Lecture 1 — Jan 6, 2022

Linear Optimization

Variables $x_1, x_2, x_3, \ldots, x_n \in \mathbb{R}$

Given constraints on these variables.

Linear constraints

ex $x_3 \geq 0$, $2x_1 + x_3 - x_4 \geq 10$

$x_1 + 5x_4 = 6$

Want to minimize/maximize an objective function $f(x_1, x_2, \ldots, x_n): \mathbb{R}^n \to \mathbb{R}$ subject to the given constraints.

Maximize $3x_1 + 5x_2 - 2x_3 \cdots$

Example 1:

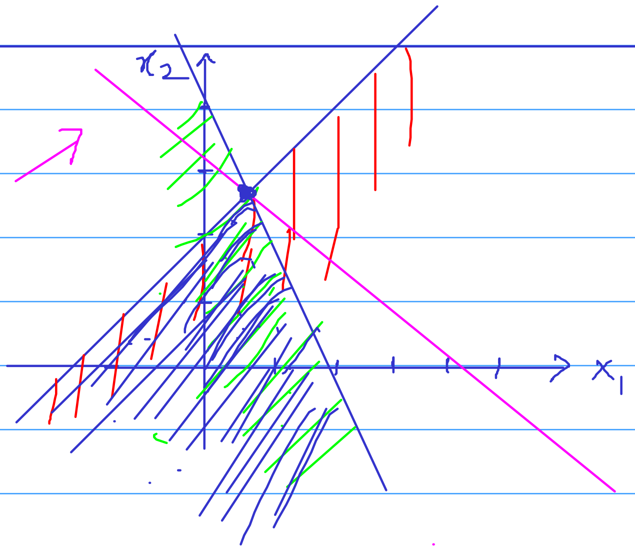
Max $x_1 + x_2$

s.t. $x_2 - x_1 \leq 2$

$2x_1 + x_2 \leq 4$

$x_1, x_2 = \left(\frac{2}{3}, \frac{8}{3}\right)$

$f(x_1, x_2) = 10/3$



Integer Linear Programming (ILP)

Allowed to restrict some variables to integers.

$\to$ ILP is NP-hard.

Simple reduction from 3-SAT

Example 2 : Mid-day meal plan

	Rice	Dal	Egg ..
Calories	130	115	75
Carb	25	20	1
Protein	3.5	8	7
⋮			
cost	5	20	6

Requirements : Calories $\in [800, 1000]$ ✓
Carbs $\in [75, 100]$ ✓
Protein $\in [10, 17]$ ✓

$x_E \in \mathbb{R}$

$x_R, x_D, x_E \geq 0$

$\text{Min } 5 x_R + 20 x_D + 6 x_E$

s.t.

$800 \leq 130 x_R + 115 x_D + 75 x_E \leq 1000$

$75 \leq 25 x_R + 20 x_D + x_E \leq 100$

$10 \leq 3.5 x_R + 8 x_D + 7 x_E \leq 17$

Combinatorial Optimization

Feasible solutions $\leftarrow$ discrete set of objects.

examples : paths, trees, cuts, independent sets, matchings

Example 1

Can we write an integer linear program (ILP) for maximum independent set problem ?

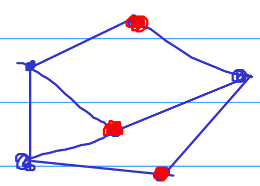
Maximum independent set : Given a graph $G(V, E)$, find a maximum size subset $S \subseteq V$ of vertices s.t. no two vertices in S have an edge between them.

$\text{Max} \sum_{v \in V} x_v$ subject to

x_v for $v \in V$

$x_v \in \{0, 1\}$

for $(u,v) \in E$ $\quad x_u + x_v \leq 1$



HW Show that every independent set corresponds to a feasible solution and every feasible solution corresponds to an independent set.

Linear programming relaxation

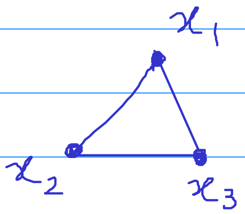
$$\text{Max} \sum_{v \in V} x_v$$

subject to
x_v for $v \in V$
$\cancel{x_v \in \{0,1\}}$ $\quad 0 \leq x_v \leq 1$
for $(u,v) \in E$ $\quad x_u + x_v \leq 1$

$0 \leq x_v \leq 0.1$
or
$0.9 \leq x_v \leq 1$

Not a linear program.

Can you think of a graph where the optimal value of the above LP is different from maximum size of an ind set ?



Max Ind set size = 1

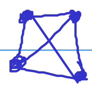
$$\text{Max} \quad x_1 + x_2 + x_3$$

$\to x_1 + x_2 \leq 1$ $\quad 0 \leq x_1, x_2, x_3 \leq 1$
$\to x_2 + x_3 \leq 1$
$\to x_3 + x_1 \leq 1$

opt value = 1.5
(0.5, 0.5, 0.5)

can we fix the situation with adding more constraints ?

for any triangle (x_u, x_v, x_w) $\quad x_u + x_v + x_w \leq 1$
for any clique C $\quad \sum_{u \in C} x_u \leq 1$



[HW]
Que: With the additional constraints, is the $\text{Opt}(LP) = \text{Max Ind set}$?

Example 2: Max weight odd subset

Given a set of n objects with positive/negative weights, find an odd subset with maximum total weight.

Straightforward Algorithm.

ILP ? $\quad$ need exponentially many constraints.
[HW]

Matrix representation of a linear program

$$\begin{aligned} &\text{Max } 4x_1 + 3x_2 \\ &-1x_1 + 2x_2 \le 2 \\ &2x_1 + 1x_2 \le 4 \\ &3x_1 - 2x_2 \le 5 \end{aligned}$$

$$w = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$$

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ 3 & -2 \end{bmatrix}_{3\times 2} \qquad b = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\begin{aligned} &\text{Max } w^T x \\ &\text{s.t. } x \in \mathbb{R}^2 \\ &\underbrace{Ax}_{3\times 1} \le \underset{3\times 1}{b} \end{aligned}$$

Each row of the matrix corresponds to a constraint.

Say n variables and k constraints.

$A \in \mathbb{R}^{k\times n}$ (A is $k \times n$)

$b \in \mathbb{R}^k$

$x \in \mathbb{R}^n$ $\quad x = (x_1, x_2, \dots, x_n) \longleftarrow$ $n\times 1$ matrix (conventionally)

$w \in \mathbb{R}^n$

Conversion between various kinds of constraints and objectives

$\ge \longrightarrow \le$ $\qquad x_1 + x_3 \ge 5 \iff -x_1 - x_3 \le -5$

$= \longrightarrow \le$ $\qquad x_1 + 2x_2 = 1 \iff \begin{cases} x_1 + 2x_2 \le 1 \\ x_1 + 2x_2 \ge 1 \end{cases}$

Min $\longrightarrow$ Max $\qquad \min\ 2x_1 - x_2 \iff \max\ x_2 - 2x_1$

$\le \longrightarrow \underset{x_i \ge 0}{=}$ $\qquad x_1 + x_2 \le 5 \iff \begin{cases} x_1 + x_2 + x_3 = 5 \\ x_3 \ge 0 \end{cases}$

$\qquad x_3 \le 0 \iff \begin{cases} x_3 = -x_4 \\ x_4 \ge 0 \end{cases}$

Some standard forms of LP

1. Max $\omega^T x$ s.t. $Ax \le b$

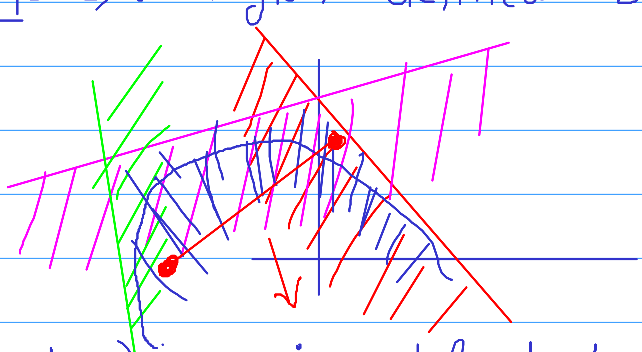
2. Max $\omega^T x$ s.t. $Ax = b$ and $x \ge 0$ $\left(x_1 \ge 0 \quad x_2 \ge 0 \quad \dots \quad x_n \ge 0\right)$

3. Min $\omega^T x$ s.t. $Ax \ge b$

4. Min $\omega^T x$ s.t. $Ax = b$ and $x \ge 0$

Geometric Perspective :

Def (Halfspace): region defined by one linear inequality.



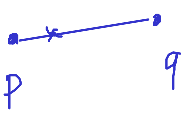
$$a_1 x_1 + a_2 x_2 + \cdots a_n x_n \le b$$

Def (polyhedron): region defined by a set of linear inequalities or equivalently intersection of halfspaces.

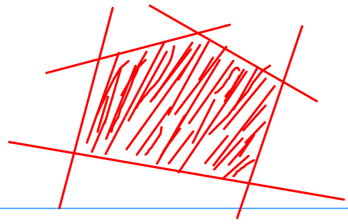
→ Feasible region of an LP is a polyhedron.

[HW] **Claim (convexity):** A polyhedron is a convex set.

