Jan 25

Last week Recap.

(1) corners of a Polytope, 3 definitions.

(2) Polytope (finitely many linear constraints, bounded)

= convex hull of finitely many points.

Two LP questions

(1) Feasibility : Given $Ax \leq b$, $\exists x$? satisfying $Ax \leq b$

(2) Optimization: Max $w^T x$ s.t. $Ax \leq b$.

The two questions are equivalent.

Optimization $\leq$ Feasibility (binary search)

Feasibility $\leq$ Optimization

does there exist $x \in \mathbb{R}^n$ s.t.

$$a_1^T x \leq b_1$$
$$\begin{bmatrix} a_2^T x \leq b_2 \\ \vdots \\ a_m^T x \leq b_m \end{bmatrix}$$

Min $a_1^T x$

s.t. $\begin{bmatrix} a_2^T x \leq b_2 \\ \vdots \\ a_m^T x \leq b_m \end{bmatrix}$

OPT $> b_1 \Rightarrow$ not feasible

OPT $\leq b_1 \Rightarrow$ feasible.

Note that we have assumed that the optimization oracle works correctly only when the given constraints are feasible.

The constraints we have taken above (2 to m) might not be feasible. In that case the optimal value we get will be useless.

To ensure that we always give a feasible set of constraints to the optimization oracle, we start by a single constraint as below.

$$\left.\begin{aligned} &\min \quad a_{m-1}^T x \\ &\text{s.t.} \quad a_m^T x \le b_m \end{aligned}\right\} \rightarrow \begin{aligned} &> b_{m-1} \Rightarrow \text{not feasible} \\ &\le b_{m-1} \Rightarrow \text{feasible} \end{aligned}$$

$$\left.\begin{aligned} &\min \quad a_{m-2}^T x \\ &\text{s.t.} \quad a_{m-1}^T x \le b_{m-1} \\ &\qquad\ \ a_m^T x \le b_m \end{aligned}\right\} \rightarrow \begin{aligned} &> b_{m-2} \Rightarrow \text{not feasible} \\ &\le b_{m-2} \Rightarrow \text{feasible.} \end{aligned}$$

We keep adding constraints one by one. When we have constraints (k to m), we ask the oracle to minimize $a_{k-1}^T x$. This opt value will tell us whether the set of constraints (k-1 to m) is feasible. We move further ahead accordingly.

Towards a polynomial time algorithm

"Duality" "Easily verifiable proofs"

"NP ∩ coNP" $\Longleftrightarrow$ "Good Characterization"

Feasibility:

If there exists an $x \in \mathbb{R}^n$ satisfying $Ax \leq b$, is there an easily verifiable proof for this fact?

(NP) Proof : x

easily test $Ax \leq b$.

only if x has a short description.

Feasibility $\in$ NP

Show there always exists an x s.t. number of bits in x is poly bounded (n, m)

HW

Claim if x is a corner then it has a small description.

$$A' x = b'$$

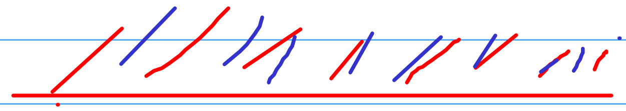
rational

$$x = A'^{-1} b'$$

bounded. rational rational

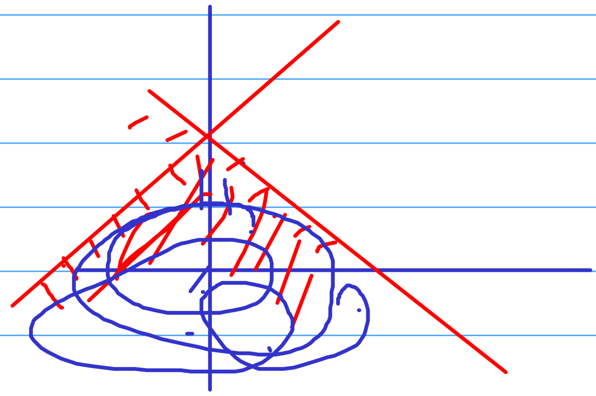
bounded.

