# AdWords - online ad auction

User searches for a word. x. (online)

Advertisers bid for the word x.

One of them gets allocated the ad slot.

Each advertiser has a daily budget.

Objective: maximize the total revenue generated from advertizers.

Mehta Saberi Vazirani Vazirani 2005 introduced this model. and gave a strategy with competitive ratio $\frac{e-1}{e} \approx 0.63$

Variations:

- Advertiser Pays only when ad is clicked.
- Multiple ad slots.
- Charge the second highest bid.

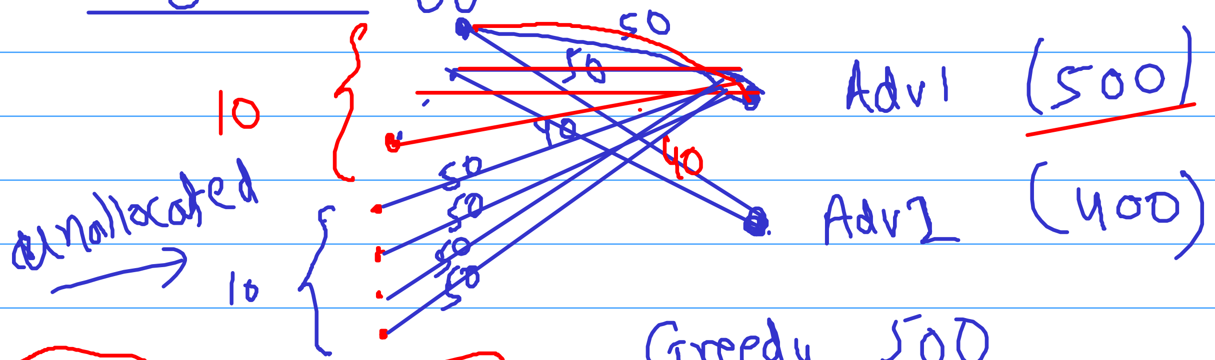
Primal Dual Scheme
Buchbinder Jain Naor 2007 $\left(\frac{e-1}{e}\right)$

Assumption: Single bids are much smaller than daily budgets.

min $\quad ALG \leq \alpha \cdot OPT \quad \alpha \geq 1$.

Max $\quad ALG \geq \beta \cdot OPT \quad \beta \leq 1$

**Greedy Strategy**: allocate the word to highest bidder.




Greedy 500

offline $400 + 500 = 900$

$\boxed{\text{comp ratio} = 1/2 \quad \text{HW}}$

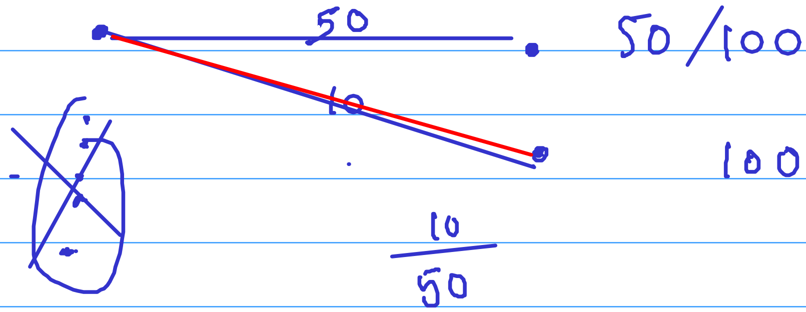
**Simple Case**: When all bids are same [KP, 2000]

Allocate to bidder with highest fraction of their remaining daily budget.

achieves competitive ratio $\frac{e-1}{e} \approx 0.63$.

**Def** $\text{comp-ratio}(ALG) = \min_{I} \frac{ALG(I)}{OPT(I)}$

The same strategy will not work when bids are different.

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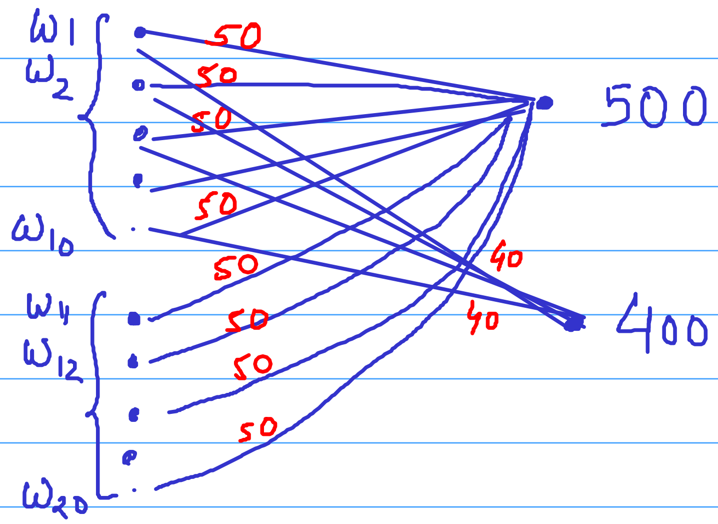
Maximization Problem

$$\text{Comp ratio}(ALG) = \min_{I} \frac{ALG(I)}{Off\text{-}OPT(I)} \leq 1$$

Minimization Problem

$$\text{Comp ratio}(ALG) = \max_{I} \frac{ALG(I)}{Off\text{-}OPT(I)} \geq 1$$

Greedy gives $\approx \frac{1}{2}$ on this example




Greedy 500

OffOPT 900

We can assign a few initial words to the first bidder. Then as the budgest of the first bidder goes down we can switch to the second bidder. Then keep switching between them depending on the budget.