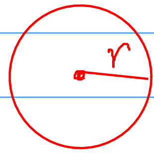
Convex set : if $p \in S$ and $q \in S$ then $\lambda p + (1-\lambda) q \in S$ for $\lambda \in [0,1]$



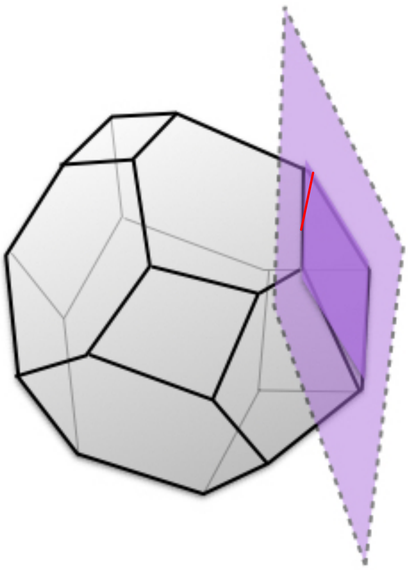
Lecture 2 . Jan 10



Def (Polytope) : A bounded polyhedron

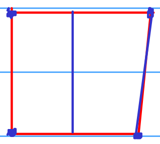
($\exists\ \underline{r}$ s.t. $P \subseteq S_r$ )

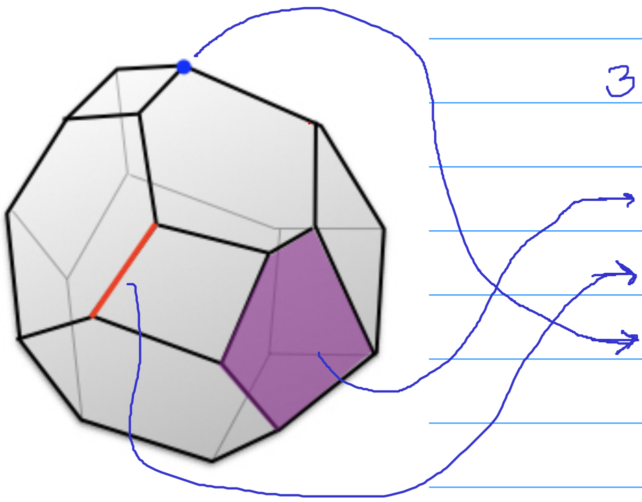
Examples : Cube, Pyramid, Tetrahedron (3D)
Polygon (2D)



$a_1x_1 + a_2x_2 + a_3x_3 = C$

Face : For a polytope (or polyhedron), a face is a set of points on its boundary, which itself is a polytope (polyhedron).





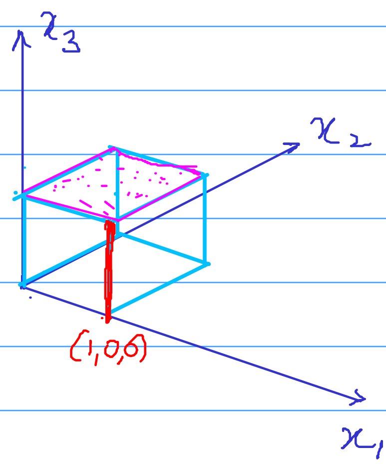
3D polytope

2D face

1D face

0D face

Def (Face): A face of a polyhedron is a set of points obtained by replacing some of the inequalities with equalities (if you get a nonempty region)



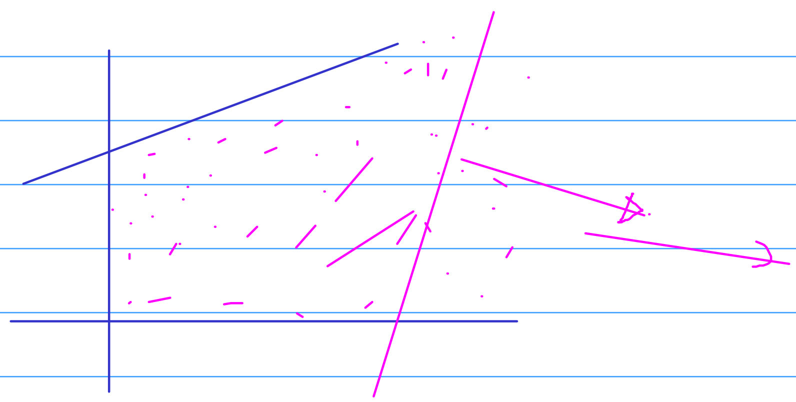
$$\boxed{\begin{array}{ll} 0 \le x_1, & x_1 \le 1 \\ 0 \le x_2, & x_2 \le 1 \\ 0 \le x_3, & x_3 \le 1 \end{array}}$$

$$\boxed{\begin{array}{ll} 0 \le x_1, & x_1 = 1 \\ 0 = x_2, & x_2 \le 1 \\ 0 \le x_3, & x_3 \le 1 \end{array}}$$

$$\boxed{\begin{array}{ll} 0 = x_1, & x_1 = 1 \\ 0 \le x_2, & x_2 \le 1 \\ 0 \le x_3, & x_3 \le 1 \end{array}}$$

Empty

$$\boxed{\begin{array}{ll} 0 \le x_1, & x_1 \le 1 \\ 0 \le x_2, & x_2 \le 1 \\ 0 \le x_3, & x_3 = 1 \end{array}}$$



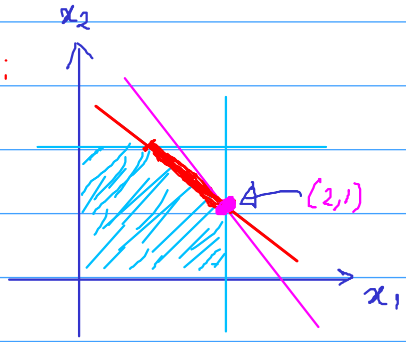
Terminology: For a point (or set of points) in P, the inequalities that are satisfied with equality by the point (or set of points) are called **tight constraints** for that point (or set of points).

Characterization of a set of optimizing points:

Lemma 1 For any polyhedron P ($Ax \le b$), and a function $\omega^T x$, the set of points in P maximizing $\omega^T x$ form a face of P.

(if P is non-empty and optimal value is bounded)

Example:



$$\boxed{\begin{array}{c} x_1 \ge 0, \quad x_2 \ge 0 \\ x_1 \le 2 \quad x_2 \le 2 \\ x_1 + x_2 \le 3 \end{array}}$$

P

① Max $2x_1 + 2x_2$
Tight constraints
$x_1 + x_2 = 3$

② Max $3x_1 + 2x_2$
Maximizing point $(2,1)$

Tight constraints $\left.\begin{array}{l} x_1 = 2, \\ x_1 + x_2 = 3 \end{array}\right\}$

$f(x) = \alpha x_1 + \alpha x_2$ for $\alpha > 0$

$f(x) = \alpha(x_1) + \beta(x_1 + x_2)$
$\alpha, \beta > 0$

proof of Lemma 1.

Let the constraints for polyhedron P be given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = a_1^T x \le b_1 \\ a_2^T x \le b_2 \\ \vdots \\ a_k^T x \le b_k \end{array}\right\}$$

Let $S \subseteq P$ be the set of points in P maximizing $\omega^T x$.

Let the first ℓ constraints be tight for all points in S and let there be no other such constraints.

I.e. for $1 \le i \le \ell$ $\quad a_i^T p = b_i$ for all $p \in S$.

$$\left.\begin{array}{l} \exists\ p' \in S \ \text{ s.t. } \ a_{\ell+1}^T p' < b_{\ell+1} \\ \exists\ p'' \in S \ \text{ s.t. } \ a_{\ell+2}^T p'' < b_{\ell+2} \end{array}\right\}$$

is there a common point $z \in S$ satisfying strict inequalities for all $\ell < i \le k$?

claim: All points in P satisfying first ℓ constraints with equality are maximizing points for $\omega^T x$.

Proof of the claim:

There should be a point $z \in S$ such that

$$a_i^T z = b_i \quad \text{for} \quad 1 \le i \le \ell$$
$$\text{and} \quad a_i^T z < b_i \quad \text{for} \quad \ell < i \le k$$

Why? for any $\ell < j \le k$

$$\exists \ \beta_j \in S \quad s.t. \quad a_j^T \beta_j < b_j$$

Define $z = (\beta_{\ell+1} + \beta_{\ell+2} + \cdots + \beta_k)/(k-\ell)$ [HW]

Now, Let y be another point in P satisfying $a_i^T y = b_i$ for $1 \le i \le \ell$

Want to show that y is maximizing point for $\omega^T x$.

For the sake of contradiction, let us assume that

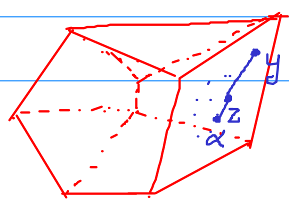
$$\Longrightarrow \omega^T z > \omega^T y \quad \Rightarrow \quad \omega^T (z - y) > 0$$

Define a new point

$$\alpha = z + \varepsilon(z - y) \qquad \text{for some small } \varepsilon > 0$$

Obs $\omega^T \alpha = \omega^T z + \varepsilon \underbrace{\omega^T (z-y)}_{>0}$

$$\Rightarrow \quad \omega^T \alpha > \omega^T z$$



We will argue that $\alpha \in P$. That will contradict the assertion that z is maximizing $\omega^T x$.

Need to show that α satisfies all the constraints for P.

for $1 \le i \le \ell$ $\quad a_i^T \alpha = a_i^T z + \varepsilon a_i^T (z-y)$

$$= b_i + \varepsilon (b_i - b_i) = b_i$$

for $\ell < i \le k$ $\quad a_i^T \alpha = \underbrace{a_i^T z}_{< b_i} + \varepsilon \underbrace{a_i^T (z-y)}$

choose ε to be sufficiently small.

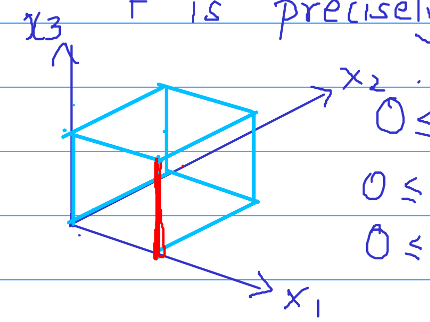
$$\Longrightarrow \quad a_i^T \alpha < b_i$$

This finishes the proof of the claim.

Conclusion: S is the face defined by taking first ℓ constraints as equalities.

Hence, the set of maximizing points form a face of P.

Lemma 2: For every face F of a polyhedron P, there
[HW] is a linear function $\omega^T x$ such that F is precisely the set of points in P maximizing $\omega^T x$.