If there is no $x \in \mathbb{R}^n$ satisfying $Ax \le b$
is there an easily verifiable proof for this fact?
(coNP)



Example 1

$$\begin{cases} x + y \le 1 \\ y - x \le 1 \\ y \ge 3 \end{cases}$$



$$2y \le 2$$

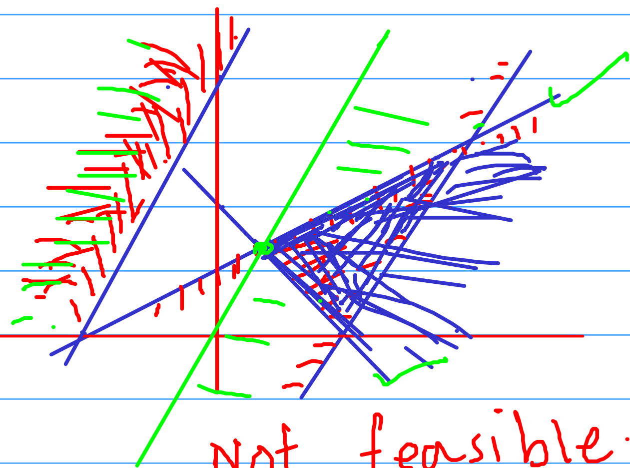
$$y \le 1$$
$$-y \le -3$$

$$\underline{0 \le -2}$$

contradiction

Example 2

$$3x - 2y \le 3$$
$$2y - x \le 1$$
$$x + y \ge 1$$
$$y - 2x \ge 2$$

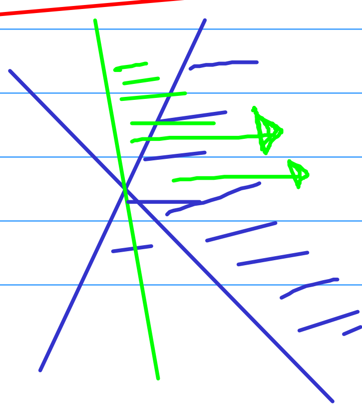


Not feasible.

$$2y - x \le 1$$
$$-x - y \le -1$$

$$y - 2x \le 0$$

$$\boxed{0 \le -2}$$



Farkas's Lemma

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ s.t.

$Ax \leq b$ is not feasible then

there exists $y = (y_1, y_2, \dots, y_m) \geq 0$ s.t.

$$\underset{\text{row}}{y^T} \underset{m\times n}{A} = 0 \quad \text{and} \quad y^T b = -1.$$

Elaboration

Not feasible.

$$\begin{aligned}
y_1 \times \{\ & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\
y_2 \times \{\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\
\vdots \quad & \qquad \vdots \\
y_m \times \{\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \\
\hline
& 0x_1 + 0x_2 + \dots + 0x_n \leq -1
\end{aligned}$$

$y_1, y_2 \dots y_m \geq 0$

Proof: Fourier Motzkin Elimination.

$$\begin{aligned}
3x - 2y &\leq 3 \\
2y - x &\leq 1 \\
x + y &\geq 1 \\
y - 2x &\geq 2
\end{aligned}
\quad \xrightarrow{\text{eliminating } x} \quad
\begin{aligned}
3x - 2y &\leq 3 & (1) \\
2y - x &\leq 1 & (2) \\
-x - y &\leq -1 & (3) \\
2x - y &\leq -2 & (4)
\end{aligned}$$

$(1) + (2)\times 3 \longrightarrow 4y \leq 6$

$(1) + (3)\times 3 \quad -5y \leq 0$

$$\left[\begin{aligned} 3y &\leq 0 \\ -3y &\leq -4 \end{aligned}\right] \quad (2), (4)$$

$2\times(3)$
$+4$

$0 \le -4$

$0 \le -1.$

(eliminate one variable)

In every step of FME, feasibility is preserved. HW

$$\left.\begin{array}{l} x_n \le 1 \\ x_n \le 2 \\ -x_n \le -5 \end{array}\right\}$$

$0 \le -3.$

$$\left.\begin{array}{l} x_n \le \alpha \\ -x_n \le \beta \end{array}\right\} \text{ s.t. } \alpha+\beta<0$$

$0 \le -1$

if elimination not possible $\Rightarrow$ feasible HW

description of y is small. HW

Farkas Lemma (2). HW

$Ax \le b$, $x \ge 0$ is not feasible.

