Linear Optimization

Variables \( x_1, x_2, x_3, \ldots, x_n \in \mathbb{R} \)

Given constraints on these variables.

Linear constraints

\[
\begin{align*}
& x_3 \geq 0, \\
& 2x_1 + x_3 - x_4 \geq 10 \\
& x_1 + 5x_4 = 6
\end{align*}
\]

Want to minimize/maximize an objective function

\( f(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R} \) subject to the given constraints.

Maximize \( 8x_1 + 5x_2 - 2x_3 \ldots \)

Example 1:

Max \( x_1 + x_2 \)

s.t. \( x_2 - x_1 \leq 2 \)

\( 2x_1 + x_2 \leq 4 \)

\( x_1, x_2 = \left( \frac{2}{3}, \frac{8}{3} \right) \)

\( f(x_1, x_2) = \frac{10}{3} \)

---

Integer Linear Programming (ILP)

Allowed to restrict some variables to integers.

\[ \rightarrow \text{ILP is NP-hard.} \]

Simple reduction from 3-SAT
Example 2: Mid-day meal plan

<table>
<thead>
<tr>
<th></th>
<th>Rice</th>
<th>Dal</th>
<th>Egg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calories</td>
<td>130</td>
<td>115</td>
<td>75</td>
</tr>
<tr>
<td>Carb</td>
<td>25</td>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>Protein</td>
<td>3.5</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>cost</td>
<td>5</td>
<td>20</td>
<td>6</td>
</tr>
</tbody>
</table>

Requirements: Calories ∈ [800, 1000] ✓
Carbs ∈ [75, 100] ✓
Protein ∈ [10, 17] ✓

\[ x_R \in \mathbb{R}, x_D, x_E \geq 0 \]

\[ \text{Min } 5x_R + 20x_D + 6x_E \]
\[ \text{s.t.} \]
\[ 800 \leq 130x_R + 115x_D + 75x_E \leq 1000 \]
\[ 75 \leq 25x_R + 20x_D + x_E \leq 100 \]
\[ 10 \leq 3.5x_R + 8x_D + 7x_E \leq 17 \]

Combinatorial Optimization

Feasible solutions ← discrete set of objects.
Examples: paths, trees, cuts, independent sets, matchings

Example 1

Can we write an integer linear program (ILP) for maximum independent set problem?

Maximum independent set: Given a graph \(G(V,E)\), find a maximum size subset \(S \subseteq V\) of vertices s.t.
no two vertices in \(S\) have an edge between them.

\[ \text{Max } \sum_{v \in V} x_v \]
\[ \text{subject to } \]
\[ x_v \text{ for } v \in V \]
\[ x_v \in \{0, 1\} \]
for \((u,v) \in E\) \(x_u + x_v \leq 1\)

HW Show that every independent set corresponds to a feasible solution
and every feasible solution corresponds to an independent set.
Linear programming relaxation
\[
\begin{align*}
\text{Max} & \quad \sum_{v \in V} \chi_v \\
\text{subject to} & \quad \chi_v \text{ for } v \in V \\
& \quad \chi_v \in \{0,1\}, \quad 0 \leq \chi_v \leq 1 \\
& \quad \chi_u + \chi_v \leq 1 \\
\text{for } (u,v) \in E
\end{align*}
\]

\[0 \leq \chi_v \leq 0.1 \text{ or } 0.9 \leq \chi_v \leq 1\]

Not a linear program.

Can you think of a graph where the optimal value of the above LP is different from maximum size of an ind set?

\[
\begin{align*}
\chi_1 \\
\chi_2 \\
\chi_3
\end{align*}
\]

\[
\begin{align*}
\text{Max } & \quad \chi_1 + \chi_2 + \chi_3 \\
\rightarrow & \quad \chi_1 + \chi_2 \leq 1, \quad 0 \leq \chi_1, \chi_2, \chi_3 \leq 1 \\
& \quad \chi_2 + \chi_3 \leq 1 \\
& \quad \chi_3 + \chi_1 \leq 1
\end{align*}
\]

Optimal value = 1.5

\((0.5, 0.5, 0.5)\)

\[
\text{Max Ind set size = 1}
\]

can we fix the situation with adding more constraints?

for any triangle \((v_u, v_v, v_w)\) \[\chi_u + \chi_v + \chi_w \leq 1\]

for any clique \(C\) \[\sum_{u \in C} \chi_u \leq 1\]

[HW]

Que: With the additional constraints, is the \(\text{Opt}(LP) = \text{Max Ind set}\)?

Example 2: Max weight odd subset

Given a set of \(n\) objects with positive/negative weights, find an odd subset with maximum total weight.

Straightforward Algorithm.

ILP? need exponentially many constraints.

[HW]
Matrix representation of a linear program

\[
\begin{align*}
\text{Max } & \ 4x_1 + 3x_2 \\
& \ s.t. \ \begin{cases}
-1x_1 + 2x_2 \leq 2 \\
2x_1 + x_2 \leq 4 \\
3x_1 - 2x_2 \leq 5
\end{cases} \\
\end{align*}
\]

\[w = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2 \times 1}, \quad \text{Max } w^T x \]

Each row of the matrix corresponds to a constraint.

Say \( n \) variables and \( k \) constraints.

\[A \in \mathbb{R}^{k \times n} \quad (A \text{ is } k \times n)\]

\[b \in \mathbb{R}^k\]

\[x \in \mathbb{R}^n \quad x = (x_1, x_2, \ldots, x_n) \quad \text{nxl matrix}\]

\[w \in \mathbb{R}^n \quad (\text{conventionally})\]

Conversion between various kinds of constraints and objectives

\[
\begin{align*}
\geq & \quad \implies \quad \begin{cases} x_1 + x_3 \geq 5 \implies -x_1 - x_3 \leq -5 \\
= & \quad \implies \quad \begin{cases} x_1 + 2x_2 = 1 \implies \begin{cases} x_1 + 2x_2 \leq 1 \\
x_1 + 2x_2 \geq 1
\end{cases}
\end{cases}
\end{align*}
\]

Min \quad \rightarrow \quad \text{Max} \quad \begin{align*}
\begin{cases}
\min \quad 2x_1 - x_2 \quad \iff \quad \max \quad x_2 - 2x_1
\end{cases}
\end{align*}

\[
\begin{align*}
\leq & \quad = \quad \begin{cases} x_1 + x_2 \leq 5 \iff \begin{cases} x_1 + x_2 + x_3 = 5 \\
x_3 \geq 0
\end{cases}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
x_3 \leq 0 \iff \begin{cases} x_3 = -x_4 \\
x_4 > 0
\end{cases}
\end{align*}
\]
Some standard forms of LP

1. Max $\omega^T x$ s.t. $Ax \leq b$

2. Max $\omega^T x$ s.t. $Ax = b$ and $x \geq 0$

3. Min $\omega^T x$ s.t. $Ax \geq b$

4. Min $\omega^T x$ s.t. $Ax = b$ and $x \geq 0$

---

Geometric Perspective:

**Def (Halfspace):** region defined by one linear inequality.

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b$$

**Def (polyhedron):** region defined by a set of linear inequalities or equivalently

Intersection of halfspaces.

