Online Algorithms via Primal-Dual Technique

Reference: Buchbinder and Naor
Section 3 and section 10.1

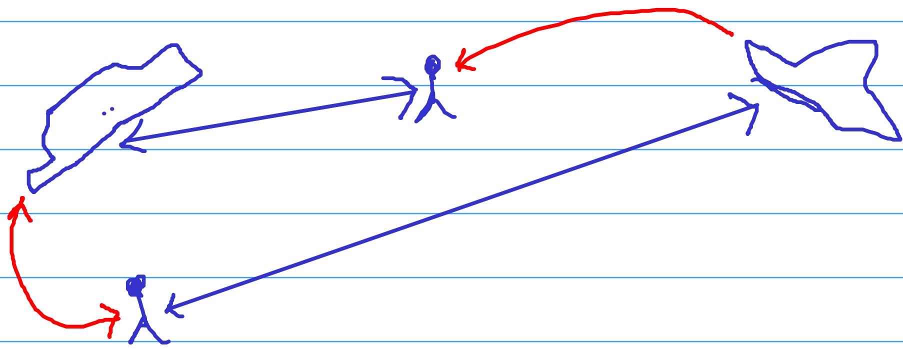
Online Problems: Input comes piece by piece.

Required to build solution in an online fashion.

Irrevocable decisions.

Example

Taxi Matching.



α - Competitive

if $\dfrac{ALG}{\text{Offline OPT}} \leq \alpha$ at every point of time.

Minimization

Example 2

TCP Acknowledgement:

packet arrive one by one

can send acknowledgement immediately

or wait and send ack for a bundle of packets.

Ski Rental Problem.

Ski

Rent per day $\rightarrow$ 100 Rs.

Buying $\longrightarrow$ 1000 Rs.

Don't know for how many days snow will be there.

k (unknown)

$$\text{Competitiveness} = \frac{\text{Cost you pay}}{\text{optimal cost paid if one knows } k} \leq \alpha$$

$\alpha \downarrow 2$

Design a strategy which is α-competitive for every value of k.

<u>Strategy 1</u> Buy on 7^{th} day.

If the 7^{th} turns out to be the last day

$$\text{Comp.} = \frac{1600}{700} \approx 2.28$$

Strategy Buy on 9^{th} day.

$$\text{Comp} = \frac{1800}{900} \approx 2.$$

Buy on 10^{th} day.

$$\text{comp} = \frac{1900}{1000} = 1.9$$

A randomized strategy.

[Expected Cost
Linearity of expectation]

On the 8th day toss a coin if heads buy the ski otherwise rent it.

Then on the 10th day definitely buy it if not bought yet.

K	Your Expected Cost	Offline optimal	α
8	$700 + \frac{1}{2}\times 100 + \frac{1}{2}\times 1000 = 1250$	800	1.5625
9	$\frac{1}{2}(700+1000) + \frac{1}{2}\times 900 = 1300$	900	1.44
10	$\frac{1}{2}(700+1000) + \frac{1}{2}(900+1000) = 1800$	1000	1.8

Best Deterministic Strategy.

Rent Price = 1 unit
Buy Price = B unit

Rent for B-1 days buy on the B-th day.

$$\text{comp ratio} = \frac{B-1+B}{B} = 2 - 1/B \approx 2$$

HW: Prove that this is the best deterministic strategy.

We will design a randomized algorithm using primal dual LP with

$$\text{comp ratio } \frac{e}{e-1} \approx 1.582$$

Writing LP.

Ignore the online nature of the problem while writing LP
Assume K is given.

LP

$$\min \sum_{j=1}^{K} z_j \cdot 1 + x \cdot B$$

$x \geq 0$ (buying.)

$\forall\ 1 \leq j \leq K \qquad z_j \geq 0$ (renting on j-th day)

$\forall\ 1 \leq j \leq k \qquad z_j + x \geq 1$

Dual LP

$$\max \sum_{j=1}^{K} y_j$$

$1 \leq j \leq K \qquad y_j \geq 0$

$$\sum_{j=1}^{K} y_j \leq B \leftarrow$$

$$y_j \leq 1 \leftarrow$$

Think as if every day we are getting a new primal constraint and a new dual variable.

We would like to maintain a primal feasible and a dual feasible solution at every day.

Ignore integrality for now.

$P \rightarrow$ primal objective value
$D \rightarrow$ dual objective value $\qquad \boxed{D \leq P}$

Always maintain $\quad P \leq \alpha \cdot D \quad$ (α decided later)

Will imply that $\quad P \leq \alpha \cdot OPT \quad$ always.

Start with everything zero. $P=0$ $D=0$

Day 1

$x + z_1 \geq 1$ $y_1 \geq 0$

To keep P as small as possible

we will make $x + z_1 = 1$

To maintain $P \leq \alpha \cdot D$ $P/D \leq \alpha$

we need to increase the dual solution.

Increase it as much as possible.

set $y_1 \leftarrow 1$

D becomes 1.

