Feb 8 Today last date for midterm feedback.

Complementary Slackness

$\max\ w^T x$	$\min\ b^T y$
$Ax \leq b$	$A^T y \geq w$.
$x \geq 0$	$y \geq 0$.
$\max \sum_{i=1}^{n} w_i x_i$	$\min \sum_{j=1}^{m} b_j y_j$
$\sum_{i=1}^{n} a_{ji} x_i \leq b_j$ for $j = 1, 2, \dots, m$	$\sum_{j=1}^{m} a_{ji} y_j \geq w_i$ for $i = 1, 2, \dots, n$
$x_i \geq 0$ for $i = 1, 2, \dots, n$	$y_j \geq 0$ for $j = 1, 2, \dots, m$

Weak duality x primal feasible
y dual feasible.

$$\underbrace{\sum_{i=1}^{n} w_i x_i}_{\text{primal}} \leq \sum_{i=1}^{n} \Big(\sum_{j} a_{ji} y_j \Big) x_i = \sum_{j} \Big(\underbrace{\sum_{i} a_{ji} x_i}_{} \Big) y_j \leq \underbrace{\sum_{j} b_j y_j}_{\text{dual}}$$

$$w^T x \leq y^T A x \leq y^T b$$

Strong Duality: If x^* and y^* are primal optimal and dual optimal then equality holds throught.

$$\Rightarrow \quad \sum_j \left(\sum_i a_{ji}\, x_i^*\right) y_j^* = \sum_j b_j\, y_j^*$$

$$\Rightarrow \quad \sum_j \underbrace{y_j^*}_{\geq 0} \underbrace{\left(b_j - \sum_i a_{ji}\, x_i^*\right)}_{\geq 0} = 0$$

$\Rightarrow$ for each j $\quad y_j^* \left(b_j - \sum_i a_{ji} x_i^*\right) = 0$

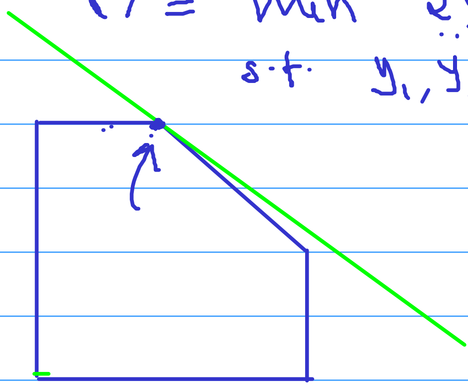
For each j either $y_j^* = 0$ or $\sum_i a_{ji}\, x_i^* = b_j$

(j-th dual variable is zero) or (j-th primal constraint is tight)

$y_j^* > 0 \quad \Rightarrow \quad$ j-th primal constraint must be tight.

j-th dual variable is nonzero only when j-th primal constraint is tight.

$17 = \min \; 2y_1 + 2y_2 + 3y_3$ $\quad$ s.t. $y_1, y_2, y_3 \geq 0,$ $\quad \big| \quad y_1 + y_3 \geq 5, \quad y_2 + y_3 \geq 6.$



$\max \; 5x_1 + 6x_2$

$$\left\{ \begin{array}{ll} x_1 \leq 2 & x_1 \geq 0 \\ x_2 \leq 2 & x_2 \geq 0 \\ x_1 + x_2 \leq 3 & \end{array} \right\}$$

$(x_1^*, x_2^*) = 1, 2$

primal tight constraints

$5x_1^* + 6x_2^* = 17$

$(y_1^*, y_2^*, y_3^*) = 0, 1, 5$

$$\left.\begin{array}{r} (x_1 + x_2 \leq 3) \times 5 \\ (x_2 \leq 2) \times 1 \\ \hline 5x_1 + 6x_2 \leq 17 \end{array}\right\}$$

Objective function can be written as

a positive combination of the constraints that are tight at an optimal point.

$$\Rightarrow \quad \sum_i w_i x_i^* = \sum_i \sum_j a_{ji} y_j^* x_i^*$$

$$\Rightarrow \quad \sum_i \underbrace{x_i^*}_{\geqslant 0} \Big(\underbrace{\sum_j a_{ji} y_j^* - w_i}_{\geqslant 0} \Big) = 0$$

for each i, Either $x_i^* = 0$ or $\sum_j a_{ji} y_j^* = w_i$

$x_i^* > 0 \Rightarrow$ i-th dual constraint is tight.

edge which participates in a shortest path the corresponding thread becomes tight.

