CS602 Combinatorial Optimization via Linear and Convex Programming **Jan-Apr 2022**

Homework (No submission)

Lecture 1 (Jan 6)

- For a graph $G(V, E)$, consider the following linear program.

$$\begin{aligned} \max \sum_{v \in V} x_v & \qquad \text{subject to} \\ x_v \geq 0 & \qquad \text{for each } v \in V, \\ \sum_{v \in C} x_v \leq 1 & \qquad \text{for each clique } C \text{ in } G. \end{aligned}$$

  Find an example of a graph G where the optimal value of the above LP is not the same as the maximum size of an independent set in G.

- Write an integer linear program for the maximum weight odd subset problem: given n objects with positive/negative weights, one has to select an odd number of objects with maximum total weight. First try it for $n = 2$ or $n = 3$.

- Prove that a polyhedron, i.e., a region defined by a set of linear inequalities, is a convex set.

Lecture 2 (Jan 10)

- Let $\beta_1, \beta_2, \ldots, \beta_r$ be some points maximizing a linear function $w^T x$ over a polyhedron P. Show that their average $z = (\beta_1 + \beta_2 + \cdots + \beta_r)/r$ is also a maximizing point.

- Let $\beta_1, \beta_2, \ldots, \beta_r$ be points which satisfy the inequality $a^T x \leq b$. Moreover, β_1 satisfies it with strict inequality, i.e., $a^T \beta_1 < b$. Show that their average point $z = (\beta_1 + \beta_2 + \cdots + \beta_r)/r$ also satisfies $a^T z < b$.

- For a face F of a polyhedron P, show that there is a linear function $w^T x$, such that the set of points in P maximizing $w^T x$ is precisely F.

Lecture 3 (Jan 13)

- Last question from previous lecture: Let the polyhedron be described by $a_i^T x \leq b_i$ for $1 \leq i \leq k$. Consider all the tight constraints for the face F, let them be $a_i^T x = b_i$ for $1 \leq i \leq \ell$. Define the linear function $w^T x$ with $w = a_1 + a_2 + \cdots + a_\ell$. Show that the points in the face F maximize $w^T x$ over the polyhedron P. Also show that other points in P outside F do not achieve the maximum objective value for $w^T x$.

- We are given a system S of linear constraints in variables $\lambda, x_1, x_2, \ldots, x_n$. We want to construct another system S' of linear constraints in variables $x_1, x_2, \ldots, x_n$ with following property:

  $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is feasible for S' **if and only if** there exists $\beta \in \mathbb{R}$ such that $(\beta, \alpha_1, \alpha_2, \ldots, \alpha_n)$ is feasible for S.

  Describe a procedure to construct S'.

Lecture 4 (Jan 17)

- We are given a system S of linear constraints in variables $y, x_1, x_2, \ldots, x_n$. We construct another system S' of linear constraints in variables $x_1, x_2, \ldots, x_n$ by eliminating variable y as described by Fourier Motzkin elimination. Prove that

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ is feasible for } S'$$

if and only if

$$\text{there exists } \beta \in \mathbb{R} \text{ such that } (\beta, \alpha_1, \alpha_2, \ldots, \alpha_n) \text{ is feasible for } S.$$

- Let P be the convex hull of points $q_1, q_2, \ldots, q_r \in \{0,1\}^n$. Show that each q_i is a corner of P.

- Find an example of a graph $G(V,E)$ such that the following system of constraints **does not** describe the independent set polytope of G (convex hull of the independent set points). That is, there is a feasible point for the below system of constraints which is not in the convex hull of independent set points.

$$\begin{aligned} x_v &\geq 0 && \text{for each } v \in V, \\ \sum_{v \in C} x_v &\leq 1 && \text{for each clique } C \text{ in } G. \end{aligned}$$

- Find an example of a **bipartite** graph $G(V,E)$ such that the following system of constraints **does not** describe the perfect matching polytope of G (convex hull of the perfect matching points). That is, there is a feasible point for the below system of constraints which is not in the convex hull of perfect matching points.

$$\begin{aligned} x_e &\geq 0 && \text{for each } e \in E, \\ \sum_{e \text{ incident on } v} x_e &= 1 && \text{for each vertex } v \text{ in } V. \end{aligned}$$

  Or, equivalently, find a bipartite graph G with weights on edges $\{w_e : e \in E\}$ such that maximum weight of a perfect matching in G is not the same as max $\sum_{e\in} w_e x_e$ over the above system of constraints.