$0 \le x_1 \le 1$

$0 \le x_2 \le 1$

$0 \le x_3 \le 1$

$a^T x \le b$

$x_1 = 1,\ x_2 = 0,\ 0 \le x_3 \le 1$ $\leftarrow$ Face

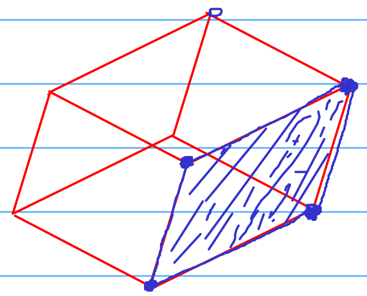
Optimizing function $f(x) = x_1 - x_2$

$= 2x_1 - 3x_2$

Proof Idea: Start with the set of tight constraits for F.
Try to define ω in terms of these tight constraints.
Define $\omega^T x$ to be a positive combination of tight constraints.

Charaterization of a Face: A set of points maximizing a linear function over the Polyhedron.

Corollary of Lemma 1: For any function $\omega^T x$, there are some corner points in P, which maximize $\omega^T x$.



How do we formally define a corner of a polyhedron?

Three definitions:

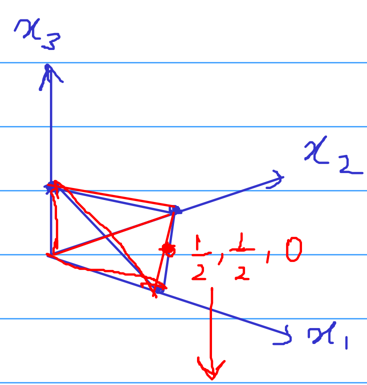
① A corner is a zero dimensional face. (single point)

I.e., a point in the polyhedron which satisfies n linearly independent constraints with equality.

$1 \geq x_1 \geq 0$ $\quad x_1 = 0,\ x_2 = 1,\ x_3 = 1$

$1 \geq x_2 \geq 0$

$1 \geq x_3 \geq 0$



$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$

$x_1 + x_2 + x_3 \leq 1$

$x_1 + x_2 \leq 1$

redundant

tight $\rightarrow x_3 = 0$

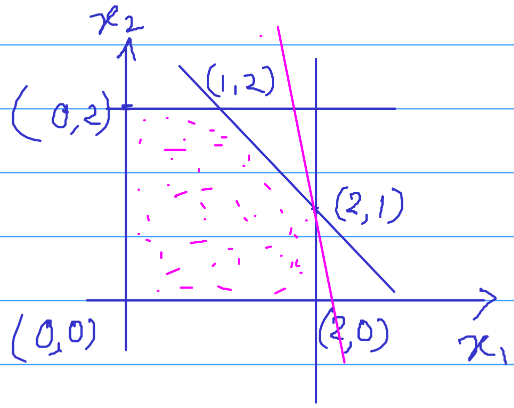
$\rightarrow x_1 + x_2 + x_3 = 1$

$x_1 + x_2 = 1$

Not linearly independent

② q is a corner of polyhedron P, if there is a linear function $\omega^T x$ s.t.

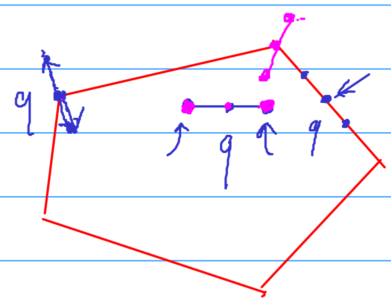
q is the unique point in P maximizing $\omega^T x$.



$(2,1) \leftarrow 3x_1 + x_2$

③ q is a corner of P, if it is not a midpoint of two other points in P.

Equivalently, for any $\alpha \in \mathbb{R}^n$
if $q+\alpha \in P$ then $q-\alpha \notin P$.



Lecture 3 $\boxed{\text{Jan 13}}$

Equivalence of the three definitions

$$
\begin{array}{ccc}
① & \Longrightarrow & ② \\
\Uparrow & & \Downarrow \\
& ③ &
\end{array}
$$

① $\Longrightarrow$ ② Follows from Lemma 2.

2 $\Rightarrow$ 3

Point $q \in P$.

From 2, There is a function $\omega^T x$ for which q is the unique point in P maximizing $\omega^T x$.

We want to show that q cannot be a mid-point of two other points in P.

for the sake of contradiction $q = \dfrac{p+r}{2}$ where $p, r \in P$

$$\omega^T q = \frac{\omega^T p + \omega^T r}{2}$$

Say, $\omega^T p < \omega^T q \implies \omega^T r > \omega^T q \to$ contradiction.

Hence $\omega^T p = \omega^T q = \omega^T r \implies p = q = r \longrightarrow$ Contradiction.

($\because$ uniqueness)

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3 $\Rightarrow$ 1 Assume $q \in P$ cannot be the mid-point of two other points in P.

We want to show there are n linearly independent equality constraints for q.

Let P be described by

$$a_1^T x \le b_1$$
$$a_2^T x \le b_2$$
$$\vdots$$
$$a_k^T x \le b_k$$

(l could be zero)

Let q satisfy first l constraints with equality

for $1 \le i \le l$ $\quad a_i^T q = b_i$

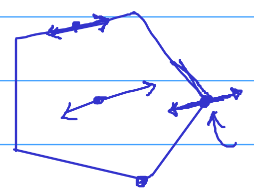
for $l < i \le k$ $\quad a_i^T q < b_i$

$a_1, a_2, \cdots, a_l \in \mathbb{R}^n$

If $\quad \text{rank}(a_1, a_2, \ldots, a_l) = n$, we are done.

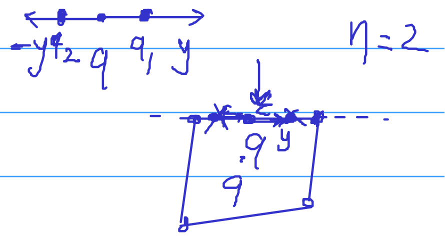
Suppose $\quad \text{rank}(a_1, a_2, \ldots a_l) < n$.

Then there exists a solution $0 \neq y \in \mathbb{R}^n$ for



$$\left\{ \begin{array}{l} a_1^T y = 0 \\ a_2^T y = 0 \\ \vdots \\ a_l^T y = 0 \end{array} \right\}$$

Homogeneous system $\Rightarrow$ Infinitely many nonzero solution



Now, consider two points,

$$q_1 = q + \varepsilon y$$
$$q_2 = q - \varepsilon y \qquad \text{for small enough } \varepsilon > 0$$

Claim $\quad q_1$ and q_2 are inside P.

proof: $\quad q \pm \varepsilon y \in P$

$1 \leq i \leq l \qquad a_i^T(q \pm \varepsilon y) = a_i^T q \pm \varepsilon \underbrace{a_i^T y}_{0} = a_i^T q = b_i$

$l < i \leq k \quad a_i^T(q \pm \varepsilon y) = \underbrace{a_i^T q}_{< b_i} \pm \varepsilon \underbrace{a_i^T y} < b_i$

for small enough ε.

But $\quad q = \dfrac{q_1 + q_2}{2}$

which is a contradiction.

Hence $\quad \underline{\text{rank}(a_1, a_2, \cdots, a_l) = n}$

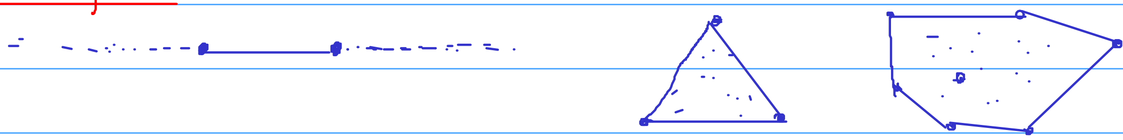
$\Rightarrow$ n linearly independent equality constraints for q.

Def (Convex Hull): For a set of points $q_1, q_2, \cdots q_r \in \mathbb{R}^n$ their convex hull is the set of all points which can be written as a convex combination of $q_1, q_2, \dots, q_r$.

$$\operatorname{conv}(q_1, q_2, \cdots q_r)$$

$$= \{\lambda_1 q_1 + \lambda_2 q_2 + \cdots + \lambda_r q_r : \lambda_1, \lambda_2 \cdots, \lambda_r \geq 0, \ \lambda_1 + \lambda_2 + \cdots + \lambda_r = 1\}$$

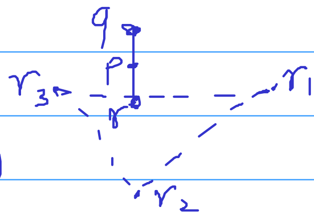
Examples:



Observation: $p \in \operatorname{conv}(q, r)$

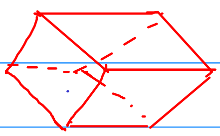
and $r \in \operatorname{conv}(r_1, r_2, \cdots r_\ell)$

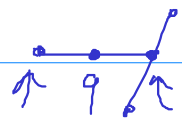
[HW] then $p \in \operatorname{conv}(q, r_1, r_2, \dots, r_\ell)$



Claim: Any polytope is the convex hull of its finitely many corners.

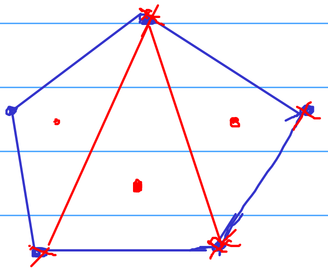
finitely many combinations of n tight constraints

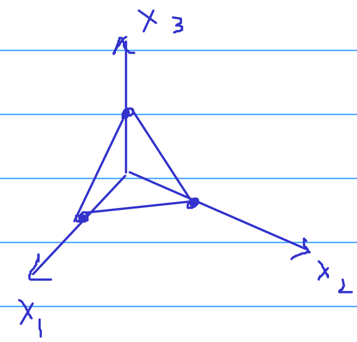




Proof: Use definition 3 of corners.

Carathéodory's Theorem: Any point in an n-dimensional polytope is a convex combination of at most $n+1$ corners.