Theorem: x^*, y^* are optimal solutions

if and only if

x^* and y^* are feasible solutions and

satisfy both the complementary slackness conditions

$$(1)\quad y_j^* > 0 \Rightarrow \sum_i a_{ji} x_i^* = b_j$$

$$(2)\quad x_i^* > 0 \Rightarrow \sum_j a_{ji} y_j^* = w_i$$

Complementary slackness for other standard forms?

Approximate complementary slackness

$\Rightarrow$ approximately optimal. ?

For example, suppose you have x, y s.t.

$$\left.\begin{array}{lll} (1) & y_j > 0 \Rightarrow & b_j/2 \leq \sum_i a_{ji} x_i \leq b_j \\ (2) & x_i > 0 \Rightarrow & w_i \leq \sum_j a_{ji} y_j \leq 2w_i \end{array}\right\}$$

what can you conclude about the

objective values of x and y? HW

$$\sum_i w_i x_i \qquad \sum_j b_j y_j$$

Primal Dual schema (Feb 9)

Not an algorithm, but a meta-algorithm to solve LPs.

- Reduces LP to a simpler LP.
- Also useful in approximation/online algorithms

Schema

- Start with a dual feasible solution.
- Try to construct a primal feasible solution satisfying complementary slackness
  - → if succeed → optimal → done.
  - → if fail → there is a way to improve the dual.

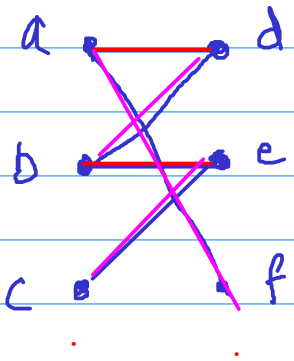
Primal-Dual schema for shortest path

$u \to v$

$y_v - y_u \le w_e$

- Start with all dual variables zero.
- At any stage, see if there is a path from s to t using only tight edges. (complementary slackness)
  - → if yes then we already have a shortest path
  - → if No then you can increase some of the dual variables, and make some more edges tight.

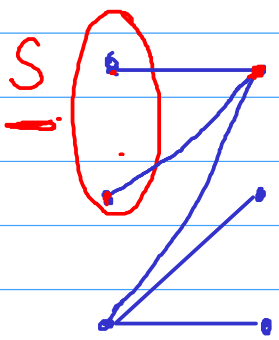
Matching in Bipartite Graphs.



perfect Matching

Reading Assignment:
Augmenting path algorithm to find maximum matching and Hall's theorem.

Hall's theorem: If there is no perfect matching then there exist a subset S of left vertices s.t. $|N(S)| < |S|$

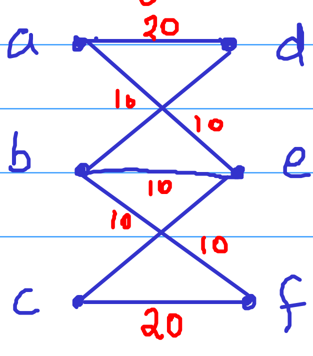


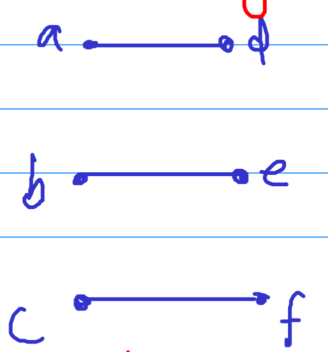
Assume we have an algorithm s.t. Given a graph it

→ either gives a perfect matching

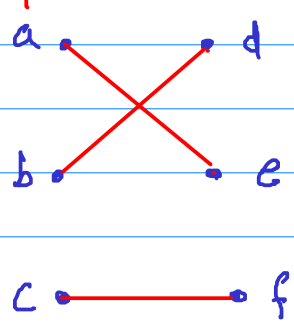
→ or gives a set S of left vertices with $|N(S)| < |S|$.

Min weight Perfect Matching in Bipartite Graphs.





weight = 50



weight = 40.

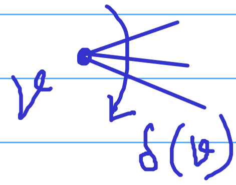
LP for min weight perfect matching

x_e for $e \in E$

$$\min \sum_{e \in E} w_e x_e \qquad \begin{cases} x_e = 1 & e \in PM \\ x_e = 0 & e \notin PM \end{cases}$$

s.t.

$$\left.\begin{array}{l} \displaystyle\sum_{e \in \delta(v)} x_e = 1 \\ 0 \leq x_e \end{array}\right\}$$



Is this a correct LP?

can there be a weight vector w s.t. the OPT is less than the min weight of a PM?

that is, there are only fractional optimal solutions?