$\Rightarrow$ Feasible region of an LP is a polyhedron.

**Claim (convexity):** A polyhedron is a convex set.

Convex set: if $p \in S$ and $q \in S$ then $\lambda p + (1-\lambda)q \in S$ for $\lambda \in [0,1]$
Def (Polytope): A bounded polyhedron

\[ \exists r \text{ s.t. } P \subseteq S_r \]

Examples: Cube, Pyramid, Tetrahedron (3D)
Polygon (2D)

\[ a_1 x_1 + a_2 x_2 + a_3 x_3 = C \]

Face: For a polytope (or polyhedron), a face is a set of points on its boundary, which itself is a polytope (polyhedron).
**Def (Face):** A face of a polyhedron is a set of points obtained by replacing some of the inequalities with equalities (if you get a nonempty region).

\[
\begin{array}{c}
0 \leq x_1, x_1 \leq 1 \\
0 \leq x_2, x_2 \leq 1 \\
0 \leq x_3, x_3 \leq 1
\end{array}
\quad
\begin{array}{c}
0 \leq x_1, x_1 = 1 \\
0 = x_2, x_2 \leq 1 \\
0 \leq x_3, x_3 \leq 1
\end{array}
\quad
\begin{array}{c}
0 = x_1, x_1 = 1 \\
0 \leq x_2, x_2 \leq 1 \\
0 \leq x_3, x_3 = 1
\end{array}
\quad
\begin{array}{c}
0 \leq x_1, x_1 \leq 1 \\
0 \leq x_2, x_2 \leq 1 \\
0 \leq x_3, x_3 = 1
\end{array}
\quad
\text{Empty}
\]

**Terminology:** For a point (or set of points) in \( P \), the inequalities that are satisfied with equality by the point (or set of points) are called tight constraints for that point (or set of points).
Characterization of a set of optimizing points:

Lemma 1: For any polyhedron \( P(Ax \leq b) \), and a function \( \omega^T x \), the set of points in \( P \) maximizing \( \omega^T x \) form a face of \( P \).

(if \( P \) is non-empty and optimal value is bounded)

Example:

\[
\begin{align*}
\text{Maximize} & \quad 2x_1 + 2x_2 \\
\text{Constraints} & \quad x_1 \geq 0, \ x_2 \geq 0 \\
& \quad x_1 \leq 2, \ x_2 \leq 2 \\
& \quad x_1 + x_2 \leq 3 \\
& \quad x_1 + x_2 = 3
\end{align*}
\]

\[
f(x) = \alpha x_1 + \beta x_2 \text{ for } \alpha > 0
\]

\[
f(x) = \alpha (x_1) + \beta (x_1 + x_2)
\]

\[\alpha, \beta > 0\]

proof of Lemma 1:

Let the constraints for polyhedron \( P \) be given by

\[
a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = a_1^T x \leq b_1 \\
a_2^T x \leq b_2
\]

Let \( S \subseteq P \) be the set of points in \( P \) maximizing \( \omega^T x \).

Let the first \( l \) constraints be tight for all points in \( S \) and let there be no other such constraints.

I.e. for \( 1 \leq i \leq l \), \( a_i^T p = b_i \) for all \( p \in S \).

\[
\exists \ p' \in S \text{ s.t. } a_{l+1}^T p' < b_{l+1} \quad \text{is there a common point } z \in S \text{ satisfying strict inequalities for all } l < i \leq k?
\]
claim: All points in \( P \) satisfying first \( l \) constraints with equality are maximizing points for \( w^T x \).

Proof of the claim:

There should be a point \( z \in S \) such that

\[
\begin{align*}
    a_i^T z &= b_i \quad \text{for } 1 \leq i \leq l \quad \text{and}\quad \\
    a_i^T z &< b_i \quad \text{for } 1 \leq i \leq k
\end{align*}
\]

Why? for any \( l < j \leq k \)

\[ j \beta_j \in S \quad \text{s.t.} \quad a_j^T \beta_j < b_j \]

Define \( z = (\beta_{l+1} + \beta_{l+2} + \cdots + \beta_k)/(k-l) \) \( [\text{HW}] \)

Now, let \( y \) be another point in \( P \) satisfying

\[
    a_i^T y = b_i \quad \text{for } 1 \leq i \leq l
\]

Want to show that \( y \) is maximizing point for \( w^T x \).

For the sake of contradiction, let us assume that

\[
    w^T z > w^T y \quad \Rightarrow \quad w^T (z - y) > 0
\]

Define a new point

\[
    \alpha = z + \epsilon (z - y) \quad \text{for some small } \epsilon > 0
\]

\[
    w^T \alpha = w^T z + \epsilon w^T (z - y) > 0
\]

\[
    \Rightarrow \quad w^T \alpha > w^T z
\]
We will argue that \( x \in P \). That will contradict the assertion that \( z \) is maximizing \( w^T x \).

Need to show that \( x \) satisfies all the constraints for \( P \):

for \( 1 \leq i \leq l \)
\[ a_i^T x = a_i^T z + \varepsilon a_i^T (z-y) = b_i + \varepsilon (b_i - b_i) = b_i \]

for \( l < i \leq k \)
\[ a_i^T x = a_i^T z + \varepsilon a_i^T (z-y) < b_i \]

Choose \( \varepsilon \) to be sufficiently small.

\[ \Rightarrow a_i^T x < b_i \]

This finishes the proof of the claim.

**Conclusion:** \( S \) is the face defined by taking first \( l \) constraints as equalities.

Hence, the set of maximizing points form a face of \( P \).

---

**Lemma 2:** For every face \( F \) of a polyhedron \( P \), there is a linear function \( w^T x \) such that \( F \) is precisely the set of points in \( P \) maximizing \( w^T x \).

```
0 \leq x_1 \leq 1
0 \leq x_2 \leq 1
0 \leq x_3 \leq 1
```

**Proof Idea:** Start with the set of tight constraints for \( F \). Try to define \( w \) in terms of these tight constraints. Define \( w_x \) to be a positive combination of tight constraints.
Characterization of a Face: A set of points maximizing a linear function over the polyhedron.

Corollary of Lemma 1: For any function $\omega^T x$, there are some corner points in $P$, which maximize $\omega^T x$.

How do we formally define a corner of a polyhedron?

Three definitions:

1. A corner is a zero dimensional face (single point)

I.e., a point in the polyhedron which satisfies $n$ linearly independent constraints with equality:

$$
\begin{align*}
1 & \geq \lambda_1 \geq 0 \\
1 & \geq \lambda_2 \geq 0 \\
1 & \geq \lambda_3 \geq 0
\end{align*}
$$

$$
\begin{align*}
\lambda_1 &= 0, \\
\lambda_2 &= 1, \\
\lambda_3 &= 1
\end{align*}
$$

$$
\begin{align*}
x_1 &\geq 0 \\
x_3 &\geq 0 \\
x_2 &\geq 0
\end{align*}
$$

$$
\Rightarrow x_1 + x_2 \leq 1
$$

$$
\Rightarrow x_1 + x_3 \leq 1
$$

$$
\Rightarrow x_1 + x_2 + x_3 \leq 1
$$

$$
\Rightarrow x_1 + x_2 + x_3 = 1
$$

$$
\Rightarrow x_1 + x_2 = 1
$$

Not linearly independent
2. \( q \) is a corner of polyhedron \( P \), if there is a linear function \( \omega^T x \) s.t. \( q \) is the unique point in \( P \) maximizing \( \omega^T x \).

\[
\begin{align*}
(0,0) & \quad (1,0) \\
(0,2) & \quad (1,2) \\
(2,1) & \quad (2,1) \\
(0,0) & \quad (0,0)
\end{align*}
\]

\( (2,1) = 3x_1 + x_2 \)

3. \( q \) is a corner of \( P \), if it is not a midpoint of two other points in \( P \).