$$\text{Revenue} = 5 \times 50 + 5 \times 40 + 5 \times 50 = 700$$

The strategy should use a combination of the bid and the remaining budget.

Which Combination will be optimal ?
will get from Primal Dual.

## Primal Dual

Words come online $w_1, w_2, \ldots, w_m$

$n$ bidders with daily budget $B_1, B_2, \ldots, B_n$

$b_{ij} \leftarrow$ bid for $w_j$ from $i$-th bidder

**LP**

$$\text{Max} \sum_{i,j} y_{ij} \cdot b_{ij}$$

for every $j$ $\quad y_{1j}, y_{2j}, \cdots, y_{nj}$

for every $j$ $\quad \sum_{i=1}^{n} y_{ij} \le 1 \qquad y_{ij} \ge 0$

Indicates $j$-th item goes to $i$-th bidder

for every $i$ $\quad \sum_{j=1}^{m} y_{ij} \cdot b_{ij} \le B_i$

**Dual LP**

$$\min \sum_i B_i x_i + \sum_j z_j$$

$\forall i \quad x_i \ge 0 \qquad \forall j \quad z_j \ge 0$

for each $i, j$ $\quad b_{ij} x_i + z_j \ge b_{ij}$

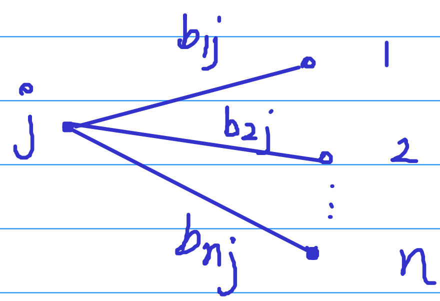
Will maintain primal feasible and dual feasible.
Will ensure that

$$\Delta D \ge \Delta P \ge \frac{e-1}{e} \Delta D$$

will make $\Delta P$ as large as possible.
Will make $\Delta D$ as small as possible.

Initialize all variables to zero.

$j$-th word comes




$y_{1j} + y_{2j} + \cdots + y_{nj} \le 1$

$z_j + b_{ij} x_i \ge b_{ij}$ for each $i$

$z_j \ge b_{ij}(1 - x_i)$

corresponding $y_{ij}$ should increase. to $= 1$

$$z_j = \max_i b_{ij}(1 - x_i)$$

Assign j-th word to the bidder maximizing $b_{ij}(1-x_i)$

$y_{ij} \leftarrow 1$ for this $i$.

$$x_i \leftarrow x_i + \Delta x_i$$

$$\Delta P = b_{ij}$$

$$\Delta D = b_{ij}(1-x_i) + B_i \Delta x_i$$

for comp ratio $\beta$, we want

$$1 \geq \frac{\Delta P}{\Delta D} \geq \beta$$

and we want $\beta$ as close to 1 as possible

$$b_{ij} = \beta\left[(1-x_i)b_{ij} + \Delta x_i B_i\right]$$

$$B_i \Delta x_i = b_{ij} x_i + b_{ij}\left(\frac{1}{\beta} - 1\right)$$

$$x_i \leftarrow \left(1 + \frac{b_{ij}}{B_i}\right)x_i + \frac{b_{ij}}{B_i}\left(\frac{1-\beta}{\beta}\right)$$

How to find $\beta$?

When i-th bidder exhausts the budget then $x_i$ should become 1.

If $i$-th bidder gets words $j_1, j_2, \dots, j_q$ and it budget exhausts.

then

$$b_{ij_1} + b_{ij_2} + \dots + b_{ij_q} = B_i$$

$$x_i = \left(\frac{1-\beta}{\beta}\right)\left[\underbrace{\left(1+\frac{b_{ij_1}}{B_i}\right)\left(1+\frac{b_{ij_2}}{B_i}\right)\cdots\left(1+\frac{b_{ij_q}}{B_i}\right)}_{\simeq e \simeq 2.72} - 1\right]$$

HW

$$\left(1+\frac{1}{B_i}\right)^{B_i} \simeq e$$

We want this to be 1.

$$x_i = \left(\frac{1-\beta}{\beta}\right)(e-1) = 1$$

$$\beta = \frac{e-1}{e} = 1 - \frac{1}{e} \simeq 0.63$$

It is known that no other Deterministic Strategy can beat this comp ratio.