Week 2 Jan 18, Monday.

So far what we have seen:

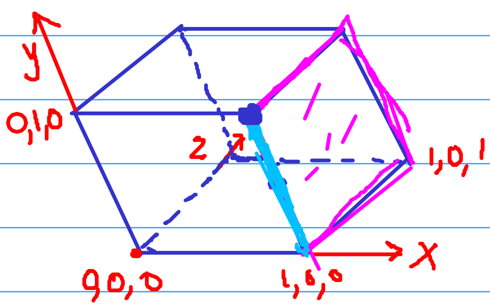
- The feasible region of a Linear Program, that is the set of points satisfying a given set of linear inequalities is called a polyhedron.

- A polyhedron is convex

- Face: points lying on a boundary of a polyhedron. Formally points in the polyhedron for which a set of constraints is tight form a face.

- Optimizing Points: for any linear function, the set of points in a polyhedron maximizing/minimizing the function form a face.

- Every face maximizes some linear function.



$0 \le x \le 1$

$0 \le y \le 1$

$0 \le z \le 1$

Face
$x = 1$
$f(x) = x$

Face 2 $x=1,\ z=0$
$f(x) = x - z$

Face3 $x=1,\ z=0$, $y=1$
$f(x) = x+y-z$

- Characterization of a face: a set of points maximizing a linear function.

Corollary: there is always a corner point that is one of the maximizing points.

What is a Corner/vertex?

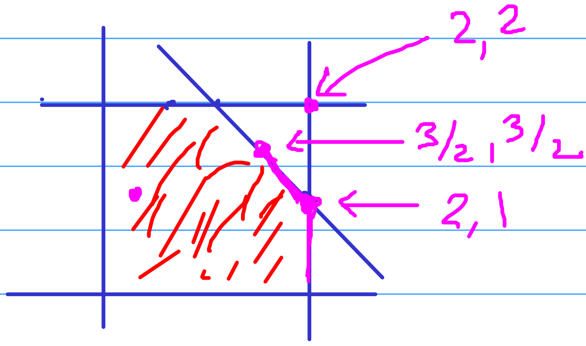
Corners / vertices — Multiple Definitions.

1) Corner is a zero dimensional face.

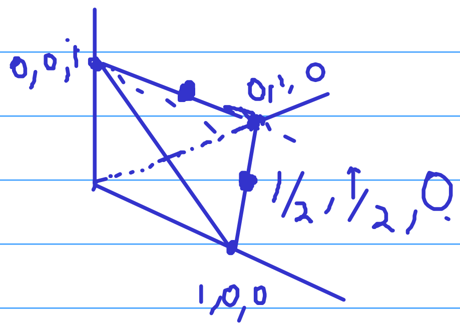
To elaborate, a point in the polyhedron for which there are n linearly independent tight constraints.

$\uparrow$ dimension / no. of variables

$x \geq 0$
$y \geq 0$
$\boxed{x \leq 2}$ ✓
$y \leq 2$
$\boxed{x + y \leq 3}$ ✓



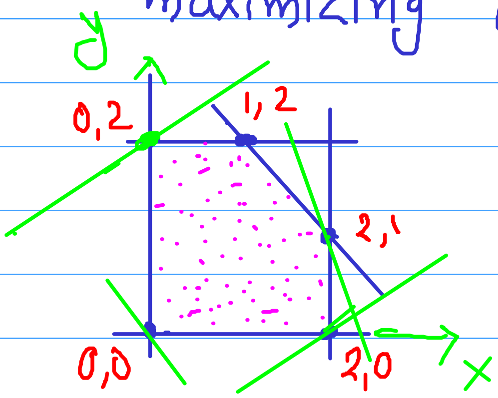
$x, y, z \geq 0$
$x + y + z \leq 1$
$x + y \leq 1$ (redundant)



Not independent

$$\begin{cases} x + y + z = 1 \\ z = 0 \\ x + y = 1 \end{cases}$$

2) A point z is a corner of a polyhedron P if there is a linear function $w^T x$, such that z is the unique point maximizing $w^T x$ over P.