Lecture 5 (Jan 20)

- Consider the below graph $G(V,E)$ with 9 edges and a point $\alpha \in \mathbb{R}^E$ whose coordinates for each edge are shown in the figure. Prove that this point is not in the perfect matching polytope of this graph, that is, it cannot be written as a convex combination of perfect matching points.

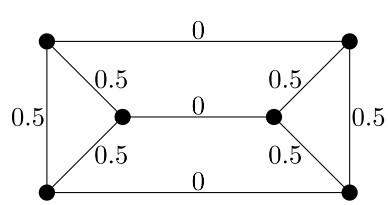

Figure 1: A graph with a point in the edge space.

- Show that the following system of constraints describe the matching polytope for a bipartite graph.

$$\begin{aligned} x_e &\geq 0 && \text{for each } e \in E, \\ \sum_{e \text{ incident on } v} x_e &\leq 1 && \text{for each vertex } v \text{ in } V. \end{aligned}$$

- For LP with bounded optimal value, show that the optimal value is bounded exponentially in number of variables, number of constraints, and the bit size of coefficients in the constraints.
- Suppose we have an optimization oracle that outputs the LP optimal value for a given feasible system of constraints and an objective function. If the given system is infeasible than it outputs an arbitrary answer. Using this optimization oracle, design an algorithm for the LP feasibility problem.
- For any polytope described by a system of constraints, show that any of its corners can be described using only polynomially many bits (polynomial in number of variables, number of constraints, and the bit size of coefficients in the constraints).

Lecture 6 (Jan 24)

- Given $a_{i,j}$'s and b_i's, prove that if the system S_1 is not feasible then the system S_2 is not feasible, where

$$S_1 \rightarrow \begin{cases} \text{for } 1 \leq i \leq k_1, & a_{i,1}x_1 + \cdots + a_{i,n-1}x_{n-1} + x_n = b_i \\ \text{for } k_1 < j \leq k_2, & a_{j,1}x_1 + \cdots + a_{j,n-1}x_{n-1} - x_n = b_j \\ \text{for } k_2 < \ell \leq k, & a_{\ell,1}x_1 + \cdots + a_{\ell,n-1}x_{n-1} = b_\ell \end{cases}$$

$$S_2 \rightarrow \begin{cases} \text{for } 1 \leq i \leq k_1 \text{ and } k_1 < j \leq k_2, & (a_{i,1} + a_{j,1})x_1 + \cdots + (a_{i,n-1} + a_{j,n-1})x_{n-1} = b_i + b_j \\ \text{for } k_2 < \ell \leq k, & a_{\ell,1}x_1 + \cdots + a_{\ell,n-1}x_{n-1} = b_\ell \end{cases}$$

- Assume the form of Farkas' Lemma proved in the class

$$Ax \geq b \text{ is not feasible} \implies \exists y \geq 0 \text{ s.t. } y^T A = 0, y^T b > 0.$$

  And prove the following forms of Farkas' Lemma.

  - $Ax \leq b$ is not feasible $\implies \exists y \geq 0$ s.t. $y^T A = 0, y^T b < 0$.
  - $Ax \leq b, x \geq 0$ is not feasible $\implies \exists y \geq 0$ s.t. $y^T A \geq 0, y^T b < 0$.
  - $Ax = b, x \geq 0$ is not feasible $\implies \exists y$ s.t. $y^T A \geq 0, y^T b < 0$.

Lecture 8 (Jan 31)

- Prove strong duality theorem for various forms of LPs and their dual LPs.
- Show that taking dual of dual gives you the original LP back.