← 2 dim



$$\left.\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \\ x_1 + x_2 + x_3 = 1 \end{array}\right]$$

Proof (Induction based on the dimension of the polytope)

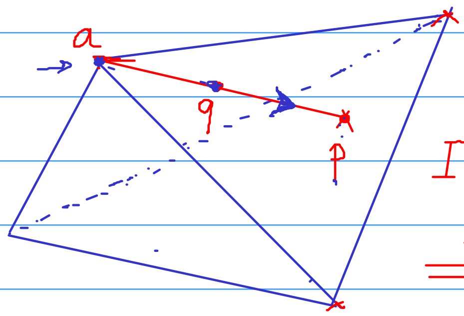
↓

n – no. of linearly independent equality constraints.

Base Case: dim = 0 (trivial)

Induction hypothesis: assume that the statement is true for polytopes with dimension upto $n-1$.

Induction Step: want to show it for dim = n.



$q \in \text{Conv}(a, p)$

IH $\quad p \in \text{Conv}(p_1, p_2, p_3)$

$\Rightarrow q \in \text{Conv}(a, p_1, p_2, p_3)$

Let q be the given point in the polytope

Let a be an arbitrary corner of polytope.
(s.t. the equality constraints for q are also satisfied by a)

Define a new point

$$p = q + \alpha (q - a)$$

$$\alpha > 0$$

$a^T x < b$

$a^T x = b$

$a^T x > b$

start from $\alpha = 0$
and keep increasing it till an additional constraint becomes tight. (we hit some face)

$\Rightarrow$ p satisfies one more equality constraint than q and hence, p lies in a $\underbrace{\text{smaller dimension}}_{\leq n-1}$ polytope.

By induction hypothesis

$$p \in \text{conv}(p_1, p_2, \dots, p_n)$$

$\hookrightarrow$ corners

$$q \in \text{conv}(a, p) \quad \left(\because q = \frac{1}{\alpha+1} p + \frac{\alpha}{\alpha+1} a \right)$$

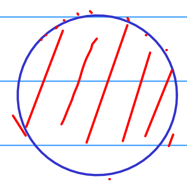
$$\alpha \geq 0$$

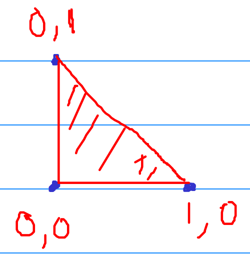
$$\Longrightarrow q \in \text{Conv}(a, p_1, p_2, \dots, p_n)$$

q is a convex combination of $n+1$ corners.

Claim: Any convex hull of finitely many points in $\mathbb{R}^n$ is a polytope.

(i.e., can be described by finitely many linear inequalities in variables $x_1, x_2, \ldots, x_n$)





$-\mathrm{Conv}\big((0,0),(1,0),(0,1)\big)$

$$\boxed{\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \\ x_1 + x_2 \leq 1 \end{array}}$$

Proof: $\mathrm{Conv\,hull}(q_1, q_2, \ldots, q_r) \subseteq \mathbb{R}^n$

$$\left\{ x \in \mathbb{R}^n : \begin{array}{l} x = \lambda_1 q_1 + \lambda_2 q_2 + \cdots + \lambda_r q_r \\ \lambda_1 + \lambda_2 + \cdots + \lambda_r = 1 \\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \ldots,\ \lambda_r \geq 0 \end{array} \right\} \quad n+r \text{ variables}$$

$$= \{ x : Ax \leq b \} \quad n \text{ variables}$$

We want to construct a system of inequalities $Ax \leq b$ s.t.

Remove all λ_i s one by one via Fourier Motzkin Elimination

Removing λ_i

$$\begin{array}{l} \lambda_i \geq E_1 \\ \lambda_i \geq E_2 \\ \lambda_i \leq E_3 \\ \lambda_i \leq E_4 \end{array}$$

The two sets of constraints have the "same" feasible solutions.

FM Elimination (Fourier Motzkin) — Lecture 4, Jan 17

Let S be a system of constraints in variables $(x_1, x_2, \dots, x_n, y)$

Let S' be the system of constraints in variables $(x_1, x_2, \dots x_n)$ which is obtained from S by eliminating y via FM elimination

Want: a point $(x_1, x_2, \dots x_n)$ is feasible for S'

[HW] $\Updownarrow$ $\Uparrow$ easy $\Downarrow$ somewhat easy

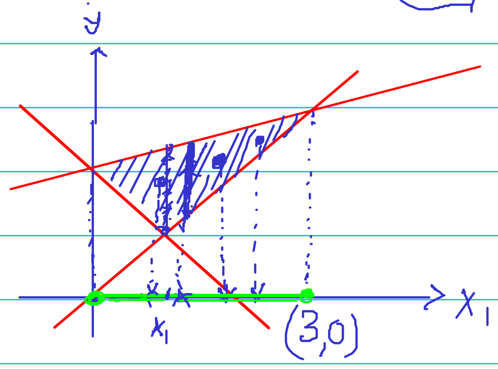
$\exists\, y \in \mathbb{R}$ $(x_1, x_2, \dots x_n, y)$ is feasible for S.

$$\underbrace{\begin{array}{l} y \le E_1 \\ y \ge E_2 \\ y \ge E_3 \\ y = E_4 \\ y = E_5 \end{array}}_{S} \Longrightarrow \underbrace{\begin{array}{l} E_4 \le E_1 \\ E_4 \ge E_2 \\ E_4 \ge E_3 \\ E_4 = E_5 \end{array}}_{S'}$$

$$\underbrace{\begin{array}{l} y \le E_1 \\ y \ge E_2 \\ y \ge E_3 \\ y \le E_4 \\ y \le E_5 \end{array}}_{S} \xrightarrow{\;n_l \quad n_g\;} \underbrace{\begin{array}{l} E_1 \ge E_2 \\ E_1 \ge E_3 \\ E_2 \le E_4 \\ E_3 \le E_4 \\ E_2 \le E_5 \\ E_3 \le E_5 \end{array}}_{S'} \quad n_l * n_g$$

Example

$$\left.\begin{array}{l} x_1 - y \le 0 \\ x_1 + y \ge 2 \\ 3y - x_1 \le 6 \end{array}\right\} S$$



$y \ge x_1$
$y \ge 2 - x_1$
$y \le \dfrac{6 + x_1}{3}$

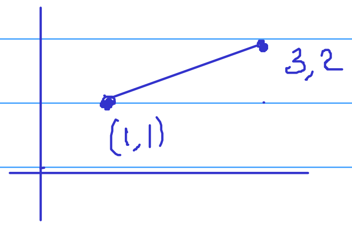
$\dfrac{6 + x_1}{3} \ge x_1 \implies x_1 \le 3$

$\dfrac{6 + x_1}{3} \ge 2 - x_1 \implies x_1 \ge 0$

Geometrically, Polyhedron (S') is a projection of Polyhedron (S) on to the coordinates $(x_1, \dots, x_n)$

S $\quad y \le X_1 + X_2 \longrightarrow$ No constraints $\mathbb{R}^2$

Getting a description of conv(P_1, P_2) via Fourier Motzkin Elimination (Example 1)



$$\to (x_1, x_2) = \lambda_1 (1,1) + \lambda_2 (3,2)$$

$$\lambda_1 + \lambda_2 = 1$$
$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0 \leftarrow$$

$$x_1 = \lambda_1 + 3\lambda_2$$
$$x_2 = \lambda_1 + 2\lambda_2$$

Use $\lambda_2 = 1 - \lambda_1$

$$\to x_1 = \lambda_1 + 3 - 3\lambda_1 = -2\lambda_1 + 3$$
$$x_2 = \lambda_1 + 2 - 2\lambda_1 = -\lambda_1 + 2$$
$$\lambda_1 \geq 0$$
$$1 - \lambda_1 \geq 0 \implies \lambda_1 \leq 1$$

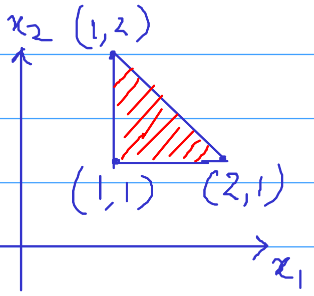
Use $\lambda_1 = 2 - x_2$

$$2 - x_2 \geq 0 \implies x_2 \leq 2$$
$$2 - x_2 \leq 1 \implies x_2 \geq 1$$

$$x_1 = -2(2 - x_2) + 3$$

$$\boxed{\begin{aligned} 1 &= 2x_2 - x_1 \\ 1 \leq\ &x_2 \leq 2 \end{aligned}}$$

Getting a description of conv (P_1, P_2, P_3) via FM elimination



$$(x_1, x_2) = \lambda_1(1,1) + \lambda_2(2,1) + \lambda_3(1,2)$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$\downarrow$ Eliminating λ_3 (put $\lambda_3 = 1 - \lambda_1 - \lambda_2$)

$$x_1 = \lambda_1 + 2\lambda_2 + 1 - \lambda_1 - \lambda_2 = \lambda_2 + 1$$
$$x_2 = \lambda_1 + \lambda_2 + 2 - 2\lambda_1 - 2\lambda_2 = 2 - \lambda_1 - \lambda_2$$
$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0$$
$$1 - \lambda_1 - \lambda_2 \geq 0 \Leftrightarrow \lambda_1 + \lambda_2 \leq 1$$

$\downarrow$ Eliminating λ_2 (put $\lambda_2 = x_1 - 1$)

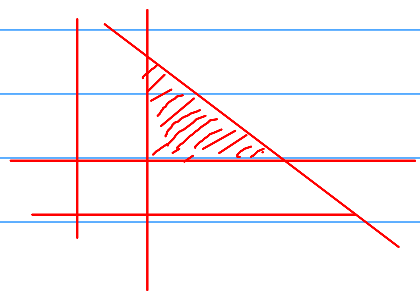
$$x_2 = 3 - \lambda_1 - x_1$$
$$\lambda_1 \geq 0$$
$$x_1 - 1 \geq 0$$
$$\lambda_1 + x_1 \leq 2$$

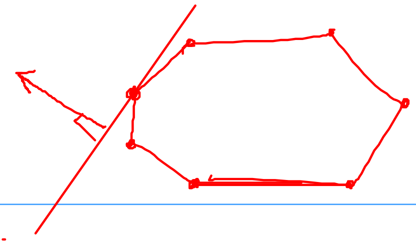
$\downarrow$ Eliminating λ_1 (put $\lambda_1 = 3 - x_1 - x_2$)

$$3 - x_1 - x_2 \geq 0$$
$$x_1 - 1 \geq 0$$
$$3 - x_2 \leq 2$$

$\downarrow$

$$\boxed{\begin{array}{c} x_1 + x_2 \leq 3 \\ x_1 \geq 1 \\ x_2 \geq 1 \end{array}}$$





Facts So far :

① Set of optimizing points form a face of the polyhedron.