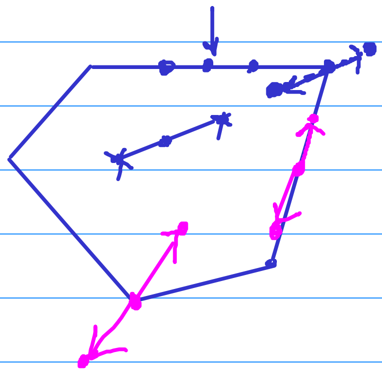


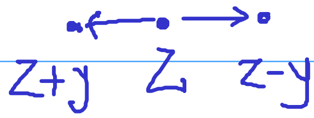
0, 2	$f(x) = y - x$
2, 0	$f(x) = x - y$
0, 0	$f(x) = -x - y$
2, 1	$f(x) = 2x + y$

3)

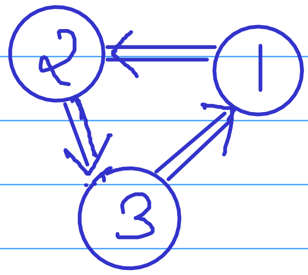
z is a corner point if it is not a mid-point of two distinct points in P.

Equivalently, for every $y \in \mathbb{R}^n$, if $z+y \in P$ then $z-y \notin P$.





Equivalence between the three definitions.



2 $\Rightarrow$ 3

$\exists\, w$ s.t. z is the unique point maximizing $w^T x$.

for some $y \in \mathbb{R}^n$, $z+y \in P$

$$w^T(z+y) < w^T z$$

$$w^T y < 0$$

$$\Rightarrow w^T(z-y) = w^T z - w^T y > w^T z$$

$$z-y \notin P.$$

① $\Rightarrow$ ②

(i) z is corner if there are n lin Ind tight constraints.

<u>P</u>

$$a_1^T x \le b_1$$
$$a_2^T x \le b_2$$
$$\vdots$$
$$a_m^T x \le b_m$$

<u>say</u>, z satisfies

$$\left.\begin{array}{l} a_1^T z = b_1 \\ a_2^T z = b_2 \\ \vdots \\ a_n^T z = b_n \end{array}\right]$$

<u>To show</u> $\exists\ w^T x$ s.t. z is the unique point maximizing.

$$w = a_1 + a_2 + a_3 + \dots + a_n$$

$$w^T z = a_1^T z + a_2^T z \cdots + a_n^T z = \underline{b_1 + b_2 + \dots + b_n}$$

for any $x \in P$

$$w^T x = a_1^T x + a_2^T x \cdots + a_n^T x \le b_1 + b_2 \dots + b_n$$

(from the constraints)

Thus, z maximizes $w^T x$.

Why is z unique?

n lin independent equations $\Longrightarrow$ unique solution (which is z)

Thus, for any point x different from z we must have one of the first n constraints with <u>strict inequality</u>.

$$\Longrightarrow \quad w^T x < \underline{b_1 + b_2 \cdots + b_n}$$

3 ⇒ 1

3 says that z is a corner of P if for any $y \in \mathbb{R}^n$ $z+y \in P \not\Rightarrow z-y \in P$

for 1 we would like to show that there are n linearly independent constraints that are tight for z.

For the sake of contradiction, Suppose this is not true.

Let the following constraints are tight for z.

$$a_i^T z = b_i, \text{ for } 1 \leq i \leq k$$

and the rest are not tight i.e.,

$$a_i^T z < b_i \quad \text{for } k < i \leq m$$

By assumption, rank $(a_1, a_2, \dots, a_k) < n$.

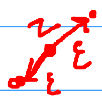
Consider the system

$y = (y_1, y_2, \dots, y_n)$

$$\left.\begin{aligned} a_1^T y &= 0 \\ a_2^T y &= 0 \\ &\vdots \\ a_k^T y &= 0 \end{aligned}\right\}$$

There exists a nonzero solution. (because rank $< n$)

Now we claim that for a small enough ε,



$$z + \varepsilon y \in P, \quad z - \varepsilon y \in P$$

This will contradict the criterion in 3.

$z+\varepsilon y \in P$? Need to verify the constraints

for $1 \leq i \leq k$ $\qquad (a_i^T y = 0) \leftarrow$

$$a_i^T (z \pm \varepsilon y) = a_i^T z = b_i \checkmark$$

for $i > k$

$$a_i^T (z+\varepsilon y) = \underbrace{a_i^T z}_{\boxed{<b_i}} + \underbrace{\varepsilon a_i^T y}_{\text{small enough}} < b_i \checkmark$$

In conclusion,
$z+\varepsilon y \in P$ as it satisfies all the constraints

By the same argument $z-\varepsilon y \in P$

As mentioned above, this leads to a contradiction.

