \mid x_1=1, x_2=0]$

We set $x_2=1$

Now we want to decide about x_3

We will compute

$E[W \mid x_1=1, x_2=1, x_3=0]$ and $E[W \mid x_1=1, x_2=1, x_3=1]$

Everything is fixed in these computations.

No variable is random.

$E[W \mid x_1=1, x_2=1, x_3=0]$

$= 1 \times 16 + 0 \times 20 + 1 \times 12 + 1 \times 18$

$= 46$

$E[W \mid x_1=1, x_2=1, x_3=1]$

$= 1 \times 16 + 1 \times 20 + 0 \times 12 + 1 \times 18$

$= 54$

Since $E[W \mid x_1=1, x_2=1, x_3=1] > E[W \mid x_1=1, x_2=1, x_3=0]$

we set $x_3 = 1$

Final assignment $x_1=1$ $x_2=1$ $x_3=1$

weight of satisfied clauses $= 54$

Clearly greater than the expectation $E[W]=45$

Coincidentally, 54 is the optimal value.

There was a question in the class :

Can we set x_1, x_2, x_3 simultaneously ?

The idea was to compare

$E[W|x_1=0]$ with $E[W|x_1=1]$, $\Rightarrow$ decide x_1

$E[W|x_2=0]$ with $E[W|x_2=1]$ $\Rightarrow$ decide x_2

and $E[W|x_3=0]$ with $E[W|x_3=1]$ $\Rightarrow$ decide x_3

Let's try this with above example.

$$\left.\begin{aligned} E[W|x_1=0] &= 43 \\ E[W|x_1=1] &= 47 \end{aligned}\right\} \Rightarrow x_1=1$$

$$E[W|x_2=0] = \frac{1}{2}\times 16 + 1\times 20 + \frac{3}{4}\times 12 + \frac{1}{2}\times 18$$
$$= 46$$

$$E[W|x_2=1] = 1\times 16 + \frac{1}{2}\times 20 + \frac{3}{4}\times 12 + \frac{1}{2}\times 18$$
$$= 44$$

$$\Rightarrow x_2 = 0$$

$$E[W|x_3=0] = \frac{3}{4}\times 16 + \frac{1}{2}\times 20 + 1\times 12 + \frac{1}{2}\times 18$$
$$= 43$$

$$E[W|x_3=1] = \frac{3}{4}\times 16 + 1\times 20 + \frac{1}{2}\times 12 + \frac{1}{2}\times 18$$
$$= 47$$

$$\Rightarrow x_3 = 1$$

We get $x_1=1, x_2=0, x_3=1$

Total weight of Satisfied Clauses $= 20+18 = 38$

Less than $E[W]=45$. Bad Strategy.

March 30

Can we improve $1/2$ factor?

Yes, by LPs. $1 - 1/e \approx 0.63$

LP Variable z_j for clause C_j

$$\max \sum_{j=1}^{m} w_j z_j$$

$0 \le z_j \le 1$

$0 \le y_i \le 1$ $\quad y_i$ for each Boolean x_i

$$z_j \le \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i)$$

Def $P_j \leftarrow \{ i : x_i \text{ appears in clause } C_j \}$

$N_j \leftarrow \{ i : \bar{x}_i \text{ appears in clause } C_j \}$

Rounding Solve this LP.

Let (y^*, z^*) be the optimal solution for the LP.

set x_i to TRUE with prob. y_i^*

and FALSE with prob $1 - y_i^*$

expected weight of satisfied clauses $E[W]$?

$$E[W] = \sum_{j=1}^{m} w_j \times (\text{Prob of } C_j \text{ being satisfied})$$

$$E[W] = \sum_{j=1}^{m} w_j \underbrace{\left(1 - \left[\prod_{i\in P_j}(1-y_i^*)\right]\left[\prod_{i\in N_j} y_i^*\right]\right)}_{\geq \beta \cdot z_j^* \text{ (to show)}}$$