② Thus, there is always some corner among the optimizing points.

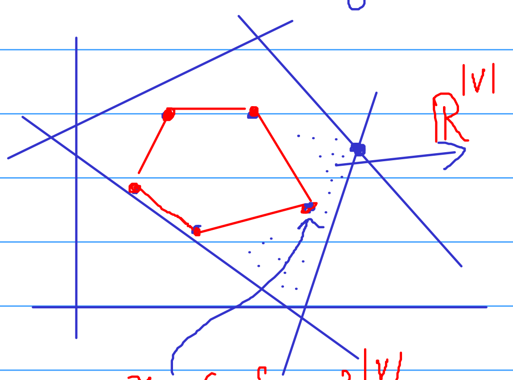
③ For every corner of the polyhedron, we can construct a linear function that is optimized at that corner.

④ Every polytope is the convex hull of its corners.

⑤ Convex hull of finitely many points is a polytope.

Que: What is the right linear program or polyhedron for a combinatorial optimization problem?

For example, consider the problem of finding a maximum weight independent set.

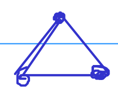


$G(V, E)$

$$\max \sum_{v \in V} w_v x_v$$

for each edge $e = (u,v)$, $x_u + x_v \leq 1$

$0 \leq x_u \leq 1$

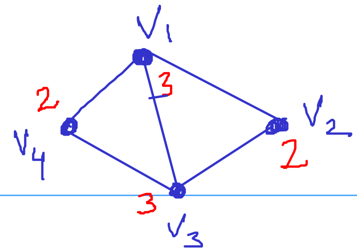


$S \subseteq V \longrightarrow x_S \in \{0,1\}^{|V|}$

(1) Each independent set point should be feasible (i.e., inside the polyhedron)

Hence, the convex hull of independent set points should be contained in the polyhedron

Example G



Max $3x_1 + 2x_2 + 3x_3 + 2x_4$

$0 \le x_1, x_2, x_3, x_4 \le 1$
$x_1 + x_2 \le 1$
$x_2 + x_3 \le 1$
$x_3 + x_4 \le 1$
$x_4 + x_1 \le 1$
$x_1 + x_3 \le 1$

Max weight Ind set = 4

Conv (Ind set points) $\neq$

$0 \to \{v_2, v_4\} \to (0,1,0,1)$
$\frac{1}{2}$ $\{v_1\} \to (1,0,0,0)$
$\frac{1}{2}$ $\{v_3\} \to (0,0,1,0)$
$0 \to \{v_2\} \to (0,1,0,0)$
$0 \to \{v_4\} \to (0,0,0,1)$
$0 \to \phi \to (0,0,0,0)$

$f(x) = 4 \leftarrow (x_1, x_2, x_3, x_4) = (0,1,0,1)$

OPT = 2

$f(x) = 5 \leftarrow (x_1, x_2, x_3, x_4) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

not a convex combination.

[HW]: Any 0-1 point can't be a convex combination of other 0-1 points.

Remember that any corner of the polyhedron is an optimizing point for some linear function.

Hence, we do not want any corners other than the independent set points.

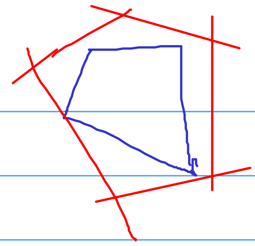
In other words, the polyhedron described by the constraints should be exactly equal to convex hull of independent set points.

Que If we write a linear program with all the clique constraints, does that give the independent set polytope?

$0 \le x_u \le 1$

for every clique C $\quad \sum_{u \in C} x_u \le 1$

[HW] example where this LP fails.



Perfect Matching in Bipartite Graphs

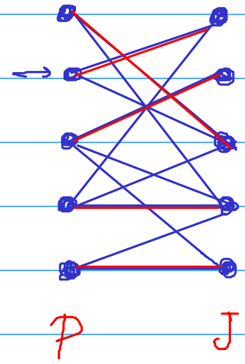
$\{x_e\}_{e \in E}$

P(BPM)

$$0 \le x_e \le 1$$

for every vertex $v \in V$

$$\sum_{\substack{e \text{ incident} \\ \text{on } v}} x_e = 1$$



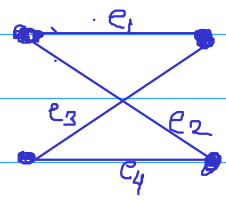
PM: a subset of edges s.t. every vertex has exactly one edge on it.

$0 \le \alpha \le 1$

$(\alpha, 1-\alpha, 1-\alpha, \alpha)$

$\in$ Convhull

$$1 \ge x_{e_1}, x_{e_2}, x_{e_3}, x_{e_4} \ge 0$$
$$x_{e_1} + x_{e_2} = 1$$
$$x_{e_1} + x_{e_3} = 1$$
$$x_{e_3} + x_{e_4} = 1$$
$$x_{e_2} + x_{e_4} = 1$$



PM points

$\alpha\,\vec{(1, 0, 0, 1)}$ $\quad 1/2$

$(1-\alpha)\,\vec{(0, 1, 1, 0)}$ $\quad 1/2$

$(1/2, 1/2, 1/2, 1/2)$

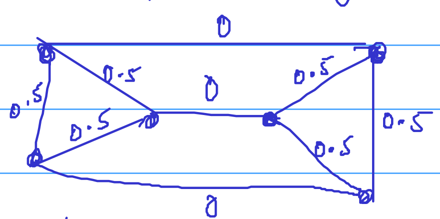
Does this system of constraints describe the perfect matching polytope for bipartite graphs?

Can you find an example where

the above polytope contains a point which is not a convex combination of perfect matching points?

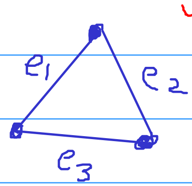
equivalently there is a linear weight function such that optimal value of LP is larger than maximum weight of a perfect matching?

Example on general graphs:



$\notin$ convhull(PM)

[HW]



No PMs

Conv hull (PM) is empty

$$0 \le x_{e_1}, x_{e_2}, x_{e_3} \le 1$$
$$x_{e_1} + x_{e_2} = 1$$
$$x_{e_2} + x_{e_3} = 1$$
$$x_{e_3} + x_{e_1} = 1$$
$$(1/2, 1/2, 1/2)$$

Lemma: For bipartite graphs,

$P(BPM)$ is the convex hull of perfect matching points.

Or, equivalently each corner of $P(BPM)$ is a perfect matching point.

Proof: We will show that each corner of $P(BPM)$ is integral. $\in \{0,1\}^{|E|}$

$\Rightarrow$ each corner is PM point.

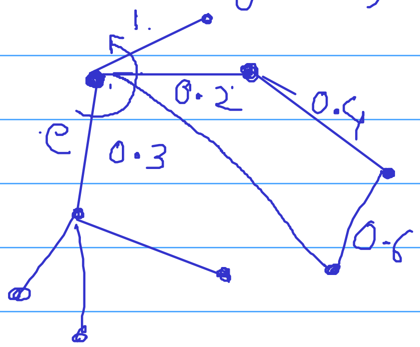
Take $\alpha \in \mathbb{R}^E$ which is feasible

$$\alpha_e \geq 0 \quad \forall e$$

$$\sum_{\substack{e \text{ incident} \\ \text{on } v}} \alpha_e = 1 \quad \forall v$$

Suppse α is fractional, i.e, for some edge e,

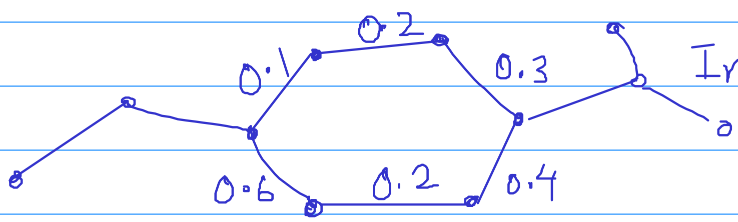
$$0 < \alpha_e < 1$$



Summation around a vertex is 1.

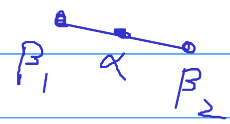
$\Rightarrow$ at least one of the neighboring edges e' $0 < \alpha_{e'} < 1$

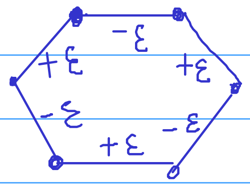
keep finding neighboring edges till we get a cycle.



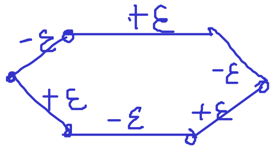
In a bipartite graph, cycles are all even length

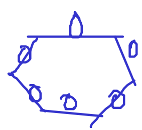
Want to argue that α is not a corner.