Jan 19

Def (convex hull): For a set of points $\{P_1, P_2, P_3, \ldots, P_K\} \subseteq \mathbb{R}^n$, its convex hull is the set of all points that can be expressed as a convex combination of $P_1, P_2, P_3, \ldots, P_K$.

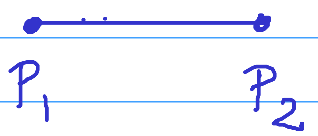
Formally,

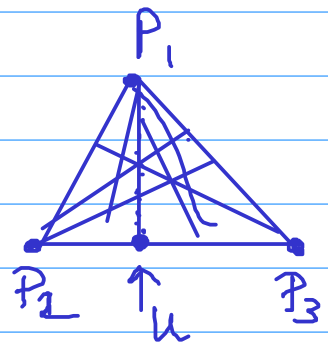
$(1-\lambda) P_1 + \lambda P_2$, $\lambda \geq 0$

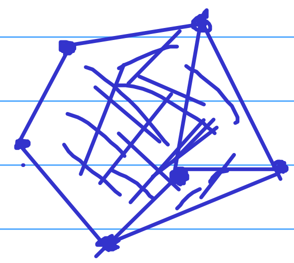
$$\text{Convex.hull}\{P_1, P_2, P_3, \ldots, P_K\}$$

$$= \{\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_K P_K : \lambda_1, \lambda_2, \ldots \lambda_K \geq 0, \ \lambda_1 + \lambda_2 + \cdots + \lambda_K = 1\}$$

Examples (2-dim)

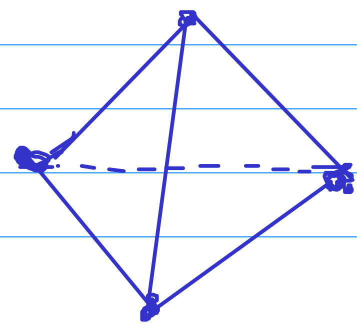




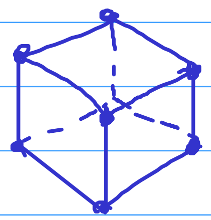


$$\lambda_1 P_1 + (1-\lambda_1)\Big[\underbrace{\lambda_2 P_2 + (1-\lambda_2) P_3}_{u}\Big] = \lambda_1 P_1 + \lambda_2 (1-\lambda_1) P_2 + (1-\lambda_1)(1-\lambda_2) P_3$$

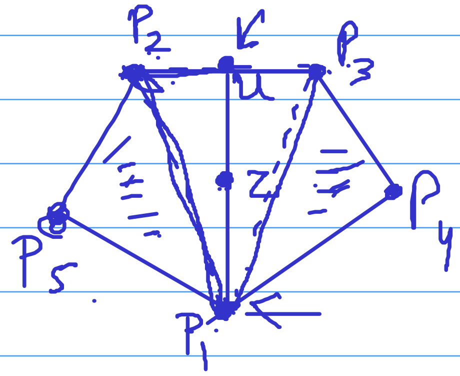
Examples (3-dim)



Claim: Any polytope is the convex hull of its (finitely many) corners.



2-dim



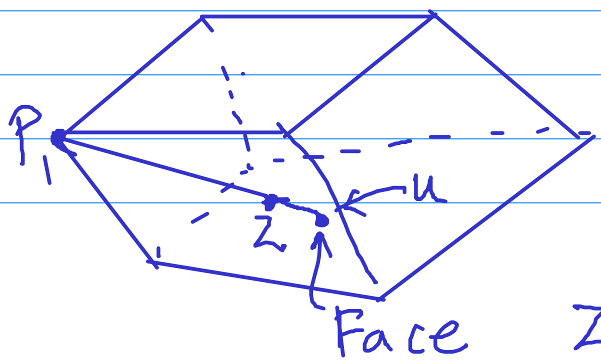
Theorem (Carathéodory)

Any point in a polytope is convex combination of at most $n+1$ corners. (n = dimension).

Proof by Induction:

Base Case (dim = 1)

Induction Step: Assume that the statement is true for $n-1$ dim polytopes.