Equivalently, for any \( \alpha \in \mathbb{R}^n \)
if \( q + \alpha \in P \) then \( q - \alpha \not\in P \).

Lecture 3  Jan 13

Equivalence of the three definitions

1 \( \Rightarrow \) 2

2 \( \Rightarrow \) 3

3 \( \Rightarrow \) 1

Follows from Lemma 2.
Point \( q \in \mathcal{P} \).

From (2), there is a function \( \mathbf{w}^T \mathbf{x} \) for which \( q \) is the unique point in \( \mathcal{P} \) maximizing \( \mathbf{w}^T \mathbf{x} \).

We want to show that \( q \) cannot be a mid-point of two other points in \( \mathcal{P} \).

For the sake of contradiction

\[
q = \frac{\mathbf{p} + \mathbf{r}}{2}
\]

where \( \mathbf{p}, \mathbf{r} \in \mathcal{P} \).

\[
\mathbf{w}^T q = \frac{\mathbf{w}^T \mathbf{p} + \mathbf{w}^T \mathbf{r}}{2}
\]

Say, \( \mathbf{w}^T \mathbf{p} < \mathbf{w}^T q \Rightarrow \mathbf{w}^T r > \mathbf{w}^T q \Rightarrow \) contradiction.

Hence \( \mathbf{w}^T \mathbf{p} = \mathbf{w}^T q = \mathbf{w}^T r \Rightarrow \mathbf{p} = q = r \Rightarrow \) contradiction (\( \therefore \) uniqueness)

(3) \( \Rightarrow \) (1) Assume \( q \in \mathcal{P} \) cannot be the mid-point of two other points in \( \mathcal{P} \).

We want to show there are \( n \) linearly independent equality constraints for \( q \).

Let \( \mathcal{P} \) be described by

\[
\begin{align*}
\mathbf{a}_1^T \mathbf{x} &\leq b_1 \\
\mathbf{a}_2^T \mathbf{x} &\leq b_2 \\
\vdots \\
\mathbf{a}_k^T \mathbf{x} &\leq b_k
\end{align*}
\]

(\( l \) could be zero)

Let \( q \) satisfy first \( l \) constraints with equality

\[
\text{for } 1 \leq i \leq l \quad \mathbf{a}_i^T q = b_i
\]

for \( 1 < i \leq k \quad \mathbf{a}_i^T q < b_i \).
If \( \text{rank}(a_1, a_2, \ldots, a_k) = n \), we are done.

Suppose \( \text{rank}(a_1, a_2, \ldots, a_k) < n \).

Then there exists a solution \( \mathbf{y} \neq \mathbf{0} \in \mathbb{R}^n \) for
\[
\begin{align*}
\mathbf{a}_i^T \mathbf{y} &= 0 \quad \text{Homogeneous system} \\
\mathbf{b}_i^T \mathbf{y} &= 0
\end{align*}
\]

\( \Rightarrow \) infinitely many nonzero solution

Now, consider two points,
\[
\begin{align*}
q_1 &= q + \varepsilon \mathbf{y} \\
q_2 &= q - \varepsilon \mathbf{y}
\end{align*}
\]
for small enough \( \varepsilon > 0 \)

Claim \( q_1 \) and \( q_2 \) are inside \( P \).

Proof: \( q + \varepsilon \mathbf{y} \in P \)
\[
\begin{align*}
1 \leq i \leq l & \quad a_i^T (q + \varepsilon \mathbf{y}) = a_i^T q + \varepsilon a_i^T \mathbf{y} = a_i^T q = b_i \\
1 \leq i \leq k & \quad a_i^T (q + \varepsilon \mathbf{y}) = a_i^T q + \varepsilon a_i^T \mathbf{y} < b_i \quad \text{for small enough } \varepsilon
\end{align*}
\]

But \( q = \frac{q_1 + q_2}{2} \)

which is a contradiction.

Hence \( \text{rank}(a_1, a_2, \ldots, a_k) = n \)

\( \Rightarrow n \) linearly independent equality constraints for \( q \).
Def (Convex Hull): For a set of points \( q_1, q_2, \ldots, q_r \in \mathbb{R}^n \) their convex hull is the set of all points which can be written as a convex combination of \( q_1, q_2, \ldots, q_r \):

\[
\text{conv}(q_1, q_2, \ldots, q_r) = \left\{ \lambda_1 q_1 + \lambda_2 q_2 + \cdots + \lambda_r q_r : \lambda_1, \lambda_2, \ldots, \lambda_r \geq 0, \lambda_1 + \lambda_2 + \cdots + \lambda_r = 1 \right\}
\]

Examples:

![Examples Diagram](image)

Observation: \( p \in \text{conv}(q_1, q_2, \ldots, q_r) \) and \( r \in \text{conv}(r_1, r_2, \ldots, r_s) \) then \( p \in \text{conv}(q_1, q_2, \ldots, q_r, r_1, r_2, \ldots, r_s) \)

Claim: Any polytope is the convex hull of its finitely many corners.

Proof: Use definition 3 of corners.
Carathéodory's Theorem: Any point in an \( n \)-dimensional polytope is a convex combination of at most \( n+1 \) corners.

Proof (Induction based on the dimension of the polytope)

\[ n = \text{no. of linearly independent equality constraints} \]

Base case: \( \dim = 0 \) (trivial)

Induction hypothesis: assume that the statement is true for polytopes with dimension up to \( n-1 \).

Induction step: want to show it for \( \dim = n \).

\[ q \in \text{conv}(a, p) \quad \text{IH} \quad p \in \text{conv}(p_1, p_2, p_3) \]

\[ \Rightarrow q \in \text{conv}(a, p_1, p_2, p_3) \]
Let $q$ be the given point in the polytope.

Let $a$ be an arbitrary corner of polytope (s.t. the equality constraints for $q$ are also satisfied by $a$).

Define a new point

$$p = q + \alpha (q - a)$$

$s.t.$ $\alpha > 0$

Start from $\alpha = 0$ and keep increasing it till an additional constraint becomes tight. (We hit some face)

$\Rightarrow p$ satisfies one more equality constraint than $q$ and hence, $p$ lies in a smaller dimension polytope $\leq n-1$.

By induction hypothesis

$$p \in \text{conv}(p_1, p_2, \ldots, p_n)$$

$\Rightarrow q \in \text{conv}(a, p)$

$$q = \frac{1}{\alpha + 1} p + \frac{\alpha}{\alpha + 1} a$$

$\Rightarrow q \in \text{conv}(a, p_1, p_2, \ldots, p_n)$

$q$ is a convex combination of $n+1$ corners.
Claim: Any convex hull of finitely many points in \( \mathbb{R}^n \) is a polytope.

