Feb 1 LP duality.

Last week

Feasibility $\in NP \cap coNP$

If $Ax \le b$ not feasible then a positive combination of the constraints gives you

$$0 \le -1 \quad \text{(Farkas' Lemma)}$$

Optimization

$$\max w^T x \quad s.t. \quad Ax \le b$$

Any feasible point z gives a lower bound

$$w^* \geq w^T z$$

A positive combination of the constraints may give an upper bound on the optimal value w^*.

Max $5x + 6y$

$x \geq 0$

$y \geq 0$

$x \le 2$

$y \le 2$

$x + y \le 3$

point $(1, 2)$

value $f(1,2) = 17 \le w^*$

Upper bound from the constraints.

$$\begin{aligned} x+y &\le 3 \quad \times 5 \\ y &\le 2 \quad \times 1 \\ \hline 5x + 6y &\le 17 \end{aligned}$$

$w^* \le 17$

Dual LP

LP 1

$$\text{Max } f(x) = w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$$

s.t. $y_1 \times \{ a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1$

$y_2 \times \{ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2$

$\vdots$

$y_m \times \{ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m$

$$\longrightarrow \quad w_1 x_1 + w_2 x_2 + \cdots + w_n x_n \le \sum_{i=1}^{m} y_i b_i$$

$$\text{Min } g(y) = y_1 b_1 + y_2 b_2 + \cdots + y_m b_m$$

Dual LP

$$\begin{cases} y_1, y_2, \cdots y_m \ge 0 \\ y_1 a_{11} + y_2 a_{21} + \cdots + y_m a_{m1} = w_1 \\ y_1 a_{12} + y_2 a_{22} + \cdots + y_m a_{m2} = w_2 \\ \vdots \\ y_1 a_{1n} + y_2 a_{2n} + \cdots + y_m a_{mn} = w_n \end{cases}$$

LP1	LP 2
Max $w^T x$	min $b^T y$
s.t. $Ax \le b$	s.t. $A^T y = w$
	$y \ge 0$

Weak Duality Theorem

$$\left.\begin{array}{l} x \text{ feasible for LP 1} \\ y \text{ feasible for LP2} \end{array}\right\} \Rightarrow f(x) \leq g(y)$$

$$\boxed{OPT(LP1) \leq OPT(LP2)}$$

Strong Duality Theorem:

$$OPT(LP1) = OPT(LP2) \quad (\text{if they exist})$$

$$\text{Max}\{w^T x : Ax \leq b\} = \text{Min}\{b^T y : A^T y = w, y \geq 0\}$$

proof: FM Elimination.

$$\left.\begin{array}{l} \text{Max } z \\ \begin{cases} w^T x - z = 0 \\ Ax \leq b \end{cases} \end{array}\right] \quad w^* \leftarrow \text{Opt.}$$

Eliminate all the variables one by one except z.

Claim optimal value does not change HW

$$\left.\begin{array}{l} z \leq \alpha \\ z \geq \beta \\ z \leq \gamma \end{array}\right\} \quad w^* = \min(\alpha, \gamma) \quad \boxed{z \leq w^*}$$

$$Z \leq w^*$$

$$y_1 \times \{ \; a_{11}x_1 + a_{12}x_2 \; \cdots \; + a_{1n}x_n \leq b_1$$

$$\vdots \qquad \vdots$$

$$y_m \times \{ \; a_{m1}x_1 + a_{m2}x_2 \; \cdots \; + a_{mn}x_n \leq b_m$$

$$-1 = y_{m+1} \; \{ \; w_1 x_1 + w_2 x_2 \cdots + w_n x_n - Z \leq 0$$

$$\boxed{y_i \geq 0}$$

$$0 x_1 + 0 x_2 \; \cdots \; + 0 x_n + Z \leq w^*$$

$$\left\{ \begin{array}{l} y_1 b_1 + y_2 b_2 \cdots + y_m b_m = w^* \\ y_1 a_{11} + y_2 a_{21} + \cdots y_m a_{m1} - w_1 = 0 \\ \vdots \\ y_1 a_{1n} + y_2 a_{2n} + \cdots y_m a_{mn} - w_n = 0 \end{array} \right.$$

$$\left. \begin{array}{l} x^* \rightarrow \text{feasible} \\ y^* \rightarrow \text{feasible} \\ f(x^*) = g(y^*) \end{array} \right\} \Rightarrow \left[\begin{array}{l} x^* \text{ is opt point} \\ \text{for LP1} \end{array} \right.$$

y^* is opt point for dual LP.

Primal		Dual	
unbounded	$\implies$	Infeasible	(HW)
Infeasible	$\Longleftarrow$	unbounded.	(HW)
Infeasible		Infeasible.	example. (HW).
		possible.	

Dual LP of various forms.

