AdWords - online ad auction

User searches for a word $x$. (online)

Advertisers bid for the word $x$.

One of them gets allocated the ad slot.

Each advertiser has a daily budget.

Objective: maximize the total revenue generated from advertisers.

Mehta Saberi Vazirani Vazirani 2005 introduced this model and gave a strategy with competitive ratio $\frac{e-1}{e} \approx 0.63$

Variations:

- Advertiser Pays only when ad is clicked.
- Multiple ad slots.
- Charge the second highest bid.

Primal Dual Scheme

Buchbinder Jain Naor 2007 ($\frac{e-1}{e}$)

Assumption: single bids are much smaller than daily budgets.

$\min_{\eta} \quad \text{ALG} \leq \alpha \cdot \text{OPT} \quad \alpha \geq 1$

$\max_{\eta} \quad \text{ALG} \geq \beta \cdot \text{OPT} \quad \beta \leq 1$
Greedy Strategy: allocate the word to highest bidder.

\[
\text{Adv1 (500)} \quad \text{Adv2 (400)}
\]

\[
\text{comp ratio} = \frac{1}{2}
\]

HW

Simple Case: When all bids are same [KP, 2000]

Allocate to bidder with highest fraction of their remaining daily budget achieves competitive ratio \( \frac{e-1}{e} \approx 0.63 \).

Definition: \( \text{comp. ratio}(\text{ALG}) = \min_{I} \frac{\text{ALG}(I)}{\text{OPT}(I)} \)

The same strategy will not work when bids are different.

Maximization Problem

\[
\text{comp. ratio}(\text{ALG}) = \min_{I} \frac{\text{ALG}(I)}{\text{OPT}(I)} \leq 1
\]

Minimization Problem

\[
\text{comp. ratio}(\text{ALG}) = \max_{I} \frac{\text{ALG}(I)}{\text{OPT}(I)} \geq 1
\]
Greedy gives $\frac{1}{2}$ on this example

![Diagram showing weight assignments to bidders with values]

Greedy: 500
Offopt: 900

We can assign a few initial words to the first bidder. Then as the budget of the first bidder goes down we can switch to the second bidder. Then keep switching between them depending on the budget.

Revenue = 5 \times 50 + 5 \times 40 + 5 \times 50 = 700

The strategy should use a combination of the bid and the remaining budget.

Which combination will be optimal?

will get from Primal Dual

**Primal Dual**

Words come online $w_1, w_2, \ldots, w_m$

$n$ bidders with daily budget $B_1, B_2, \ldots, B_n$

$\text{bij} \leftarrow \text{bid for wj from i-th bidder}$
Max \( \sum_{i,j} Y_{ij} \cdot b_{ij} \)

for every \( j \) \( Y_{ij}, Y_{2j}, \ldots, Y_{nj} \)

for every \( j \) \( \sum_{i=1}^{n} Y_{ij} \leq 1 \)

\( Y_{ij} \geq 0 \)

for every \( i \) \( \sum_{j=1}^{n} Y_{ij} \cdot b_{ij} \leq B_i \)

\( j \)-th item goes to \( i \)-th bidder.

Dual LP

\[
\begin{align*}
\min & \quad \sum_i B_i x_i + \sum_j Z_j \\
\text{subject to} & \quad x_i, z_j \geq 0 \\
& \quad y_{ij} x_i + z_j \geq b_{ij} \\
& \quad \text{for each } i, j
\end{align*}
\]

will maintain primal feasible and dual feasible.

will ensure that

\[ \Delta D \geq \Delta P \geq \frac{e-1}{e} \Delta D \]

\( \Delta D \) will make \( \Delta P \) as large as possible.

\( \Delta D \) will make \( \Delta D \) as small as possible.

Initialize all variables to zero.

\( j \)-th word comes

\( Y_{ij} + Y_{2j} + \ldots + Y_{nj} \leq 1 \)

\( Z_j + b_{ij} x_i \geq b_{ij} \) for each \( i \)

\( Z_j \geq b_{ij} (1-x_i) \)

corresponding \( Y_{ij} \) should increase.

\( b o = 1 \)

\( Z_j = \max_i b_{ij} (1-x_i) \)
Assign j-th word to the bidder maximizing bij \((1-x_i)\)

\[ y_{ij} \leftarrow 1 \text{ for this } i. \]

\[ x_i \leftarrow x_i + \Delta x_i \]

\[ \Delta p = b_{ij} \]

\[ \Delta D = b_{ij} (1-x_i) + B_i \Delta x_i \]

for comp ratio \( \beta \), we want

\[ 1 \geq \frac{\Delta p}{\Delta D} \geq \beta \]

and we want \( \beta \) as close to 1 as possible

\[ b_{ij} = \beta \left[ (1-x_i) b_{ij} + \Delta x_i B_i \right] \]

\[ B_i \Delta x_i = b_{ij} x_i + b_{ij} \left( \frac{1}{\beta} - 1 \right) \]

\[ x_i \leftarrow \left( 1 + \frac{b_{ij}}{B_i} \right) x_i + \frac{b_{ij}}{B_i} \left( \frac{1-\beta}{\beta} \right) \]

How to find \( \beta \)?

When i-th bidder exhausts the budget then \( x_i \) should become 1.
If \( i \)-th bidder gets words \( j_1, j_2, \ldots, j_q \) and its budget exhausts,

then

\[ b_{ij_1} + b_{ij_2} + \ldots + b_{ij_q} = B_i \]

\[ x_i = \left(1 + \frac{b_{ij_1}}{B_i} \right) \left(1 + \frac{b_{ij_2}}{B_i} \right) \ldots \left(1 + \frac{b_{ij_q}}{B_i} \right) - 1 \]

\[ \approx e \approx 2.72 \]

\[ \prod_{j=1}^{q} \left(1 + \frac{1}{B_i} \right)^{b_{ij}} \]

\[ \approx e \]

We want this to be 1.

\[ x_i = \left(1 - \frac{\beta}{\beta} \right) (e - 1) = 1 \]

\[ \beta = \frac{e - 1}{e} = 1 - \frac{1}{e} \approx 0.63 \]

It is known that no other Deterministic Strategy can beat this comp ratio.