(i.e., can be described by finitely many linear inequalities in variables \( x_1, x_2, \ldots, x_n \))

Proof: \( \text{conv hull} \ (q_1, q_2, \ldots, q_r) \subseteq \mathbb{R}^n \)

\[
\begin{align*}
\mathbf{x} &\in \mathbb{R}^n : \mathbf{x} = \lambda_1 q_1 + \lambda_2 q_2 + \cdots + \lambda_r q_r \\
\lambda_1 + \lambda_2 + \cdots + \lambda_r &= 1 \\
\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_r &\geq 0
\end{align*}
\]

\[\{ \mathbf{x} : A \mathbf{x} \leq b \} \text{ n variables} \]

We want to construct a system of inequalities \( A \mathbf{x} \leq b \) s.t.:

Remove all \( \lambda_i \)'s one by one via Fourier-Motzkin Elimination

Removing \( \lambda_1 \)

\[
\begin{align*}
\lambda_1 &\geq E_1 \\
\lambda_1 &\geq E_2 \\
\lambda_1 &\leq E_3 \\
\lambda_1 &\leq E_4
\end{align*}
\]

The two sets of constraints have the "same" feasible solutions.
Let \( S \) be a system of constraints in variables \((x_1, x_2, \ldots, x_n, y)\).

Let \( S' \) be the system of constraints in variables \((x_1, x_2, \ldots, x_n)\)
which is obtained from \( S \) by eliminating \( y \) via FM elimination.

Want: a point \((x_1, x_2, \ldots, x_n)\) is feasible for \( S' \).

### Example

\[
\begin{align*}
\begin{cases}
  x_1 - y &\leq 0 \\
  x_1 + y &\geq 2 \\
  3y - x_1 &\leq 6
\end{cases}
\end{align*}
\]

\( S \)

\[
\begin{align*}
\begin{cases}
  y &\geq x_1 \\
  y &\leq 2 - x_1 \\
  y &\leq \frac{6 + x_1}{3}
\end{cases}
\end{align*}
\]

\( S' \)

Geometrically, polyhedron \((S')\) is a projection of polyhedron \((S)\)
on to the coordinates \((x_1, \ldots, x_n)\).

\[
S \quad y \leq x_1 + x_2 \quad \Rightarrow \quad \text{No constraints} \quad \mathbb{R}^2
\]
Getting a description of $\text{conv}(p_1, p_2)$ via Fourier Motzkin Elimination (Example 1)

Let $(x_1, x_2) = \lambda_1 (1,1) + \lambda_2 (3,2)$

\[ \lambda_1 + \lambda_2 = 1 \]
\[ \lambda_1 \geq 0 \]
\[ \lambda_2 \geq 0 \]

\[ x_1 = \lambda_1 + 3\lambda_2 \]
\[ x_2 = \lambda_1 + 2\lambda_2 \]

Use $\lambda_2 = 1 - \lambda_1$

\[ x_1 = \lambda_1 + 3 - 3\lambda_1 = -2\lambda_1 + 3 \]
\[ x_2 = \lambda_1 + 2 - 2\lambda_1 = -\lambda_1 + 2 \]

$\lambda_1 \geq 0$

\[ 1 - \lambda_1 \geq 0 \implies \lambda_1 \leq 1 \]

Use $\lambda_1 = 2 - x_2$

\[ 2 - x_2 \geq 0 \implies x_2 \leq 2 \]
\[ 2 - x_2 \leq 1 \implies x_2 \geq 1 \]

\[ x_1 = -2(2 - x_2) + 3 \]
\[ 1 = 2x_2 - x_1 \]
\[ 1 \leq x_2 \leq 2 \]
Getting a description of Conv \((p_1, p_2, p_3)\) via FM elimination

\[(x_1, x_2) = \lambda_1(1, 1) + \lambda_2(2, 1) + \lambda_3(1, 2)\]
\[\lambda_1 + \lambda_2 + \lambda_3 = 1\]
\[\lambda_1, \lambda_2, \lambda_3 \geq 0\]

\[\downarrow\] Eliminating \(\lambda_3\) (put \(\lambda_3 = 1 - \lambda_1 - \lambda_2\))

\[x_1 = \lambda_1 + 2\lambda_2 + 1 - \lambda_1 - \lambda_2 = \lambda_2 + 1\]
\[x_2 = \lambda_1 + \lambda_2 + 2 - 2\lambda_1 - 2\lambda_2 = 2 - \lambda_1 - \lambda_2\]
\[\lambda_1 \geq 0\]
\[\lambda_2 \geq 0\]

\[1 - \lambda_1 - \lambda_2 \geq 0 \iff \lambda_1 + \lambda_2 \leq 1\]

\[\downarrow\] Eliminating \(\lambda_2\) (put \(\lambda_2 = x_1 - 1\))

\[x_2 = 3 - \lambda_1 - x_1\]
\[\lambda_1 \geq 0\]
\[x_1 - 1 \geq 0\]
\[\lambda_1 + x_1 \leq 2\]

\[\downarrow\] Eliminating \(\lambda_1\) (put \(\lambda_1 = 3 - x_1 - x_2\))

\[3 - x_1 - x_2 \geq 0\]
\[x_1 - 1 \geq 0\]
\[3 - x_2 \leq 2\]

\[x_1 + x_2 \leq 3\]
\[x_1 \geq 1\]
\[x_2 \geq 1\]
Facts So far:

1. Set of optimizing points form a face of the polyhedron.
2. Thus, there is always some corner among the optimizing points.
3. For every corner of the polyhedron, we can construct a linear function that is optimized at that corner.
4. Every polytope is the convex hull of its corners.
5. Convex hull of finitely many points is a polytope.

**Clue:** What is the right linear program or polyhedron for a combinatorial optimization problem?

For example, consider the problem of finding a maximum weight independent set:

\[
\max \sum_{(u,v) \in E} w_{uv} x_{uv}
\]

for each edge \(e = (u,v)\), \(x_u + x_v \leq 1\)

\[
0 \leq x_u \leq 1
\]

11. Each independent set point should be feasible
   
   (i.e., inside the polyhedron)

Hence, the convex hull of independent set points should be contained in the polyhedron.
Example \( G \)

\[
\begin{array}{c}
\text{Max weight Independent set} = 4 \\
\text{Convex combination} = 4 \\
\text{OPT} = 2
\end{array}
\]

\begin{align*}
0 \rightarrow \{v_2, v_4\} & \rightarrow (0, 0, 0, 0) \\
\frac{1}{2} \rightarrow \{v_1, v_3\} & \rightarrow (0, 0, 1, 0) \\
0 \rightarrow \{v_2, v_3\} & \rightarrow (0, 1, 0, 0) \\
0 \rightarrow \{v_1\} & \rightarrow (0, 0, 0, 1) \\
0 \rightarrow 0 & \rightarrow (0, 0, 0, 0)
\end{align*}

\[ f(x) = 4 \quad \leftarrow (x_1, x_2, x_3, x_4) = (0, 1, 0, 1) \]

\[ f(x) = 5 \quad \leftarrow (x_1, x_2, x_3, x_4) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]

\[ \text{not a convex combination.} \]

\[ \text{HW3: Any 0-1 point can't be a convex combination of other 0-1 points.} \]

Remember that any corner of the polyhedron is an optimizing point for some linear function.