Primal	Dual	
Max $\omega^T x$ s.t. $Ax \leq b$ $\boxed{x \geq 0}$	min $b^T y$ $A^T y \geq \omega$ $y \geq 0$	HW
Max $\omega^T x$ s.t. $Ax = b$ $x \geq 0$	min $b^T y$ $A^T y \geq \omega$ $\uparrow$	HW

Primal constraint $\rightarrow$ dual variable

$\leq \rightarrow \geq 0$

$= \rightarrow$

Dual of Dual $\underset{\text{equivalent}}{=}$ Original LP

HW.

Feb 2 Dual LP for various forms

$$\text{Max } 5x_1 + 6x_2$$
$$\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \end{cases} \quad \begin{array}{l} x_1 \leq 2 \\ x_2 \leq 2 \\ x_1 + x_2 \leq 3 \end{array}$$

$$\text{Max } 5x_1 + 6x_2$$
$$\begin{array}{rcll} x_1 & \leq & 2 & \leftarrow y_1 \\ x_2 & \leq & 2 & \leftarrow y_2 \\ x_1 + x_2 & \leq & 3 & \leftarrow y_3 \\ -x_1 & \leq & 0 & \leftarrow y_4 \\ -x_2 & \leq & 0 & \leftarrow y_5 \end{array}$$

Dual

$$\left.\begin{array}{l} \text{Min } 2y_1 + 2y_2 + 3y_3 \\ y_1, y_2, y_3, y_4, y_5 \geq 0 \\ y_1 + y_3 - y_4 = 5 \\ y_2 + y_3 - y_5 = 6 \end{array}\right\}$$

$$\boxed{\begin{array}{l} y_1 + y_3 \geq 5 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \\ \text{Min } 2y_1 + 2y_2 + 3y_3 \end{array}}$$

Writing dual for a general LP.

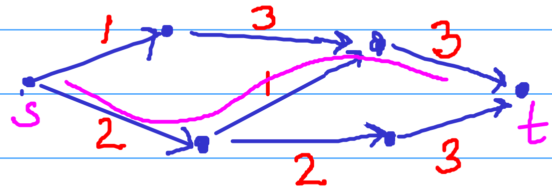
Primal:	Dual
Max $w_1 x_1 + w_2 x_2$	Min $b_1 y_1 + b_2 y_2 - b_3 y_3$
$a_{11} x_1 + a_{12} x_2 \leq b_1$	$y_1 \geq 0$
$a_{21} x_1 + a_{22} x_2 = b_2$	$y_2 \in \mathbb{R}$
$\cancel{a_{31} x_1 + a_{32} x_2 \geq b_3}$	$\cancel{y_3 \leq 0}$
$-a_{31} x_1 - a_{32} x_2 \leq -b_3$	$y_3 \geq 0$.
$x_1 \in \mathbb{R}$	$y_1 a_{11} + y_2 a_{21} - y_3 a_{31} = w_1$
$x_2 \geq 0$	$y_1 a_{12} + y_2 a_{22} - y_3 a_{32} \geq w_2$
$x_2 = -z_2$. $\boxed{x_2 \leq 0}$	$\leq w_2$
$z_2 \geq 0$	

LP for Combinatorial Problems

Shortest Path.

Directed Graph (V, E) with edge weights

Source vertex s
destination vertex t



find shortest path from s to t

a path is a subset of edges.

Not LP $\rightarrow$ $x_e \in \{0,1\}$ for $e \in E$

$x_e = 1 \rightarrow e$ is in path

$x_e = 0 \rightarrow e$ is not in the path.