$\beta_1 = \alpha +$ 

$$\alpha = \frac{\beta_1 + \beta_2}{2}$$

$\beta_2 = \alpha +$ 

$\beta_1 + \beta_2 = 2\alpha +$ 

We claim that β_1, β_2 satisfy all the constraints.

$$\sum_{e \text{ incident on } v} \beta_{1e} = \sum_{e \text{ incident on } v} \alpha_e + \varepsilon - \varepsilon = 1$$

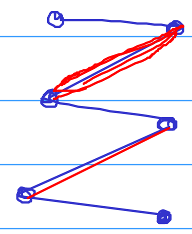
$$\beta_{1e} = \alpha_e \pm \varepsilon$$

We can small enough ε to ensure $\beta_{1e}, \beta_{2e} \geq 0$

Hence β_1, β_2 are feasible points

Matching Polytope

$$\text{Max} \sum_{e \in E} x_e$$



$$\begin{cases} x_e \geq 0 \\ \displaystyle\sum_{e \text{ incident on } v} x_e \leq 1 \end{cases}$$

Claim: The above constraints describe the convex hull of matching points.

[HW]

Two kinds of LP questions

- Optimization $\longrightarrow$ max $w^T x$ s.t. $Ax \leq b$
- Feasibility $\longrightarrow$ is there $x \in \mathbb{R}^n$ satisfying $Ax \leq b$

- The two questions are equivalent.

1. Optimization $\leq$ Feasibility

Guess w_0.

$$\left[\begin{array}{l} \max\ w^T x \\ \text{s.t. } Ax \leq b \end{array}\right.$$

Decide whether opt val $\leq w_0$?

Binary Search.

$$\left.\begin{array}{l} w^T x \geq w_0 \\ Ax \leq b \end{array}\right] \begin{array}{l} \text{is it} \\ \text{feasible ?} \end{array}$$

yes $\rightarrow$ new guess $> w_0$

No $\rightarrow$ new guess $< w_0$

Can we bound Optimal value?

If w^* is the opt value $\longrightarrow$ no. of rounds $O(\underbrace{\log w^*})$

[HW]

Claim: opt value $\leq \exp(n, k, \ell)$ poly (n, k, ℓ)?

$n \rightarrow$ var

$k \rightarrow$ constraints

$\ell \rightarrow$ no. of bits in coefficients.

2. Feasibility $\leq$ Optimization

[HW]

is $Ax \leq b$ feasible? max $1^T x$ s.t. $Ax \leq b$

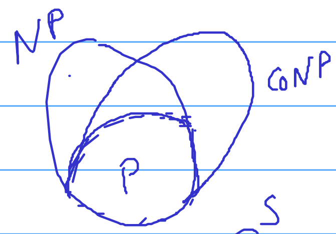
suppose we have an optimization oracle that works correctly only if the given system is feasible, otherwise it outputs garbage.

Computational Complexity of Linear Programming

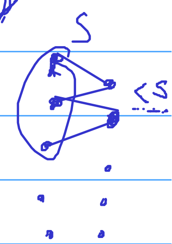
Some inefficient algorithms for LP

→ Go over all the corners.
→ Fourier Motzkin Elimination

Towards a polynomial time algorithm



LP Duality → LP is in $\underbrace{NP \cap coNP}$



Easily verifiable proofs for both Yes and No answers.
"Good Characterization"

Feasibility $Ax \leq b$

If $Ax \leq b$ is feasible, is there an easily verifiable proof for this fact.

any x satisfying $Ax \leq b$ is a proof.

Given x, it's easy to verify $Ax \leq b$

What about the size of x?

Need to guarantee $\exists$ feasible point x whose description size is poly(n, k, ℓ).
(no. of bits)

Corner ⟶ n tight constraints ⟶ $A'x = b'$ solution
$\Rightarrow x = A'^{-1} b'$ (bounded)

If $Ax \le b$ is not feasible, is there an easily verifiable proof for this fact?

What about $Ax = b$?

If $Ax=b$ is not feasible then is there a proof for this?

$$
\begin{aligned}
y_1 \times \{\, a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \,\} \\
y_2 \times \{\, a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \,\} \\
&\vdots \\
y_k \times \{\, a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n &= b_n \,\} \\
\hline
0x_1 + 0x_2 + \dots + 0x_n &= \alpha \neq 0.
\end{aligned}
$$

$$
\begin{aligned}
2 \times \{\, x_1 + x_2 &= 1 \\
2 \times \{\, x_2 + x_3 &= 3 \\
-1 \times \{\, 2x_1 + 4x_2 + 2x_3 &= 5 \\
\hline
0 \quad 0 \quad 0 &= 3
\end{aligned}
$$

Example 1

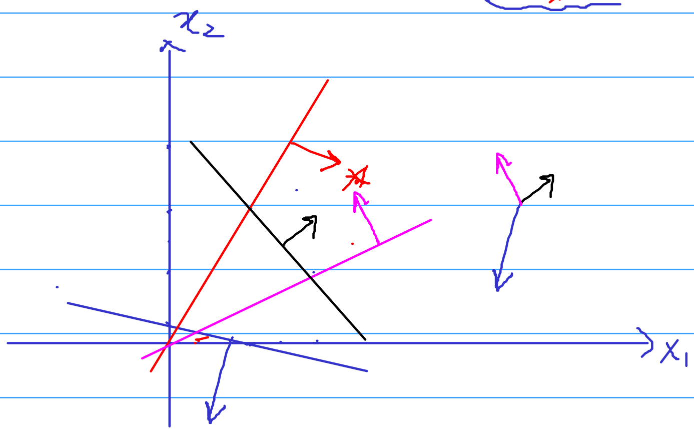
$$
\left.
\begin{aligned}
x_1 &\ge 1 \\
x_2 &\ge 1 \\
x_1 + x_2 &\le 1
\end{aligned}
\right\}
\qquad
\left.
\begin{aligned}
x_1 + x_2 &\ge 2 \\
x_1 + x_2 &\le 1
\end{aligned}
\right\} \Rightarrow 2 \le 1
$$

$$
\begin{aligned}
x_1 &\ge 1 \ \} \times 1 \\
x_2 &\ge 1 \ \} \times 1 \\
-x_1 - x_2 &\ge -1 \ \} \times 1 \\
\hline
0 &\ge 1
\end{aligned}
$$

Example 2

$$
\left.
\begin{aligned}
2x_1 - x_2 &\ge 0 \\
2x_2 - x_1 &\ge 0 \\
x_1 + x_2 &\ge 3
\end{aligned}
\right\}
$$

$$x_1 + 4x_2 \le 1$$



take last three inequalities and show a combination that leads to contradiction

$$
\begin{aligned}
1 \times \{\, 2x_2 - x_1 &\ge 0 \\
2 \times \{\, x_1 + x_2 &\ge 3 \\
1 \times \{\, -x_1 - 4x_2 &\ge -1 \\
\hline
0x_1 + 0x_2 &\ge 5
\end{aligned}
$$

Farkas Lemma

Let $A \in \mathbb{R}^{k \times n}$ and $b \in \mathbb{R}^k$ such that $Ax \geq b$ is not feasible.

Then there exists $y = (y_1, y_2, \cdots, y_k) \in \mathbb{R}^k$ such that

$$y \geq 0, \quad y^T A = 0 \quad \text{and} \quad y^T b > 0$$

$\rightarrow$ (equivalenty, $y \geq 0$, $A^T y = 0$, $b^T y > 0$)

$$0 \leq y_1 \times \{ \; a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \geq b_1$$
$$0 \leq y_2 \times \{ \; a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \geq b_2$$
$$\vdots$$
$$0 \leq y_k \times \{ \; \underline{a_{k1} x_1 + a_{k2} x_2 + \dots + a_{kn} x_n \geq b_k}$$
$$0 x_1 + 0 x_2 \; \cdots \; + 0 x_n \geq 1 \leftarrow \text{or anything} > 0$$

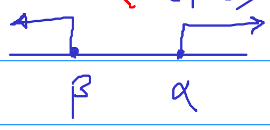
$$y^T A = 0 \equiv \{ y_1 a_{1j} + y_2 a_{2j} + \cdots + y_k a_{kj} = 0 \quad \forall \; 1 \leq j \leq n \}$$

Proof: Induction based on number of variables.

Base case: $n = 1$

$$x_1 \leq b_1$$
$$x_1 \leq b_2$$
$$x_1 \geq -b_3 \iff -x_1 \leq b_3$$
$$x_1 \geq -b_4 \iff -x_1 \leq b_4$$

infeasible only if $\{ x_1 \geq \alpha, \; x_1 \leq \beta$ while $\beta < \alpha$



$$x_1 \geq \alpha \; \} \; 1$$
$$-x_1 \geq -\beta \; \} \; 1$$
$$\overline{0 \geq \alpha - \beta}_{>0}$$

Induction hypothesis: let the statement be true for $n-1$ variables.

Induction step: Fourier Motzkin Elimination

Let's separate out constraints based on whether

$$a_{in} > 0 \quad \text{or} \quad a_{in} < 0, \quad a_{in} = 0$$

Without loss of generality, let the given system of constraints be

$$S_1 \begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots + 1\,x_n \geq b_i & \text{for } 1 \leq i \leq k_1 \\ a_{j1}x_1 + a_{j2}x_2 + \cdots - 1\,x_n \geq b_j & \text{for } k_1 < j \leq k_2 \\ a_{\ell 1}x_1 + a_{\ell 2}x_2 + \cdots + 0 \cdot x_n \geq b_\ell & \text{for } k_2 < \ell \leq k \end{cases}$$

Define another system of constraints in $n-1$ variables.

$$S_2 \begin{cases} \text{for } 1 \leq i \leq k_1, \quad k_1 < j \leq k_2 \\ (a_{i1}+a_{j1})x_1 + (a_{i2}+a_{j2})x_2 \cdots + (a_{i\,n-1}+a_{j\,n-1})x_{n-1} \geq b_i + b_j \\ a_{\ell 1}x_1 + a_{\ell 2}x_2 + \cdots a_{\ell\,n-1}x_{n-1} \geq b_\ell \quad \text{for } k_2 < \ell \leq k \end{cases}$$

Claim: If S_1 is not feasible then S_2 is not feasible.