Hence, we do not want any corners other than the independent set points.

In other words, the polyhedron described by the constraints should be exactly equal to convex hull of independent set points.

Que If we write a linear program with all the clique constraints, does that give the independent set polytope?

\[
\begin{align*}
0 & \leq x_u \leq 1 \\
\sum_{u \in C} x_u & \leq 1
\end{align*}
\]

[HW example where this LP fails.]
Perfect Matching in Bipartite Graphs

\[ \{x_e \}_{e \in E} \]

\[ 0 \leq x_e \leq 1 \]

\[ \text{for every vertex } v \in V \]
\[ \sum_{e \text{ incident on } v} x_e = 1 \]

PM \( (bpm) \)

\[ \text{a subset of edges such that every vertex has exactly one edge on it.} \]

\[ x_{e_1} + x_{e_2} = 1 \]
\[ x_{e_1} + x_{e_3} = 1 \]
\[ x_{e_3} + x_{e_4} = 1 \]
\[ x_{e_2} + x_{e_4} = 1 \]

\[ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \]

Does this system of constraints describe the perfect matching polytope for bipartite graphs?

Can you find an example where the above polytope contains a point which is not a convex combination of perfect matching points?

Equivalently, there is a linear weight function such that optimal value of LP is larger than maximum weight of a perfect matching?

Example on general graphs:

\[ 0 \leq x_{e_1}, x_{e_2}, x_{e_3} \leq 1 \]
\[ x_{e_1} + x_{e_2} = 1 \]
\[ x_{e_2} + x_{e_3} = 1 \]
\[ x_{e_3} + x_{e_1} = 1 \]

\[ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \]

\[ \notin \text{convhull}(PM) \quad \text{conv hull (PM) is empty} \]
Lemma: For bipartite graphs, \( P(bpm) \) is the convex hull of perfect matching points.

Or, equivalently each corner of \( P(bpm) \) is a perfect matching point.

**Proof:** We will show that each corner of \( P(bpm) \) is integral. \( \epsilon \in \{0, 1, \frac{1}{2}\} \)

\[ \Rightarrow \text{ each corner is PM point} \]

Take \( \alpha \in \mathbb{R}^E \) which is feasible

\[ \sum_{e \in \epsilon} \alpha e = 1 \quad \forall \epsilon \]

Suppose \( \alpha \) is fractional, i.e., for some edge \( e \),

\[ 0 < \alpha e < 1 \]

Summation around a vertex is 1.

\[ \Rightarrow \text{ at least one of the neighboring edges } e', \ 0 < \alpha e' < 1 \]

Keep finding neighboring edges till we get a cycle.

In a bipartite graph, cycles are all even length.

Want to argue that \( \alpha \) is not a corner.

\[ P_1 = \alpha + \beta_1 \]

\[ \beta_1 = \alpha + \frac{3}{2} \epsilon \]

\[ \alpha = \frac{1}{2} \beta_1 + \frac{1}{2} \beta_2 \]

\[ P_2 = \alpha + \beta_2 \]

\[ \beta_2 = \alpha + \frac{3}{2} \epsilon \]

\[ P_1 + P_2 = 2\alpha + \epsilon \]
We claim that $\beta_1, \beta_2$ satisfy all the constraints:

$$\sum_{e \text{ incident on } v} \beta_1 e = \sum_{e \text{ incident on } v} \lambda e + \varepsilon - \varepsilon = 1$$

$$\beta_1 e = \lambda e + \varepsilon$$

We can small enough $\varepsilon$ to ensure $\beta_1 e, \beta_2 e \geq 0$

Hence $\beta_1, \beta_2$ are feasible points

Matching Polytope

$$\text{Max } \sum_{e \in E} \lambda_e$$

$$\lambda_e \geq 0$$

$$\sum_{e \text{ incident on } v} \lambda_e \leq 1$$

Claim: The above constraints describe the convex hull of matching points.
Two kinds of LP questions
  * Optimization $\rightarrow \max \ w^T x \ \text{s.t.} \ Ax \leq b$
  * Feasibility $\rightarrow$ is there $x \in \mathbb{R}^n$ satisfying $Ax \leq b$

* The two questions are equivalent.

1. Optimization $\leq$ Feasibility

   \[
   \begin{array}{l}
   \text{Guess } w_0. \\
   \text{max } w^T x \\
   \text{st. } Ax \leq b \\
   \text{Binary Search } w^T x > w_0 \Rightarrow \text{feasible?} \quad \text{Yes } w_0 \text{ new guess } > w_0 \\
   \text{No } w_0 \text{ new guess } < w_0 \\
   \end{array}
   \]

   Can we bound Optimal value?

   If $w^*$ is the opt value $\Rightarrow$ no. of rounds $O(\log w^*)$

   \[[HW]\]

   Claim: opt value $\leq \exp(\eta, k, l) \ poly(n, k, l)$?

   \[
   \uparrow \quad \text{no. of bits in constraints, coefficients.}
   \]

2. Feasibility $\leq$ Optimization

   \[[HW]\]

   is $Ax \leq b$ feasible? $\max \ 1^T x \ \text{s.t.} \ Ax \leq b$

   suppose we have an optimization oracle that works correctly only if the given system is feasible, otherwise it outputs garbage.
Computational Complexity of Linear Programming

Some inefficient algorithms for LP:

- Go over all the corners
- Fourier Motzkin Elimination

Towards a polynomial time algorithm:

LP Duality → LP is in NP ∩ coNP

Easily verifiable proofs for both:

Yes and No answers:

"Good Characterization"

Feasibility: \( Ax \leq b \)

If \( Ax \leq b \) is feasible, is there an easily verifiable proof for this fact?

any \( x \) satisfying \( Ax \leq b \) is a proof.

Given \( x \), it’s easy to verify \( Ax \leq b \)

What about the size of \( x \)?

Need to guarantee at least one feasible point \( x \), whose description size is \( \text{poly}(n,k,b) \) (no. of bits)

Corner → \( n \) tight constraints → \( A'x = b' \) solution

\[ x = A'^{-1}b' \] (bounded)
If $Ax \leq b$ is not feasible, is there an easily verifiable proof for this fact?