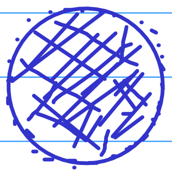
Face dim $< n$

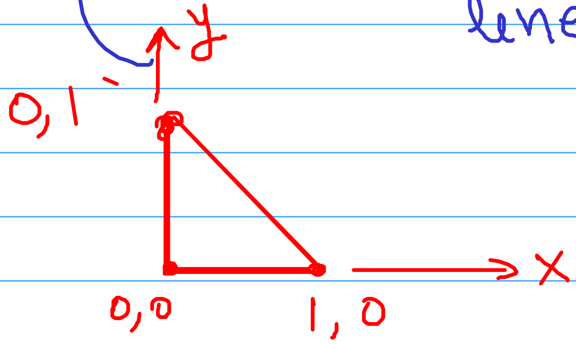
u lies on this face

$u = \underbrace{\text{conv}(n \text{ corners})}_{\text{induction}}$

$Z = \text{conv}(P_1, u)$.

Claim 2 Any convex hull of finitely many points is a polytope (described by finitely many linear inequalities)





$x \geq 0$

$y \geq 0$

$x + y \leq 1$

Proof:

$\text{convex hull}\{P_1, P_2, \cdots P_K\} \subseteq \mathbb{R}^n$

$x \in \mathbb{R}^n \rightarrow x = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_K P_K$

$$\boxed{\begin{aligned} &\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_K \geq 0 \\ &\lambda_1 + \lambda_2 + \cdots + \lambda_K = 1 \\ &x_1 = \lambda_1 P_{11} + \lambda_2 P_{21} \cdots + \lambda_K P_{K1} \\ &\vdots \\ &x_n = \lambda_1 P_{1n} + \lambda_2 P_{2n} \cdots + \lambda_K P_{Kn}. \end{aligned}}$$

$$\begin{aligned} x + y &= 1 \\ 2x + y &= 2 \\ \hline x &= 1 \end{aligned}$$

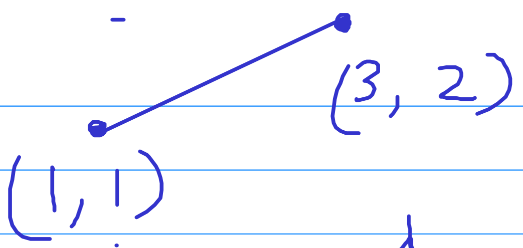
Fourier Motzkin Elimination.
(Eliminating Variables)

$$\left.\begin{aligned} \lambda_1 &\geq E_1 \\ \lambda_1 &\geq E_2 \\ \lambda_1 &\leq E_3 \\ \lambda_1 &= E_4 \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} E_4 &\geq E_1 \\ E_4 &\geq E_2 \\ E_4 &\leq E_3 \end{aligned}\right.$$

$E_1 \leq E_3$

$E_2 \leq E_3$

The set of solutions are same



$\lambda_1 + \lambda_2 = 1$

$\lambda_2 = \underline{1 - \lambda_1}$

λ_2

$$(x, y) = \overset{\lambda_1}{\lambda}(1, 1) + \overset{\lambda_2}{(1-\lambda)}(3, 2)$$

$$\boxed{\begin{aligned} x &= \lambda + (1-\lambda)\cdot 3 \\ y &= \lambda + (1-\lambda)\cdot 2 \\ \lambda &\geq 0 \end{aligned}}$$

$\lambda_1, \lambda_2, \lambda_{k-1}$

$1 - \lambda_1 - \lambda_2 \cdots - \lambda_{k-1}$

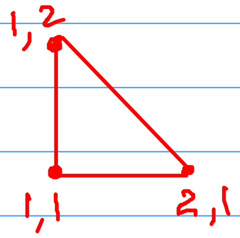
$\cancel{} = \quad x = 3 - 2\lambda$

$$\lambda = \frac{3-x}{2}$$

$$\lambda = 2 - y$$

$$\boxed{\begin{aligned} \frac{3-x}{2} = 2 - y &\Rightarrow \underline{2y - x = 1} \\ 2 - y \geq 0 &\Rightarrow \underline{y \leq 2} \end{aligned}}$$

Fourier Motzkin elimination (example 2)



$$(x,y) = \lambda_1(1,1) + \lambda_2(2,1) + \lambda_3(1,2)$$

$$\boxed{\begin{aligned} &\lambda_1+\lambda_2+\lambda_3 = 1 \\ &\lambda_1, \lambda_2, \lambda_3 \geq 0 \\ &x = \lambda_1 + 2\lambda_2 + \lambda_3 \\ &y = \lambda_1 + \lambda_2 + 2\lambda_3 \end{aligned}}$$