$\exists\ y \in \mathbb{R}^m$ $y \ge 0$ s.t.

$y^T A \ge 0$ $y^T b = -1$

Another form: If $Ax=b$, $x \ge 0$ is not feasible then there exists $y \in \mathbb{R}^m$ s.t.

(3) $y^T A \ge 0$ $y^T b = -1$.

HW

Feasibility $\in$ NP

Feasibility $\in$ coNP

<u>Optimization</u> ?

$5x+6y=17$

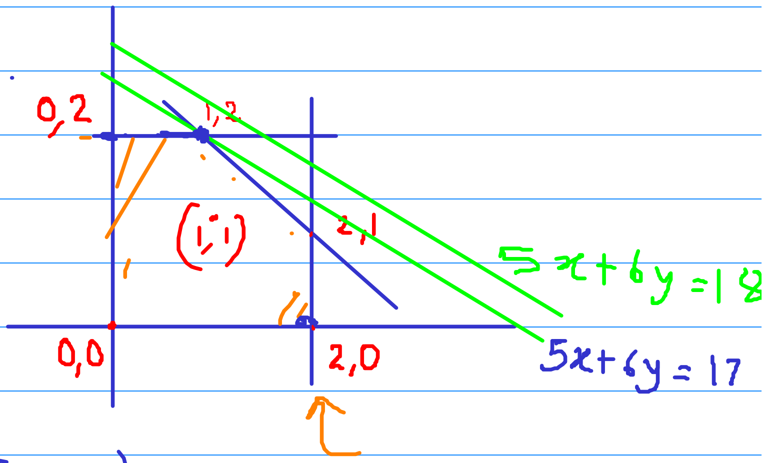
$x \geq 0$

$y \geq 0$

$\rightarrow \quad x \leq 2$

$\rightarrow \quad y \leq 2$

$x+y \leq 3$



Maximize $5x+6y = f(x,y)$

$f(1,1) = 11 \qquad OPT \geq 11$

$f(0,2) = 12 \qquad OPT \geq 12$

Any feasible point gives a lower bound on OPT.

$$\begin{aligned} x &\leq 2 \quad \times 5 \\ y &\leq 2 \quad \times 6 \\ \hline 5x+6y &\leq 22 \end{aligned}$$

$\boxed{OPT \leq 22}$

$f(1,2) = 5\times1 + 6\times2 = 17 \qquad \boxed{OPT \geq 17}$

$$f(x,y) = 5x + 6y \underset{(x \geqslant 0)}{\leqslant} 6x + 6y \leqslant 18$$

$$x + y \leqslant 3 \quad \Big\} \times 6$$

$$6x + 6y \leqslant 18$$

$$-x \leqslant 0$$

$$f(x,y) = 5x + 6y \leqslant 18$$

$$17 \leqslant OPT \leqslant 18$$

$$y \leqslant 2 \quad \Big\} \times 1$$

$$x + y \leqslant 3 \quad \Big\} \times 5$$

$$f(x,y) = 5x + 6y \leqslant 17$$

$$17 \leqslant OPT \leqslant 17$$

$$\boxed{OPT = 17}$$

Dual LP

LP 1

$$\text{Max } w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$$

s.t.

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$$

Weak Duality Theorem

Strong Duality Theorem.

$Ax \leq b$

$-w^T x \leq -w^* \cdot \varepsilon$

$y \geq 0$

$\lambda \geq 0$

$y^T A - \lambda w^T \leq 0$

$y^T A = w^T$

$y^T b = w^* + \varepsilon - 1/\lambda$