What about $Ax = b$?
If $Ax = b$ is not feasible then is there a proof for this?

$$
egin{align*}
 y_1 x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &= b_1 \\
 y_2 x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &= b_2 \\
 &\vdots \\
 y_n x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n &= b_n \\
 0 x_1 + 0 x_2 + \ldots + 0 x_n &= 0
\end{align*}
$$

0 $x_1 + 0 x_2 + \ldots + 0 x_n = 0 \\
\neq 0$

Example 1

$x_1 \geq 1$  
$x_2 \geq 1$  
$x_1 + x_2 \geq 2$  
$x_1 + x_2 \leq 1$  
$x_1 \geq 1$  
$x_2 \geq 1$  
$x_1 + x_2 \leq 1$  
$x_1 \geq 1$  
$x_2 \geq 1$  
$x_1 + x_2 \leq 1$  
$x_1 \geq 1$  
$x_2 \geq 1$  
$x_1 + x_2 \leq 1$  
$x_1 \geq 1$  
$x_2 \geq 1$  
$x_1 + x_2 \leq 1$  
$x_1 \geq 1$  
$x_2 \geq 1$  
$x_1 + x_2 \leq 1$  
$x_1 \geq 1$  
$x_2 \geq 1$  
$x_1 + x_2 \leq 1$  
$x_1 \geq 1$  
$x_2 \geq 1$  
$x_1 + x_2 \leq 1$

Example 2

$$
2 x_1 - x_2 \geq 0 \\
2 x_2 - x_1 \geq 0 \\
\quad x_1 + x_2 \geq 3 \\
\quad x_1 + 4 x_2 \leq 1
$$

take last three inequalities and show a combination that leads to contradiction

$$
1 \times \begin{cases} 
2 x_2 - x_1 \geq 0 \\
2 x_2 - x_1 \geq 0 \\
\quad x_1 + x_2 \geq 3 \\
\quad x_1 + 4 x_2 \leq 1
\end{cases} \\
2 x_1 + 0 x_2 \geq 5
$$
Farkas’ Lemma

Let \( A \in \mathbb{R}^{k \times n} \) and \( b \in \mathbb{R}^k \) such that \( Ax \geq b \) is not feasible.

Then there exists \( y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k \) such that
\[
y > 0, \quad y^TA = 0 \quad \text{and} \quad y^Tb > 0
\]

\((\text{equivalently}, \quad y > 0, \quad A^Ty = 0 \quad \text{and} \quad b^Ty > 0)\)

\[
0 \leq y_1 x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \geq b_1
\]
\[
0 \leq y_2 x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \geq b_2
\]
\[
\vdots
\]
\[
0 \leq y_k x_1 + a_{k2} x_2 + \cdots + a_{kn} x_n \geq b_k
\]

\(0 x_1 + O x_2 + \cdots + O x_n \geq 1 \quad \text{or anything} \geq 0\)

\(y^TA = 0 \equiv \{y_1 a_{1j} + y_2 a_{2j} + \cdots + y_k a_{kj} = 0 \quad \forall 1 \leq i \leq n\}\)

Proof: Induction based on number of variables.

Base case: \(n = 1\)
\[
x_1 \leq b_1
\]
\[
\text{infeasible only if} \quad x_1 \leq b_2, \quad \{x_1 \geq \alpha, \quad x_1 \leq \beta
\]
\[
1 \quad x_1 \geq -by \iff -x_1 \leq by
\]
\[
\text{while} \quad \beta \leq \alpha
\]
\[
-x_1 \geq -b \quad \frac{0 > \alpha - \beta}{\alpha - \beta > 0}
\]

Induction hypothesis: let the statement be true for \(n-1\) variables.

Induction step: Fourier Motzkin Elimination

Let’s separate out constraints based on whether
\[
a_{in} > 0 \quad \text{or} \quad a_{in} < 0, \quad a_{in} = 0
\]
Without loss of generality, let the given system of constraints be

\[
\begin{align*}
S_1 \quad & \rightarrow a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i \quad \text{for } 1 \leq i \leq k_1 \\
& \rightarrow a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n \geq b_j \quad \text{for } k_1 < j \leq k_2 \\
& \rightarrow a_{l1}x_1 + a_{l2}x_2 + \cdots + a_{ln}x_n \geq b_l \quad \text{for } k_2 < l \leq k 
\end{align*}
\]

Define another system of constraints in n-1 variables.

\[
\begin{align*}
S_2 \quad & \rightarrow (a_{i1} + a_{i1})x_1 + (a_{i2} + a_{j2})x_2 + \cdots + (a_{in} + a_{jn})x_{n-1} \geq b_i + b_j \\
& \rightarrow a_{l1}x_1 + a_{l2}x_2 + \cdots + a_{ln-1}x_{n-1} \geq b_l \quad \text{for } k_2 < l \leq k 
\end{align*}
\]

**Claim:** If \( S_1 \) is not feasible then \( S_2 \) is not feasible.

**Proof:** (HW)

Now assume \( S_2 \) is not feasible. By induction hypothesis there exist

\[
\begin{align*}
& \text{for } 1 \leq i \leq k_1, \quad k_1 < j \leq k_2 \\
& \text{for } k_2 < l \leq k \\
& \sum_{i,j} Y_{ij} (a_{ip} + a_{jp}) + \sum_{k} Y_{lk} a_{kp} = 0 \quad \text{for } 1 \leq p \leq n-1 \\
& \sum_{i,j} Y_{ij} (b_i + b_j) + \sum_{l} Y_{lk} b_l > 0
\end{align*}
\]

Argue that any non-negative combination of constraints in \( S_2 \) is also a non-negative combination of constraints in \( S_1 \).
The corresponding $y_{>0}$ coefficients for $S_1$ can be defined as

for $1 \leq i \leq k_1$ \hspace{1cm} y_i = \sum_{j=k_1+1}^{k_2} y_{ij}$

for $k_1 < j \leq k_2$ \hspace{1cm} y_j = \sum_{i=1}^{k_1} y_{ij}$

for $k_2 < l \leq k$ \hspace{1cm} y_l = y_l$

Show that

for $1 \leq p \leq n$ \hspace{1cm} \sum_{i=1}^{k} y_i a_{ip} = 0 \hspace{1cm} \text{for } 1 \leq p \leq n$

\sum_{l=1}^{k} y_l b_l > 0

In other words, $y^T A = 0$, $y^T b > 0$ as desired. \hspace{1cm} \Box
Farkas Lemma in different forms

1) \( Ax \leq b \) not feasible \( \iff \ y \geq 0, \ y^T A = 0, \ y^T b < 0 \) feasible

2) \( Ax \leq b, \ x \geq 0 \) not feasible \( \iff \ y \geq 0, \ y^T A > 0, \ y^T b < 0 \) feasible

3) \( Ax = b, \ x \geq 0 \) not feasible \( \iff \ y \in \mathbb{R}^k, \ y^T A > 0, \ y^T b < 0 \) feasible

Note: Above, in RHS, \( y^T b < 0 \) can be replaced with \( y^T b = -1 \).

Conclusion
- LP feasibility is in \( \text{NP} \cap \text{coNP} \).
- Feasibility of a system is equivalent to infeasibility of another.