AM $\geq$ GM

$$\left(\frac{\alpha_1+\alpha_2+\cdots+\alpha_K}{K}\right)^K \geq (\alpha_1\cdot\alpha_2\cdot\cdots\alpha_K)$$

$$E[W] \geq \sum_{j=1}^{m} w_j\left[1 - \left(\frac{\sum_{i\in P_j}(1-y_i^*) + \sum_{i\in N_j} y_i^*}{l_j}\right)^{l_j}\right]$$

where $l_j = |P_j| + |N_j|$

Recall $z_j^* \leq \sum_{i\in P_j} y_i^* + \sum_{i\in N_j}(1-y_i^*)$

$$\Rightarrow l_j - z_j^* \geq l_j - \left(\sum_{i\in P_j} y_i^* + \sum_{i\in N_j}(1-y_i^*)\right)$$

$$= \sum_{i\in P_j}(1-y_i^*) + \sum_{i\in N_j} y_i^*$$

$$E[W] \geq \sum_{j=1}^{m} w_j\left(1 - \left(\frac{l_j - z_j^*}{l_j}\right)^{l_j}\right)$$

$$= \sum_{j=1}^{m} w_j \underbrace{\left(1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}\right)} \geq \beta \sum w_j z_j^*$$

Apr 1

would like to show

$$\left(1-\left(1-\frac{z_j^*}{l_j}\right)^{l_j}\right) \geq \beta \cdot z_j^*$$

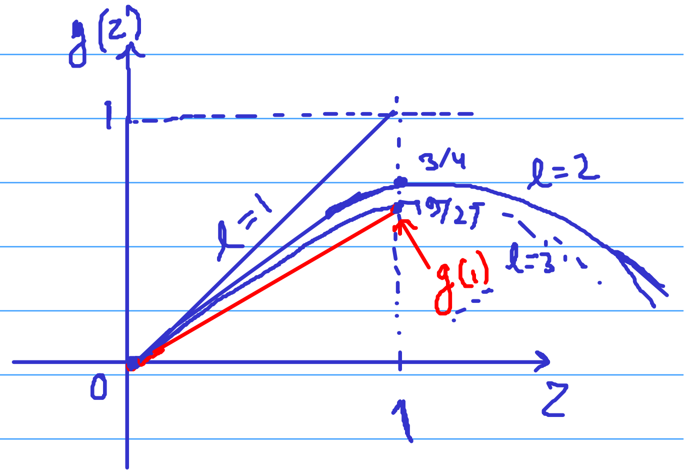
for some $\beta > 0$

Define $g(z) = 1-\left(1-\frac{z}{l}\right)^l$

$l=1 \Rightarrow g(z) = z$

$l=2 \Rightarrow z-\frac{z^2}{4}$

$l=3 \Rightarrow g(1) = \frac{19}{27}$



Claim: $g(z)$ is a concave function in $[0,1]$
you can verify that $g''(z) \leq 0$.

HW

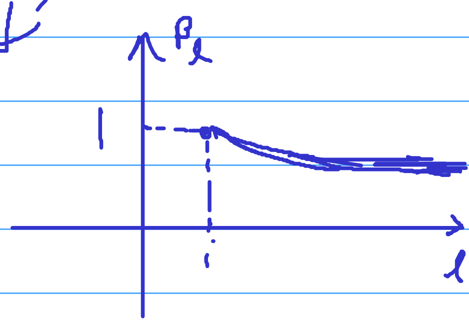
Observation $g(z) \geq g(1)z$

$$1-\left(1-\frac{z}{l}\right)^l \geq z\underbrace{\left(1-\underbrace{\left(1-\frac{1}{l}\right)^l}_{1/e}\right)}_{\beta_l} \quad \text{for } z\in[0,1]$$