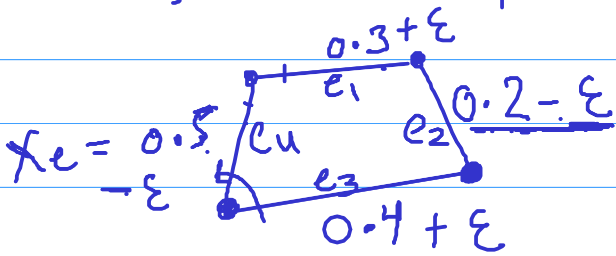
Equivalently, can this polyhedron have non-integral corners?

Claim: There is always an integral optimal solution.

Proof: HW

fractional optimal ~~solution~~

$\varepsilon > 0 \qquad \varepsilon < 0$



$$= \varepsilon\,(w_{e_1} - w_{e_2} + w_{e_3} - w_{e_4})$$

In a fractional solution, we can find a cycle with all fractional coordinates.

Modify this cycle along the cycle with alternately adding $+\varepsilon, -\varepsilon$ to the edges

You can choose $\varepsilon > 0$ or $\varepsilon < 0$ whichever decreases the total weight (or keeps it same)

Then choose ε so that at least one edge becomes integral.

Then argue inductively.

Feb 11

LP	Dual LP
$\min \sum_{e \in E} w_e x_e$	$y_v \quad v \in V$
	$y_u + y_v \leq w_e \quad e \in E$, $e = (u,v)$
$x_e \geq 0 \quad$ for $e \in E$	
$\sum_{e \in \delta(v)} x_e = 1 \quad$ for $v \in V$	$\max \sum_{v \in V} y_v$

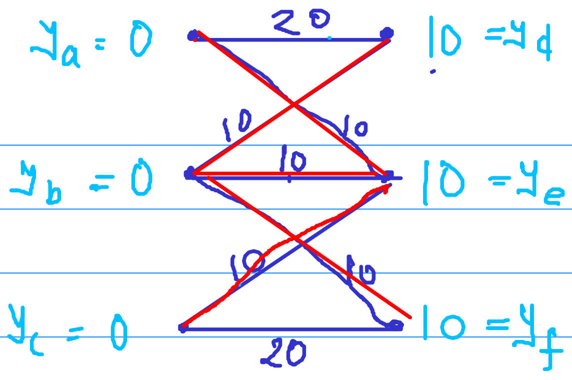
Complementary slackness:

for each edge e

$$\boxed{y_u + y_v < w_e \implies x_e = 0}$$

Start with a feasible dual solution.

{ For example, zero for left vertices and min weight among the incident edges for right vertices.



an edge e is tight

$y_u + y_v = w_e$

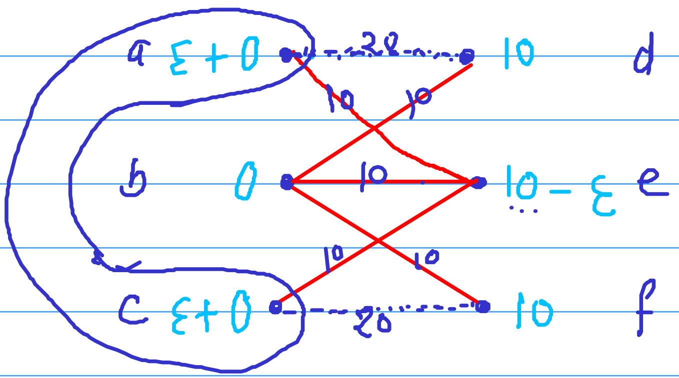
Try to construct a primal feasible solution satisfying complementary slackness.

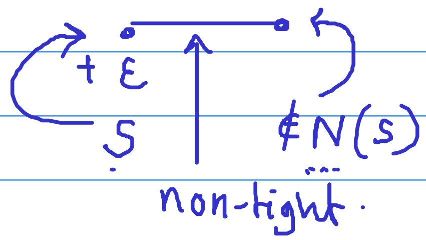
primal feasible → Perfect matching.

complementary slackness → tight edges.

Try to find a perfect matching among the tight edges.