LP Optimization

\[
\text{max } \omega^T x \\
\text{s.t. } Ax \leq b
\]

Any feasible point gives a lower bound on optimal value

Max \( x_1 + x_2 = f(x_1, x_2) \)

\[
\begin{align*}
\{ x_1 \leq 4 \} & \quad y_1 = 0 \\
\{ x_2 \leq 3 \} & \quad y_2 = 0 \\
\{ x_2 + 2x_1 \leq 7 \} & \quad y_3 = \frac{1}{3} \\
\{ 2x_2 + x_1 \leq 6 \} & \quad y_4 = \frac{1}{3} \\
\{ x_2 \geq 0 \} & \quad y_5 = 0
\end{align*}
\]

\( f(1,1) = 2 \) \quad opt value \( \geq 2 \)

Optimal dual solution
\( f(\frac{8}{3}, \frac{5}{3}) = \frac{13}{3} \) \quad opt value \( \geq \frac{13}{3} \)
can we get an upper bound

\[ x_1 \leq 4 \]
\[ x_2 \leq 3 \]

\[ f(x_1, x_2) = x_1 + x_2 \leq 7 \quad \text{opt value} \leq 7 \]

\[ 2x_2 + x_1 \leq 6 \]
\[ -x_2 \leq 0 \]
\[ x_1 + x_2 \leq 6 \quad \text{opt value} \leq 6 \]

\[ x_2 \leq 3 \]
\[ x_2 + 2x_1 \leq 7 \]
\[ 2x_2 + 2x_1 \leq 10 \]
\[ \Rightarrow x_2 + x_1 \leq 5 \quad \text{opt value} \leq 5 \]

\[ x_2 + 2x_1 \leq 7 \]
\[ 2x_2 + x_1 \leq 6 \]
\[ 3x_1 + 3x_2 \leq 13 \]
\[ \Rightarrow x_1 + x_2 \leq \frac{13}{3} \quad \text{opt value} \leq \frac{13}{3} \]

Conclude: opt value = \( \frac{13}{3} \)

Try to prove in general using Farkas' Lemma that the right upper bound on objective value can always be obtained by taking some combination of the constraints.
Dual LP

\[ \text{LP} \quad \max \quad f(x) = \sum_{i=1}^{n} \omega_i x_i + \sum_{j=1}^{n} \omega_j x_j = \sum_{i=1}^{n} \omega_i x_i + \sum_{j=1}^{n} \omega_j x_j \leq b \]

 subject to:

\[ \begin{align*}
0 \leq y_1 & \quad a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\
0 \leq y_2 & \quad a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2 \\
\vdots & \quad \vdots \quad \vdots \\
0 \leq y_n & \quad a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n \leq b_n \\
\end{align*} \]

\[ \sum_{i=1}^{n} \omega_i x_i + \sum_{j=1}^{n} \omega_j x_j \leq \sum_{i=1}^{n} \omega_i y_i b_i \quad (1) \]

\[ \text{LP}^* \quad \min \quad \sum_{i=1}^{n} \omega_i y_i b_i \leq \text{best upper bound} \]

subject to:

\[ \begin{align*}
y_1, y_2, \ldots, y_n & \geq 0 \\
\end{align*} \]

Dual LP

\[ \begin{align*}
a_{11} y_1 + a_{12} y_2 + \ldots + a_{1n} y_n & = \omega_1 \\
\vdots & \quad \vdots \quad \vdots \\
a_{n1} y_1 + a_{n2} y_2 + \ldots + a_{nn} y_n & = \omega_n \\
\end{align*} \]

Weak Duality Theorem

If \( x \) is a feasible point for \( \text{LP} \) and \( y \) is a feasible point for \( \text{LP}^* \) then

\[ f(x) = \sum_{i=1}^{n} \omega_i x_i \leq \sum_{j=1}^{n} \omega_j y_j b_j = g(y) \quad \text{follows from } (1) \]

In particular,

\[ f(x^*) \leq g(y^*) \]

\[ \text{opt} (\text{LP}) \leq \text{opt} (\text{LP}^*) \leq \text{best upper bound of opt (LP)} \]
Strong Duality Theorem

\[ \text{Opt}(LP) = \text{Opt}(LP^*) \]

\[ \max \{ w^T x : A x \leq b \} = \min \{ b^T y : y \geq 0 \text{, } A^T y = w \} \]

(if they exist) \( \{ \text{if LP is feasible and } \text{opt (LP) is bounded} \} \)

We will prove if \( \max \{ w^T x : A x \leq b \} \leq U \) then there exists \( y \geq 0 \text{ s.t. } A^T y = w \text{ and } b^T y \leq U \)

Equivalently, if there is no \( y \in \mathbb{R}^k \) with \( y \geq 0 \), \( A^T y = w \) and \( b^T y \leq U \) then \( \exists x \text{ s.t. } A x \leq b \text{ but } w^T x > U \) (or \( A x \leq b \) is infeasible)

**Proof** Suppose \( y \geq 0 \), \( A^T y = w \), \( b^T y \leq U \) is infeasible.

From Farkas Lemma

\[ \exists \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n \]

and \( \alpha_0 \geq 0 \)

s.t.

\[ (A^T)^T \alpha + \alpha_0 b \geq 0 \]

\[ w^T x + \alpha_0 \leq 0 \]

**Case 1** \( \alpha_0 = 0 \) \( \Rightarrow \) \( A x \geq 0 \) and \( w^T x < 0 \)

Assuming \( A x \leq b \) is feasible let \( x_0 \) be a feasible point.

Consider another point \( x_0 - N \alpha \) for some \( N > 0 \)

(1) \( x_0 - N \alpha \) is feasible because \( A(x_0 - N \alpha) = A x_0 - N A \alpha \leq b - 0 \)

The objective value at \( x_0 - N \alpha \) is

\[ w^T(x_0 - N \alpha) = w^T x_0 - N w^T \alpha \]

Since \( w^T x < 0 \), we can choose \( N \) to achieve arbitrary large objective value.

Hence \( x_0 - N \alpha \) can be a feasible point with \( w^T(x_0 - N \alpha) > U \)
Case 2 \( \alpha_0 > 0 \)

\[ A \alpha \geq -\alpha_0 b \]

Define \( \alpha' = \frac{\alpha}{-\alpha_0} \)

\[ w^\top \alpha' < -u \alpha_0 \]

\[ w^\top \alpha' \leq b \]

\[ w^\top \alpha' > u \]

Some conclusions:

- If \( x \) is a feasible solution for LP and \( y \) is a feasible solution for LP*,
  
  \[ f(x) = g(y) \]

  then \( x \) is an optimal solution for LP
  
  and \( y \) is an optimal solution for LP*.

- Strong LP duality means the optimality can always be proved via a dual solution.

- LP optimization is in NP \& coNP.  (Is the opt value \( \geq w^\top \alpha \))

  \[
  \begin{array}{c|c}
  \text{LP} & \text{LP*} \\
  \text{feasible and bounded} & \iff \text{feasible and bounded} \\
  \text{Infeasible} & \rightarrow \text{Infeasible} \\
  \text{Unbounded optimal} & \Rightarrow \text{unbounded} \\
  \hline
  \text{Infeasible} & \iff \text{unbounded} \\
  \text{Both LP and LP* can be infeasible} \\
  
  \end{array}
  \]

- Duality can be useful for approximation algorithms.

  \[
  \begin{array}{c}
  f(x) \quad f(x^*) \quad g(y) \\
  \uparrow \quad \downarrow \quad \uparrow \\
  \end{array}
  \]
Max \( f(x_1, x_2) = w_1 x_1 + w_2 x_2 \)

- \( x_1 \leq 4 \)
- \( x_2 \leq 3 \)
- \( x_2 + 2x_1 \leq 7 \)
- \( 2x_2 + x_1 \leq 6 \)
- \( x_2 \geq 0 \)
- \( -x_2 \leq 0 \)

\( f(x_1, x_2) \) \( x^* \) \( f(x^*) \) \( y^* \) \( g(y^*) \)