Proof: HW

Now assume S_2 is not feasible. By induction hypothesis there exist

$$y_{ij} \geq 0 \text{ for } 1 \leq i \leq k_1 \quad k_1 < j \leq k_2$$
$$y_\ell \geq 0 \text{ for } k_2 < \ell \leq k$$

S.t.

$$\sum_{i,j} y_{ij}(a_{ip} + a_{jp}) + \sum_\ell y_\ell a_{\ell p} = \underline{0} \text{ for } 1 \leq p \leq n-1$$

$$\sum_{i,j} y_{ij}(b_i + b_j) + \sum_\ell y_\ell b_\ell > \underline{0}$$

Argue that any non-negative combination of constraints in S_2 is also a non-negative combination of constraints in S_1.

The corresponding $y \geq 0$ coefficients for S_1 can be defined as

for $1 \leq i \leq k_1$ $\quad y_i = \sum_{j=k_1+1}^{k_2} y_{ij}$

for $k_1 < j \leq k_2$ $\quad y_j = \sum_{i=1}^{k_1} y_{ij}$

for $k_2 < l \leq k$ $\quad y_l = y_l$

Show that

for $1 \leq p \leq n$ $\quad \sum_{i=1}^{k} y_i a_{ip} = 0$ for $1 \leq p \leq n$

$$\sum_{i=1}^{k} y_i b_i > 0$$

In other words, $y^T A = 0$, $y^T b > 0$

as desired. $\square$

Farkas Lemma in different forms

1) $Ax \le b$ not feasible $\iff$ $y \ge 0$, $y^TA = 0$, $y^Tb < 0$ feasible

2) $Ax \le b$, $x \ge 0$ not feasible $\iff$ $\underbrace{y \ge 0}$, $y^TA \ge 0$, $y^Tb < 0$ feasible

3) $Ax = b$, $x \ge 0$ not feasible $\iff$ $\underbrace{y \in \mathbb{R}^k}$, $y^TA \ge 0$, $y^Tb < 0$ feasible

Note: Above, in RHS. $y^Tb < 0$ can be replaced with $y^Tb = -1$.

Conclusion

- LP feasibility is in NP $\cap$ coNP.
- Feasibility of a system is equivalent to infeasibility of another.

LP Optimization

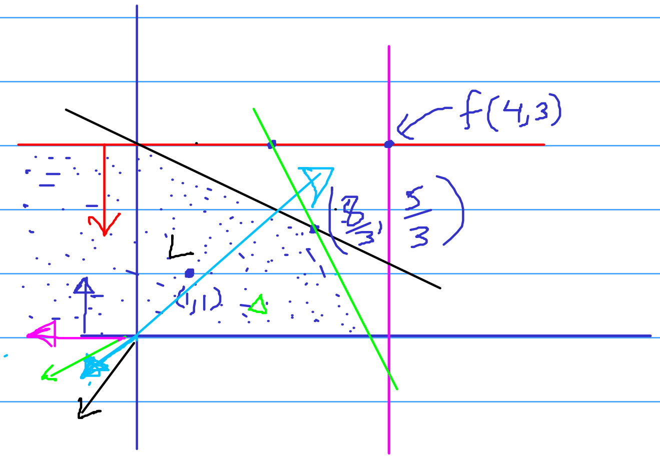
$$\max \ w^T x$$
$$\text{s.t.} \quad Ax \le b$$

Any feasible point gives a lower bound on optimal value

Max $x_1 + x_2 = f(x_1, x_2)$

$x_1 \le 4$	$y_1 = 0$
$x_2 \le 3$	$y_2 = 0$
$x_2 + 2x_1 \le 7$	$y_3 = 1/3$
$\to$ $2x_2 + x_1 \le 6$	$y_4 = 1/3$
$\to$ $x_2 \ge 0$	$y_5 = 0$

optimal dual solution.



$f(1,1) = 2$ opt value ≥ 2.

$f\left(\frac{8}{3}, \frac{5}{3}\right) = \frac{13}{3}$ opt value $\ge \frac{13}{3}$

Can we get an upper bound

$$\begin{array}{r} x_1 \le 4 \\ x_2 \le 3 \\ \hline f(x_1,x_2) = x_1 + x_2 \le 7 \end{array}$$

opt value ≤ 7

$$\begin{array}{r} 2x_2 + x_1 \le 6 \\ -x_2 \le 0 \\ \hline x_1 + x_2 \le 6 \end{array}$$

opt value ≤ 6

$$\begin{array}{r} x_2 \le 3 \\ x_2 + 2x_1 \le 7 \\ \hline 2x_2 + 2x_1 \le 10 \\ \Rightarrow x_2 + x_1 \le 5 \end{array}$$

Opt value ≤ 5

$$\begin{array}{r} x_1 \le 4 \\ x_2 + 2x_1 \le 7 \\ \hline x_1 + x_2 \end{array}$$

$$\begin{array}{r} x_2 + 2x_1 \le 7 \\ 2x_2 + x_1 \le 6 \\ \hline 3x_1 + 3x_2 \le 13 \\ \Rightarrow x_1 + x_2 \le 13/3 \end{array}$$

opt value $\le 13/3$

Conclude: opt value $= 13/3$

Try to prove in general using Farkas Lemma that the right upper bound on objective value can always be obtained by taking some combination of the constraints.

Dual LP

LP $\max \ f(x) = \omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n$

$$
\begin{aligned}
0 \le y_1 \times \quad & a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1 \\
0 \le y_2 \times \quad & a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2 \\
& \vdots \\
0 \le y_k \times \quad & a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kn} x_n \le b_k
\end{aligned}
$$

$$\Rightarrow \underbrace{\sum a_{i1} y_i}_{\omega_1} x_1 + \underbrace{\sum a_{i2} y_i}_{\omega_2} x_2 \cdots + \underbrace{\sum a_{in} y_i}_{\omega_n} x_n \le \sum_{i=1}^{k} y_i b_i \quad \text{--- } 1$$

LP* / Dual LP

$$
\begin{cases}
\min \underbrace{\sum_{i=1}^{k} y_i b_i}_{g(y)} \longleftarrow \text{best upper bound} \\
y_1, y_2, \ldots, y_k \ge 0 \\
a_{11} y_1 + a_{21} y_2 + \cdots + a_{k1} y_k = \omega_1 \\
\vdots \\
a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{kn} y_k = \omega_n
\end{cases}
$$

LP	LP*
constraint $\to$	variable
variable $\to$	constraint
$\max \ \omega^T x$	$\min \ b^T y$
$Ax \le b$	$y \ge 0$
	$A^T y = \omega$

$A \leftarrow k \times n$ $\qquad A^T \leftarrow n \times k$ (A^T: $n \times k$, y: $k \times 1$)

Weak Duality Theorem

If x is a feasible point for LP and y is a feasible point for LP* then

$$f(x) = \sum_i \omega_i x_i \le \sum_j y_j b_j = g(y) \qquad \text{follows from } 1$$

In particular,

$$f(x^*) \le g(y^*)$$

$$\mathrm{Opt}(LP) \le \mathrm{Opt}(LP^*) \longleftarrow \text{best upper bound of opt (LP).}$$

Strong Duality theorem

$$\mathrm{Opt}(LP) = \mathrm{Opt}(LP^*)$$

$$\max\{\omega^T x : Ax \le b\} = \min\{b^T y : y \ge 0,\ A^T y = \omega\}$$

(if they exist) (if LP is feasible and opt (LP) is bounded)

We will prove if $\max\{\omega^T x : Ax \le b\} \le u$ then there exists $y \ge 0$ s.t. $A^T y = \omega$ and $b^T y \le u$

Equivalently, if there is no $y \in \mathbb{R}^k$ with $y \ge 0$, $A^T y = \omega$ and $b^T y \le u$ then $\exists x$ s.t. $Ax \le b$ but $\omega^T x > u$ (or $Ax \le b$ is infeasible)

Proof Suppose $y \ge 0$, $A^T y = \omega$, $b^T y \le u$ is infeasible.

$\uparrow$ n constraints — 1 constraint

$$\begin{aligned} A^T y &\le \omega \\ -A^T y &\le -\omega \\ b^T y &\le u \end{aligned}$$

From Farkas Lemma

$\exists\ \alpha = (\alpha_1, \alpha_2 \cdots, \alpha_n) \in \mathbb{R}^n$

and $\alpha_0 \ge 0$

$$\text{s.t.}\quad (A^T)^T \alpha + \alpha_0 b \ge 0$$

$$\omega^T \alpha + u\alpha_0 < 0$$

Case 1 $\alpha_0 = 0 \Rightarrow A\alpha \ge 0$ and $\omega^T \alpha < 0$

Assuming $Ax \le b$ is feasible let x_0 be a feasible point

Consider another point $x_0 - N\alpha$ for some $N > 0$.

(i) $x_0 - N\alpha$ is feasible because $A(x_0 - N\alpha) = Ax_0 - NA\alpha \le b - 0$

The objective value at $x_0 - N\alpha$ is

$$\omega^T(x_0 - N\alpha) = \omega^T x_0 - N\omega^T \alpha$$

Since $\omega^T \alpha < 0$, we can choose N to achieve arbitrary large objective value.

Hence $x_0 - N\alpha$ can be a feasible point with $\omega^T(x_0 - N\alpha) > u$

Case 2 $\alpha_0 > 0$ $\quad A\alpha \geq -\alpha_0 b$ $\quad$ divide by $-\alpha_0$

$w^T\alpha < -u\alpha_0$ $\quad$ define $\alpha' = \dfrac{\alpha}{-\alpha_0}$

$$\boxed{\begin{array}{c} A\alpha' \leq b \\ w^T\alpha' > u \end{array}} \quad \square$$

Some conclusions:

- If x is a feasible solution for LP and y is a feasible solution for LP*

  and $f(x) = g(y)$

  then x is an optimal solution for LP and y is an optimal solution for LP*.

- Strong LP duality means the optimality can always be proved via a dual solution.