$l=1 \Rightarrow \beta_l = 1$

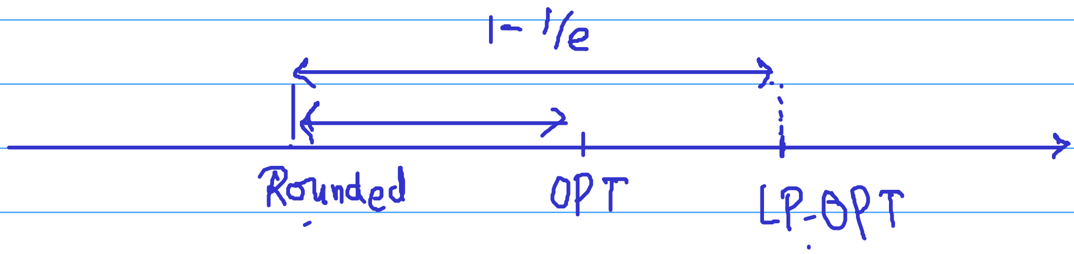
$l=2 \Rightarrow \beta_l = 3/4$

$l=3 \Rightarrow \beta_l = 19/27$



$l\to\infty \Rightarrow \beta_l \to 1-\frac{1}{e}$

$$E[W] \geq \left(1-\frac{1}{e}\right) \cdot \sum_{j=1}^{m} w_j z_j^* = \left(1-\frac{1}{e}\right) \text{LP-OPT}$$



$$E[W] \geq (1 - 1/e) \cdot OPT$$

$$E[W] \geq (1 - 1/e) \cdot OPT$$

Deterministic Implementation ?

by computing conditional expectations.

Will get something as good as $E[W]$.

Can we further improve from $(1 - 1/e)$?

First randomized scheme $\quad pr = 1/2$

$$E[W] = \sum_j \left(1 - \frac{1}{2^{l_j}}\right) w_j$$

worst case $l_j = 1$

LP based Scheme $\quad pr = y_i^*$

$$E[W] \geq \sum_j \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) w_j z_j^*$$

Best case $l_j = 1$

Bad for large clauses

Combination of the two schemes.

$x_i \leftarrow$ TRUE with prob $\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}y_i^* = \frac{1}{4} + \frac{y_i^*}{2}$

HW

Show that

$$E[W] \geq \sum_j w_j \left(1 - \left[\frac{1}{4} + \frac{1}{2}\left(1 - \frac{z_j^*}{l_j}\right)\right]^{l_j}\right)$$

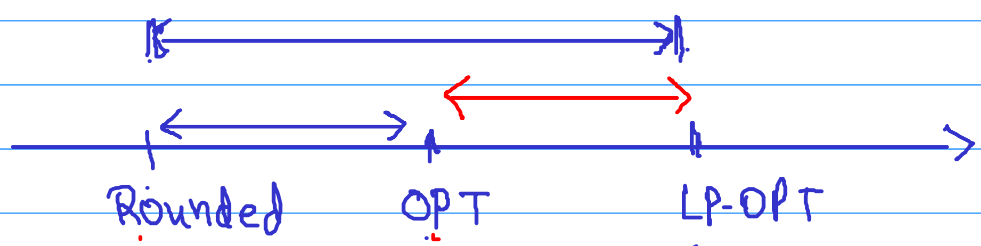
$$\Rightarrow \quad E[W] \geq \frac{3}{4} \sum_j w_j z_j^*$$

Deterministic

Is there a rounding scheme with better than 3/4 approx guarantee?

Integrality Gap of an LP formulation

$$\min_{I} \frac{OPT(I)}{LP\text{-}OPT(I)}$$



Example where integrality gap is 3/4

$$(x_1 \vee x_2),\quad (x_1 \vee \bar{x}_2),\quad (\bar{x}_1 \vee x_2),\quad (\bar{x}_1 \vee \bar{x}_2)$$

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \end{array}$$

OPT = 3 $\qquad$ LP-OPT = 4.