Objective $\quad \min \sum_{e \in E} w_e x_e$

$0 \leq x_e \leq 1$

Constraints: $\quad \sum_{e \in Out(s)} x_e = 1$

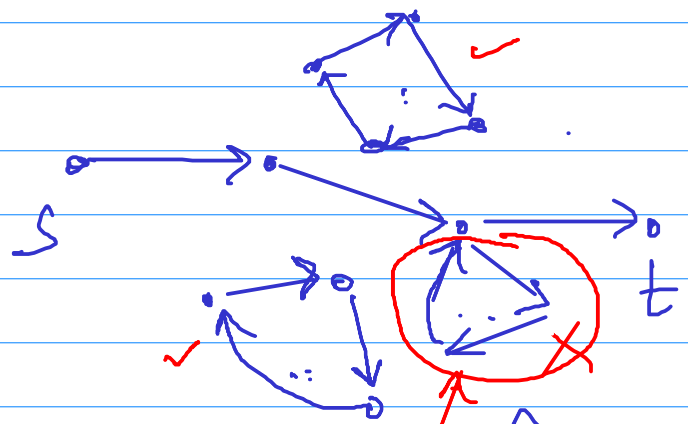
$\sum_{e \in IN(t)} x_e = 1$

for each $u \neq s, t$

$\sum_{e \in IN(u)} x_e = \sum_{e \in Out(u)} x_e \leq 1$

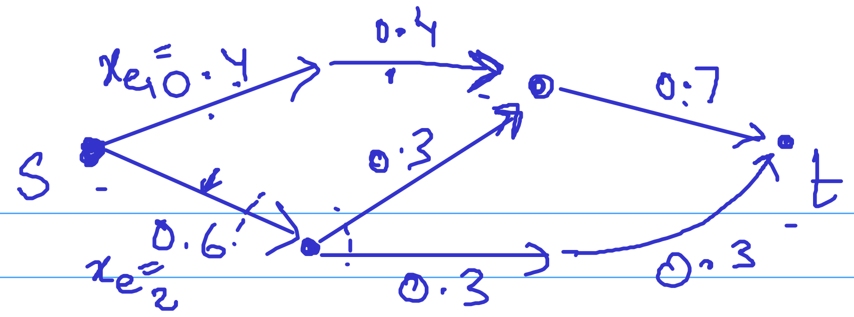
LP relaxation

redundant



optimal solution

Assumption: No negative weight cycles.



Feb 4 Shortest Path LP.

Claim: Optimal of LP is same as weight of the shortest path.

Proof via dual LP.

We will show a dual feasible solution whose objective value is same as the weight of the shortest path.

Primal

$$\text{Min} \sum_{e \in E} w_e x_e$$

$\rightarrow$ $0 \leq x_e$

$$-\sum_{e \in \text{Out}(s)} x_e = -1$$

$$\sum_{e \in \text{IN}(t)} x_e = 1.$$

for every vertex $u \neq s, t$

$$\sum_{e \in \text{IN}(u)} x_e - \sum_{e \in \text{Out}(u)} x_e = 0$$

Dual.

$y_v \in \mathbb{R}$ for $v \in V$

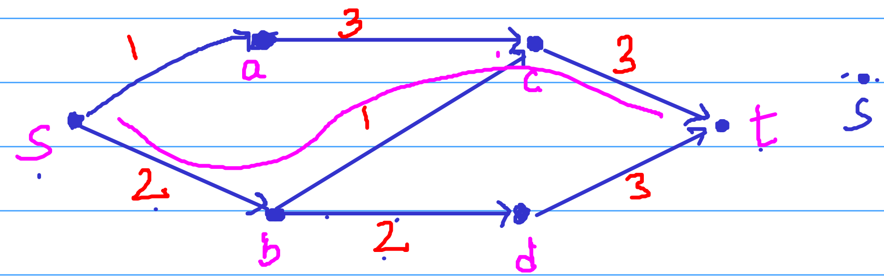
one constraint for each edge.

$e = (u, v)$ $u \rightarrow v$

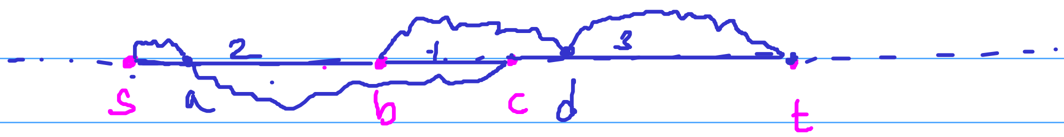
$\rightarrow$ $y_v - y_u \leq w_e$ for each edge

$$\text{Max } y_t - y_s$$

physical Interpretation of Dual.



Imagine a thread of length w_e in place of edge e. Then try to pull s and t apart as much as possible.



$y_b \quad y_c \qquad y_c - y_b \leq \underline{w_e}$

Want to show : primal OPT = length of the shortest path $s \rightsquigarrow t$

<u>Duality</u> $\quad f(x) \geq f(x^*) = g(y^*) \geq g(y)$

shortest path — primal opt. — dual opt — feasible dual

$=$

Will construct a dual feasible solution y s.t.

$g(y) = \underline{y_t - y_s}$ = length of the shortest path from s to t $= dist(s,t)$

$y_s = 0 \quad$ for u; $\quad y_u = dist(s,u)$

verify. $\quad y_v - \underline{y_u \leq w_e}$ for every edge e.

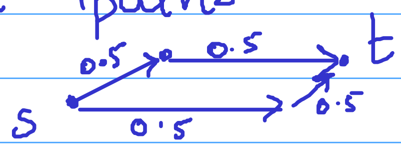
Conclusions: Shortest Path LP.

1. For any weights (with no negative weight cycles) there is always an integral optimal solution.

   It corresponds to a shortest path.

2. Any optimal solution is a convex combination of shortest paths.

   HW



HW Can you prove the above without LP dual?

Start with a fractional optimal solution and try to make it integral step by step. while maintaining the objective function value.