Eliminating λ_1

substitute $\lambda_1 = 1 - \lambda_2 - \lambda_3$ in other constraints

$$\boxed{\begin{aligned} &\lambda_2, \lambda_3 \geq 0 \qquad 1-\lambda_2-\lambda_3 \geq 0 \\ &x = 1+\lambda_2 \\ &y = 1+\lambda_3 \end{aligned}}$$

Eliminating λ_2

substitute $\lambda_2 = x-1$

$$\boxed{\begin{aligned} &x-1 \geq 0, \quad \lambda_3 \geq 0 \\ &2-x-\lambda_3 \geq 0 \\ &y = 1+\lambda_3 \end{aligned}}$$

Eliminating λ_3

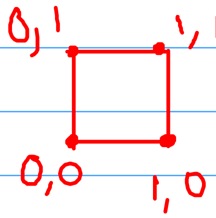
substitute $\lambda_3 = y-1$

$$\boxed{\begin{aligned} &x-1 \geq 0 \qquad y-1 \geq 0 \\ &2-x-y+1 \geq 0 \end{aligned}}$$

equivalently,

$$\boxed{\begin{aligned} &x \geq 1 \\ &y \geq 1 \\ &x+y \leq 3 \end{aligned}}$$

Example 3



$$(x,y) = \lambda_1(0,0) + \lambda_2(1,0) + \lambda_3(0,1) + \lambda_4(1,1)$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$$

$$x = \lambda_2 + \lambda_4$$

$$y = \lambda_3 + \lambda_4$$

Eliminating λ_1

substitute $\lambda_1 = 1 - \lambda_2 - \lambda_3 - \lambda_4$

$$\boxed{\begin{array}{c} \lambda_2, \lambda_3, \lambda_4 \geq 0 \\ 1 - \lambda_2 - \lambda_3 - \lambda_4 \geq 0 \\ x = \lambda_2 + \lambda_4 \\ y = \lambda_3 + \lambda_4 \end{array}}$$

Eliminating λ_2

substitute $\lambda_2 = x - \lambda_4$

$$\boxed{\begin{array}{cc} \lambda_3, \lambda_4 \geq 0 & x - \lambda_4 \geq 0 \\ & 1 - \lambda_3 - x \geq 0 \\ & y = \lambda_3 + \lambda_4 \end{array}}$$

Eliminating λ_3,

substitute $\lambda_3 = y - \lambda_4$

$$\rightarrow \boxed{\begin{array}{c} \lambda_4 \geq 0 \quad y - \lambda_4 \geq 0 \quad x - \lambda_4 \geq 0 \\ 1 - x - y + \lambda_4 \geq 0 \end{array}}$$

Eliminating λ_4

$$\begin{array}{ll} \cdot\ y \geq \lambda_4 & \cdot\ \lambda_4 \geq 0 \\ \cdot\ x \geq \lambda_4 & \lambda_4 \geq x + y - 1 \end{array}$$

4 possible combinations.

$$\left|\begin{array}{cc} y \geq 0 & y \geq x + y - 1 \\ x \geq 0 & x \geq x + y - 1 \end{array}\right| \equiv \left|\begin{array}{cc} y \geq 0 & x \leq 1 \\ x \geq 0 & y \leq 1 \end{array}\right.$$

Jan 21

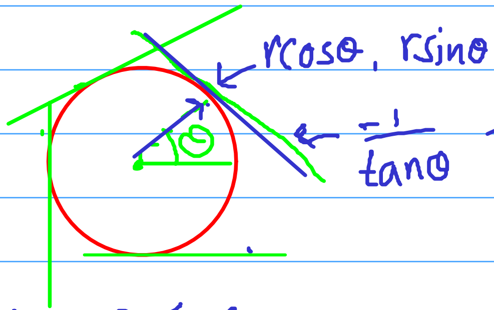
More generally any convex set is an intersection of halfspaces (possibly infinitely many).

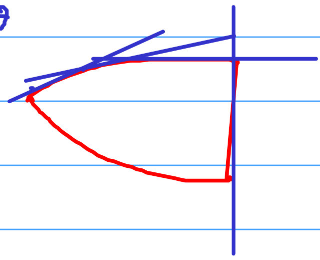
In other words, any convex set can be described by linear inequalities (possibly infinitely many).