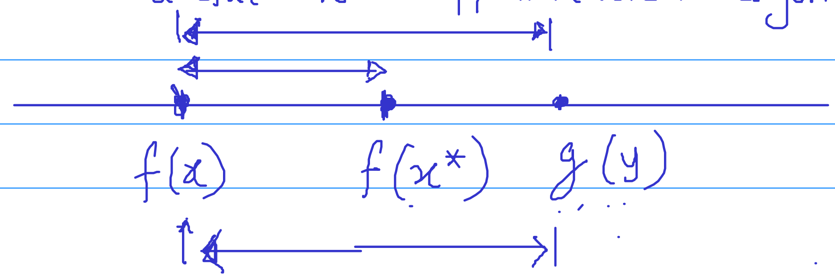
- LP optimization is in NP ∩ coNP. (Is the opt value $\geq w^*$)

-

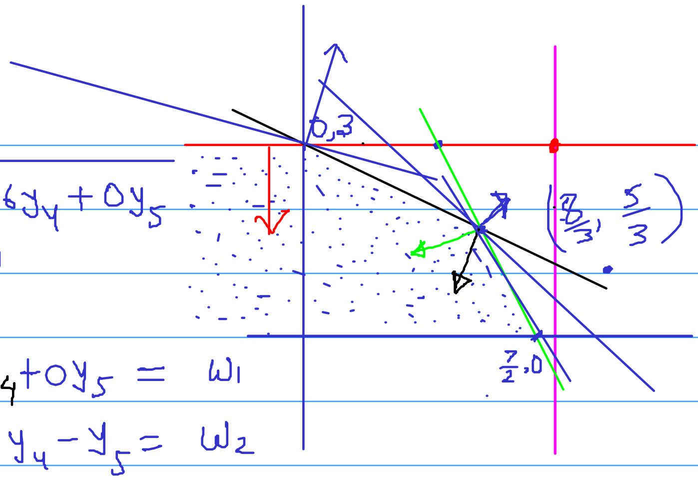
LP		LP*
feasible and bounded	$\Longleftrightarrow$	feasible and bounded
Unbounded optimal	$\Longrightarrow$	Infeasible
Infeasible	$\not\Longleftarrow$	unbounded

Both LP and LP* can be infeasible

- Duality can be useful for approximation algorithms.



Lecture 8 Jan 31



$$\text{Max } f(x_1,x_2) = w_1 x_1 + w_2 x_2$$

$x_1 \le 4$

$x_2 \le 3$

$\to x_2 + 2x_1 \le 7$

$2x_2 + x_1 \le 6$

$\cancel{x_2 \ge 0}$

$-x_2 \le 0$

$$\boxed{\begin{array}{l} \text{Min } 4y_1 + 3y_2 + 7y_3 + 6y_4 + 0y_5 \\ y_1, y_2, y_3, y_4, y_5 \ge 0 \\ \checkmark\ 1\cdot y_1 + 0\,y_2 + 2\,y_3 + 1\cdot y_4 + 0y_5 = w_1 \\ \checkmark\ 0\cdot y_1 + 1\cdot y_2 + 1\cdot y_3 + 2y_4 - y_5 = w_2 \end{array}}$$

$f(x_1, x_2)$	x^*	$f(x^*)$	y^*	$g(y^*)$
$x_1 + x_2$	$8/3, 5/3$	$13/3$	$0, 0, 1/3, 1/3, 0$	$13/3$
			$1, 1, 0, 0, 0$	7
			feasible dual (not optimal)	
$x_1 + 3x_2$	$0, 3$	9	$0, 1, 0, 1, 0$	9
$4x_1 + 2x_2$	$8/3, 5/3$	14	$0, 0, 2, 0, 0$	14
	$7/2, 0$	14		
	$1, 5$	14		

$$\text{Max } f(x_1,x_2) = w_1 x_1 + w_2 x_2$$

$x_1 \le 4$

$x_2 \le 3$

$x_2 + 2x_1 \le 7$

$2x_2 + x_1 \le 6$

$x_2 \ge 0$

$$\boxed{\begin{array}{l} \text{Min } 4y_1 + 3y_2 + 7y_3 + 6y_4 + 0y_5 \\ y_1, y_2, y_3, y_4, y_5 \ge 0 \\ 1\cdot y_1 + 0\,y_2 + 2\,y_3 + 1\cdot y_4 + 0y_5 = w_1 \\ 0\cdot y_1 + 1\cdot y_2 + 1\cdot y_3 + 2y_4 - y_5 = w_2 \end{array}}$$

equivalent

$$0\cdot y_1 + 1\cdot y_2 + 1\cdot y_3 + 2y_4 \ge w_2$$

$$\boxed{\begin{array}{l} \text{Min } 4y_1 + 3y_2 + 7y_3 + 6y_4 \\ y_1, y_2, y_3, y_4 \ge 0 \\ 1\cdot y_1 + 0\,y_2 + 2\,y_3 + 1\cdot y_4 = w_1 \\ 0\cdot y_1 + 1\cdot y_2 + 1\cdot y_3 + 2y_4 \ge w_2 \end{array}}$$

Economic Interpretation

Suppose a farmer has limited quantities of land, fertilizer and water/electricity. They need to decide on the amounts of wheat and chickpeas to grow.

$$\text{Max } \underline{2500 x_w + 6000 x_c}$$

$x_w, x_c \geq 0$ 4.93

$0.05 x_w + 0.1 x_c \leq 5$ (Land)

$0.5 x_w + 2 x_c \leq 70$ (Fertilizer)

$1 x_w + 1.1 x_c \leq 80$ (Electricity)

per unit revenue

acres per unit

Optimal Solution

$x_w = 57.24$

$x_c = 20.68$

Land is not fully utilized

Fertilizer and electricity fully utilized.

$$\text{Min } y_L 5 + y_F 70 + y_E 80$$

$y_L, y_F, y_E \geq 0$

$0.05 y_L + 0.5 y_F + 1 \cdot y_E \geq 2500$

$\underbrace{0.1 y_L + 2 y_F + 1.1 y_E}_{\text{per unit expenditure/cost for the resources}} \geq 6000$

(total cost for 1 quintal chickpeas) Rev/quin

optimal solution

$y_L = 0$

$y_F = 2241$

$y_E = 1379$

Shadow pricing: revenue generated if this resource is increased by one unit.

opportunity cost

By adding 1 kg of fertilizers, you can increase the opt value by $y_F = 2241$.

If the real cost of 1 kg fertilizers is less than 2241 then you can go ahead and buy 1 kg of fertilizers and increase your revenue.

Dual for various forms of LP

$$\text{Max } \omega^T x \quad \text{s.t. } Ax \leq b \quad \longleftrightarrow \quad \boxed{\text{min } b^T y,\ y \geq 0,\ A^T y = \omega}$$

$$\text{Max } \omega^T x \quad \text{s.t. } x \geq 0,\ Ax \leq b \qquad \text{Min } b^T y_2,\ y_1, y_2 \geq 0,\ A^T y_2 - I \cdot y_1 = \omega \qquad \boxed{\text{Min } b^T y,\ y \geq 0,\ A^T y \geq \omega}$$

$$\text{Max } \omega^T x \quad \text{s.t. } x \geq 0,\ Ax = b \qquad \boxed{\text{Min } b^T y,\ A^T y \geq \omega}$$

HW prove strong duality for other forms.

<u>Claim</u> $(LP^*)^* = LP$

Dual of an LP in general form

max $\omega_1 x_1 + \omega_2 x_2$ min $b_1 y_1 + b_2 y_2 - b_3 y_3$

✓ $\boxed{x_1 \geq 0}$ ← $a_{11} y_1 + a_{21} y_2 - a_{31} y_3 \geq \omega_1$

$x_2 \in \mathbb{R}$ ← $a_{12} y_1 + a_{22} y_2 - a_{32} y_3 = \omega_2$

✓ $a_{11} x_1 + a_{12} x_2 = b_1$ ← $y_1 \in \mathbb{R}$

✓ $a_{21} x_1 + a_{22} x_2 \leq b_2$ ← $y_2 \geq 0$

~~$a_{31} x_1 + a_{32} x_2 \geq b_3$ $y_3 \leq 0$~~

$-a_{31} x_1 - a_{32} x_2 \leq -b_3$ $y_3 \geq 0$

Max

$x_i \geq 0$

$a_1^T x = b_1$ $y_1 \in \mathbb{R}$

$\left.\begin{array}{l} a_1^T x \leq b_1 \\ -a_1^T x \leq -b_1 \end{array}\right]$ ← $y_1 = \underline{y_1'} - \underline{y_1''}$ $y_1' \geq 0$ $y_1'' \geq 0$

Shortest Path LP.

Directed graph $G(V, E)$.
Each edge $e \in E$ has a non-negative length l_e.

Source vertex $\longrightarrow$ s
destination vertex $\longrightarrow$ t

Find a shortest path from s to t.

→ Is it the right linear program?

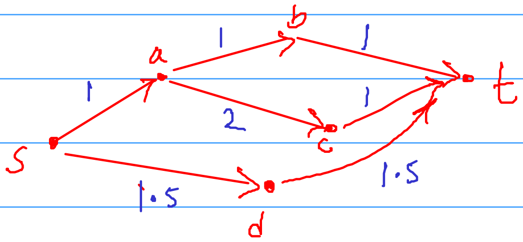
Can LP optimal be smaller than the shortest path length?

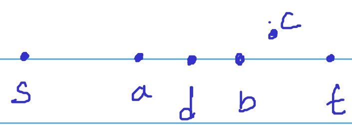
It turns out that there is always an integral optimal solution.

Dual LP for the shortest path

Physical interpretation of the dual:

Imagine a length l_e thread for edge e.
Pull t as far away from s as possible.





optimal solution

Max $y_t - y_s$

$y_a - y_s \le 1$ $\quad$ $y_t - y_c \le 1$

$y_b - y_a \le 1$ $\quad$ $y_d - y_s \le 1.5$

$y_t - y_b \le 1$ $\quad$ $y_t - y_d \le 1.5$

$y_c - y_a \le 2$

$y_a \longrightarrow$ distance of a from s.

Tight threads correspond to tight dual constraints.

Claim The optimal value of shortest path LP is equal to the length of the shortest path.

proof: We will construct a dual feasible solution whose dual objective value is equal to the length of the shortest path.

Bipartite Maximum matching Linear Program.