If doesn't exist then find $S \subseteq$ left vertices with $|N(S)| < |S|$.





$\boxed{\varepsilon = 10}$

Add $+\varepsilon$ on S and $-\varepsilon$ on $N(S)$ vertices

Change in the dual objective. $\varepsilon(|S| - |N(S)|) > 0$

Argue that the dual solution remains feasible.

Four kinds of edges.

$+\varepsilon$ ——— $-\varepsilon$ (a, e) $y_u + y_v \le w_e$ remains unchanged

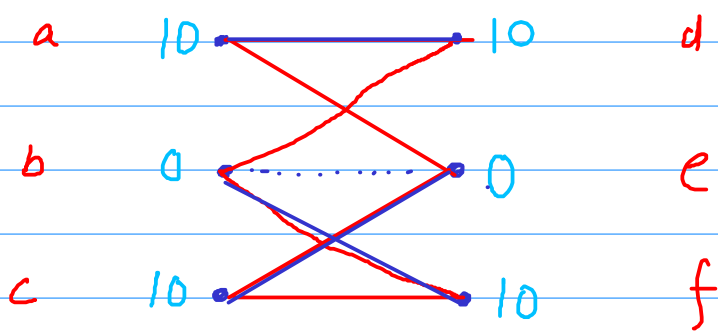
——— $-\varepsilon$ (b, e) $y_u + y_v$ decreasing.

$+\varepsilon$ ——— (a, d) Only non-tight edges will see this. $< w_e$

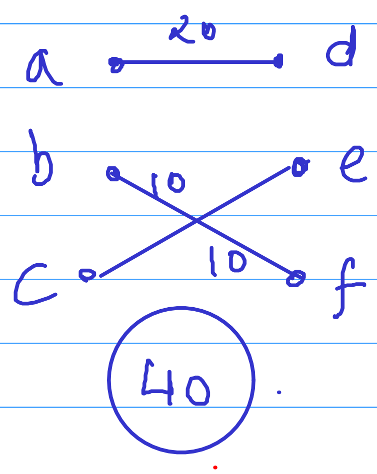
——— $y_u + y_v < w_e$

$+\varepsilon$ $y_u + y_v \le w_e$

Take ε to be the maximum value so that one of the non-tight edges become tight.



Repeat the process till you find a perfect matching using only the tight edges.



Optimal solution

Why does the algorithm terminate? Running Time?

Dual objective increases in every round.

Claim Dual objective increases by at least one in every iteration.

No. of iterations $\leq$ sum of all the edge weights.

Input parameters, n, $\log w$ poly(n, w_{max})

pseudo-polynomial time

Slight variant
Better analysis $\rightarrow$ Poly(n) time.

different progress measure.

Can we always improve the dual solution in the primal-dual schema for a general LP?

$\min\ w^T x$ s.t. $Ax = b$ $x \geq 0$	$\max\ y^T b$ $y^T A \leq w^T$ [row vector]

$(A^T y \leq w)$

Suppose y is dual feasible solution. Without loss of generality first k dual constraints are tight.

$$[y_1 \;\; y_2 \;\; \cdots \;\; y_m] \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}$$

$$* \quad * \quad * \qquad * \quad * \quad * \quad *$$

$$=w_1 \quad =w_2 \quad =w_k \quad \underbrace{<w_{k+1} \quad <w_n}$$

CS

$$\left[\; \underbrace{\begin{matrix} & & \\ & A & \\ & & \end{matrix}}_{A'} \;\middle|\; \quad \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_k \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

Want $\quad A'x = b, \quad x \geq 0$

If x doesn't exist then by Farkas' Lemma

$\exists \; u$

$$u^T A' \leq 0 \qquad u^T b > 0$$

$$[u_1 \;\; u_2 \;\; \cdots \;\; u_m] \begin{bmatrix} \\ & A & \\ \\ \end{bmatrix}$$

$$\le 0 \quad \le 0 \quad \le 0 \quad \overset{<}{*} \quad \overset{<}{*} \quad \overset{<}{**}$$

New dual solution

$$\boxed{\underline{z} = \underline{y} + \underline{u}\cdot \underline{\varepsilon}}$$

Objective   increase   HW

Feasibility.   remain

$$[y_1 \;\; y_2 \;\; \cdots \;\; y_m] \begin{bmatrix} \\ & & \\ \\ \end{bmatrix}$$

Tight constraints remain valid.

Non-tight constraints have some room.