$x^2+y^2 \le r^2$

$0 \le \theta \le 2\pi$

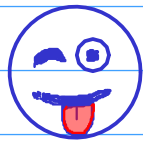
$$\boxed{\frac{y - r\sin\theta}{x - r\cos\theta} \le \frac{-1}{\tan\theta}}$$





$\Leftrightarrow \quad y\sin\theta + x\cos\theta \le r$

Proof: Fourier Motzkin elimination for infinitely many λ_i's



Solving LP via Fourier Motzkin elimination.

$$\begin{array}{l} \text{Max } \omega^T x \\ \text{s.t. } Ax \le b \end{array}$$

Write $\boxed{\begin{array}{l} \lambda = \omega^T x \\ Ax \le b \end{array}}$ and eliminate all the x variables.

You will end up with $\lambda \le 20$ $\boxed{\lambda \le 10}$ $\lambda \ge 0$ etc.

Example

$x_1 \ge 0$

$x_2 - x_1 \le 1$

$2x_1 + x_2 \le 2$

Max $x_1 + x_2$

$\lambda = x_1 + x_2$

$x_2 = \lambda - x_1$

$x_1 \ge 0$

$\lambda - 2x_1 \le 1$

$x_1 + \lambda \le 2$

$x_1 \ge 0$

$x_1 \ge \frac{\lambda - 1}{2}$

$x_1 \le 2 - \lambda$

$2 - \lambda \ge 0$

$2 - \lambda \ge \frac{\lambda - 1}{2}$

$\lambda \le 2$

$\lambda \le 5/3$

Running Time of this algorithm?

no. of inequalities can get squared in every iteration

After n iterations $\longrightarrow 2^{2^n}$ doubly exp.

We need to formally prove that the set of solutions is "preserved" under FM elimination.

(1)

$$\boxed{\begin{aligned} x_1 &\geq 0 \\ x_1 &\geq x_2 + x_3 \\ x_1 &\leq 2x_3 - x_2 \end{aligned}}$$

(2)

$$\boxed{\begin{aligned} 2x_3 - x_2 &\geq 0 \\ 2x_3 - x_2 &\geq x_2 + x_3 \\ &\cdots \end{aligned}}$$

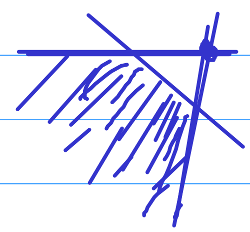
HW

$$\{(x_2, x_3) : x_2, x_3 \text{ satisfy } (2)\}$$
$$= \{(x_2, x_3) : \exists\, x_1 \text{ s.t. } x_1, x_2, x_3 \text{ satisfy } (1)\}$$

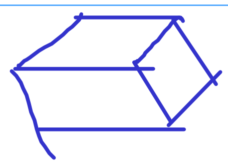
Another inefficient algorithm for finding OPT

Check each corner one by one.

Corner — n tight constraint



exponetial time



n-dim Hypercube 2n constraints 2^n corners.

Two LP questions

(1) Feasibility : Given $Ax \leq b$, $\exists x$? satisfying $Ax \leq b$

(2) Optimization: $\max w^T x$ s.t. $Ax \leq b$.

The two questions are equivalent.

Optimization $\leq$ Feasibility

$\max w^T x$
s.t. $Ax \leq b$

$A, b \longrightarrow \boxed{\text{Feasibility}}$ → yes if $\exists x$ $Ax \leq b$
→ No otherwise

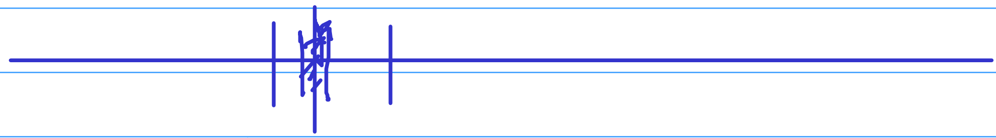
Binary search

$\left.\begin{matrix} Ax \leq b \\ w^T x \geq u \end{matrix}\right]$ feasibility → yes $OPT \geq u$
→ No $OPT < u$

$\left.\begin{matrix} Ax \leq b \\ w^T x \leq u \end{matrix}\right]$ feasibility → yes inconclusive.

Imp how many rounds.

if opt was integer log (opt value)



accuracy factor ε $OPT \pm \varepsilon$

How many iterations?

$\propto \log(1/\varepsilon) \cdot \log(\text{OPT value})$

HW bound m, n bits.