If we take

$x = 1$ $z_1 = 0$

$P = B$ $\alpha = B$

If we take

$z_1 = 1$, $x = 0$

$P = 1$ $D = 1$ very nice $\alpha = 1$

But cannot continue this because $x + z_{B+1} \geq 1$

$y_1 + y_2 + y_3 + \dots \leq B$ $D \leq B$

Will have to set $x = 1$ at the B-th day.

$\alpha = 2$ ~~Δ~~

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Compromise between the two extremes.

$$x \leftarrow x + \Delta x_1$$
$$z_1 \leftarrow 1 - x$$

x	z_j
0	0
Δx_1	$1-\Delta x_1$
$\Delta x_1 + \Delta x_2$	$1-\Delta x_1 - \Delta x_2$
$\cdots + \Delta x_j$	$1-\Delta x_1 - \Delta x_2 \cdots - \Delta x_j$

$$\sum y_j \leq B$$

Day j

set $y_j \leftarrow 1$ (if $j \leq B$)

set $\left\{ \begin{array}{l} x \leftarrow x + \Delta x_j \\ z_j \leftarrow 1 - x \end{array} \right.$

$$x + z_j \leq 1$$

Finding the right Δx_j

$$P = B \cdot x + \sum z_j$$

At day j

$$\Delta D = 1$$

$$\Delta P = B \cdot \Delta x + 1 - x - \Delta x$$

$$\Delta P \leq \alpha \cdot \Delta D$$

$$\boxed{\Delta P = \alpha \cdot \Delta D}$$

$$B \cdot \Delta x + 1 - x - \Delta x = \alpha$$
$$\Delta x (B-1) = \alpha - 1 + x$$
$$\Delta x = \frac{x}{B-1} + \frac{\alpha-1}{B-1}$$

$$x \longleftarrow x + \frac{x}{B-1} + \frac{\alpha-1}{B-1} = x\left(\frac{B}{B-1}\right) + \frac{\alpha-1}{B-1}$$

$$\frac{\alpha-1}{B-1}, \quad \left(\frac{\alpha-1}{B-1}\right)\frac{B}{B-1} + \frac{\alpha-1}{B-1}, \quad \cdots$$

After day $j \leq B$

$$x = \left(\frac{\alpha-1}{B-1}\right)\left(1 + \frac{B}{B-1} + \left(\frac{B}{B-1}\right)^2 + \cdots + \left(\frac{B}{B-1}\right)^{j-1}\right)$$
$$= (\alpha - 1)\left[\left(\frac{B}{B-1}\right)^j - 1\right]$$

To find the right α.

After day B

$$\Delta D = 0$$

Hence,

ΔP must be zero.

But we still need

$$x + z_{B+1} \geq 1$$

That means x must become 1 at day B

$$(\alpha - 1)\left(\left(\frac{B}{B-1}\right)^{B} - 1\right) = 1$$

$$\lim_{M\to\infty}\left(1+\frac{1}{M}\right)^{M} \to e$$

$$\left(1+\frac{1}{B-1}\right)^{B-1}$$

$$\alpha = \frac{\left(\frac{B}{B-1}\right)^{B-1}}{\left(\frac{B}{B-1}\right)^{B-1} - 1} \approx \frac{e}{e-1} \approx 1.58$$

Day j

$y_j \leftarrow 1$ (if $j \leq B$)

$$x \leftarrow \frac{\left(\frac{B}{B-1}\right)^{j} - 1}{\left(\frac{B}{B-1}\right)^{B} - 1} \approx \frac{e^{j/B} - 1}{e - 1}$$

$$z_j \leftarrow 1 - x$$

$$\boxed{P \leq \frac{e}{e-1} \cdot D}$$

$$\rightarrow \boxed{P \leq \frac{e}{e-1} \cdot OPT}$$

How to convert fractional primal solution to a real strategy?

Let $B = 10$

Day	x	z_j
1	0.06	0.94
2	0.13	0.87
3	0.20	0.8
4	0.28	0.72
⋮	⋮	⋮
10	1	0

Set $z_j \longrightarrow$ pr of renting on day j

x on day j will be probability of buying in one of the first j days.

Expected cost on day j $= x \cdot B + z_1 + z_2 + \cdots + z_j$

$= P$ (on day j) primal objective

$\leq e/_{e-1} \times$ (OPT on day j)

Thus, expected cost $\leq \frac{e}{e-1} \times$ offline OPT (at every day)

Implementation on next page

Implementation

Choose a number r from the range $(0,1)$ uniformly randomly.

If $x(\text{on day } j-1) \leq r \leq x(\text{on day } j)$

then we will buy on day j.

Note: r is chosen only once in the beginning; not every day.

HW Show that the above implementation will lead to the desired probabilities for buying and renting.

No strategy can beat this factor. $e/e-1$

This Strategy was first given by Karlin et al. (1994)

Buchbinder Jain Naor (2007) explained it using primal dual.

AdWords - online ad auction

User searches for a word. X.

Advertisers bid for the word X.

One of them gets allocated the ad slot.

Each advertiser has a daily budget.

Objective: maximize the total revenue generated from users.

Mehta Saberi Vazirani Vazirani 2005 introduced this model.

<u>Variations</u>:

- Advertiser Pays only when ad is clicked.
- Multiple ad slots.
- Charge the second highest bid.