\( x_1 + x_2 \) \( \frac{8}{3}, \frac{5}{3} \) \( 13/3 \) \( 0, 0, \frac{1}{3}, \frac{1}{3}, 0 \) \( 13/3 \)

feasible dual (not optimal)

\( x_1 + 3x_2 \) \( 0, 3 \) \( 9 \) \( 0, 0, 1, 0, 0 \) \( 3 \)

\( 4x_1 + 2x_2 \) \( \frac{5}{3}, \frac{5}{3} \) \( 14 \) \( 0, 0, 0, 0, 0 \) \( 14 \)

\( \frac{7}{2}, 0 \) \( 14 \)

\( 1, 5 \) \( 14 \)

Max \( f(x_1, x_2) = w_1 x_1 + w_2 x_2 \)

- \( x_1 \leq 4 \)
- \( x_2 \leq 3 \)
- \( x_2 + 2x_1 \leq 7 \)
- \( 2x_2 + x_1 \leq 6 \)
- \( x_2 \geq 0 \)

\( y_1, y_2, y_3, y_4, y_5 \geq 0 \)

Min \( 4y_1 + 3y_2 + 7y_3 + 6y_4 + 0y_5 \)

\( 1 \cdot y_1 + 0y_2 + 2y_3 + 1y_4 + 0y_5 = w_1 \)

\( 0 \cdot y_1 + 1y_2 + 1y_3 + 2y_4 - y_5 = w_2 \)

\( 0y_1 + 1y_2 + 1y_3 + 2y_4 \geq w_2 \)

Min \( 4y_1 + 3y_2 + 7y_3 + 6y_4 \)

\( y_1, y_2, y_3, y_4 \geq 0 \)

\( 1 \cdot y_1 + 0y_2 + 2y_3 + 1y_4 = w_1 \)

\( 0 \cdot y_1 + 1y_2 + 1y_3 + 2y_4 \geq w_2 \)
Economic Interpretation

Suppose a farmer has limited quantities of land, fertilizer and water/electricity. They need to decide on the amounts of wheat and chickpeas to grow.

Max. $2500 \, x_w + 6000 \, x_c$

subject to:

- $x_w, x_c \geq 0$
- $0.05 \, x_w + 0.1 \, x_c \leq 5$ (Land)
- $0.5 \, x_w + 2 \, x_c \leq 70$ (Fertilizer)
- $1 \, x_w + 1.1 \, x_c \leq 80$ (Electricity)

Per unit revenue, acres per unit

Optimal Solution

$x_w = 57.24$
$x_c = 20.68$

Min. $y_L \, y_w + y_F \, 70 + y_E \, 80$

subject to:

- $y_L, y_F, y_E \geq 0$
- $0.05 \, y_L + 0.5 \, y_F + 1 \, y_E \geq 2500$
- $0.1 \, y_L + 2 \, y_F + 1.1 \, y_E \geq 6000$

Per unit expenditure (cost) for the resources

Total cost for 1 quintal: $0.05 \, y_L + 0.5 \, y_F + 1 \, y_E$

For chickpeas:

Optimal Solution

$y_L = 0$
$y_F = 22.41$
$y_E = 13.79$

Land is not fully utilized

Fertilizer and electricity fully utilized.

Shadow pricing: revenue generated if this resource is opportunity cost increased by one unit.

By adding 1 kg of fertilizers, you can increase the opt value by $y_F = 22.41$.

If the real cost of 1 kg fertilizers is less than 22.41 then you can go ahead and buy 1 kg of fertilizers and increase your revenue.
Dual for various forms of LP

\[
\begin{align*}
\text{Max } & \quad w^T x \\
\text{s.t.} & \quad Ax \leq b
\end{align*}
\]

\[
\begin{align*}
\text{Min } & \quad b^T y \\
\text{s.t.} & \quad y \geq 0 \quad \text{A}^T y = w
\end{align*}
\]

\[
\begin{align*}
\text{Max } & \quad w^T x \\
\text{s.t.} & \quad x \geq 0 \quad \text{A}^T x - I \cdot y_1 = w
\end{align*}
\]

\[
\begin{align*}
\text{Min } & \quad b^T y \\
\text{s.t.} & \quad y \geq 0 \quad \text{A}^T y = x
\end{align*}
\]

\[
\begin{align*}
\text{Max } & \quad w^T x \\
\text{s.t.} & \quad x \geq 0 \quad \text{A}^T x = b
\end{align*}
\]

\[
\begin{align*}
\text{Min } & \quad b^T y \\
\text{s.t.} & \quad \text{A}^T y \geq w
\end{align*}
\]

\[\text{Theorem: } (LP)^* = LP\]

Dual of an LP in general form

\[
\begin{align*}
\text{Max } & \quad w_1 x_1 + w_2 x_2 \\
\text{s.t.} & \quad x \geq 0 \\
& \quad 2z_1 \in \mathbb{R} \\
& \quad a_{11} x_1 + a_{12} x_2 = b_1 \quad y_1 \in \mathbb{R} \\
& \quad a_{21} x_1 + a_{22} x_2 \leq b_2 \quad y_2 \geq 0 \\
& \quad a_{31} x_1 + a_{32} x_2 \geq b_3 \quad y_2 \leq 0 \\
& \quad -a_{31} x_1 - a_{32} x_2 \leq -b_3 \quad y_2 \geq 0 \\
\text{Max } & \quad x_1 \geq 0
\end{align*}
\]

\[
\begin{align*}
& \quad \text{Min } b_1 y_1 + b_2 y_2 - b_3 y_3 \\
& \quad a_{11} y_1 + a_{21} y_2 - a_{31} y_3 \geq w_1 \\
& \quad a_{21} y_1 + a_{22} y_2 - a_{32} y_3 = w_2 \\
& \quad a_{31} y_1 + a_{32} y_2 - a_{33} y_3 = w_3 \\
& \quad a_{1}^T x = b_1 \quad y_1 \in \mathbb{R} \\
& \quad -a_{1}^T x \leq -b_1 \quad y_1 = y_1' - y_1'' \quad y_1' \geq 0 \quad y_1'' \geq 0
\end{align*}
\]
Shortest Path LP.

Directed graph \( G(V, E) \).
Each edge \( e \in E \) has a non-negative length \( l_e \).

Source vertex \( \rightarrow s \)
Destination vertex \( \rightarrow t \)

Find a shortest path from \( s \) to \( t \).

→ Is it the right linear program?

Can LP optimal be smaller than the shortest path length?

It turns out that there is always an integral optimal solution.
Dual LP for the shortest path

Physical interpretation of the dual:

Imagine a length le thread for edge e.
Pull t as far away from s as possible.

Max \( y_t - y_s \) 
\[
\begin{align*}
y_a - y_s & \leq 1 \\
y_b - y_a & \leq 1 \\
y_t - y_b & \leq 1 \\
y_c - y_a & \leq 1.5 \\
y_t - y_d & \leq 1.5 \\
y_c - y_d & \leq 1.5 \\
y_t - y_d & \leq 1.5 \\
y_c - y_d & \leq 2
\end{align*}
\]

\( y_a \rightarrow \text{distance of } a \text{ from } s. \)

Tight threads correspond to tight dual constraints.
Claim: The optimal value of shortest path LP is equal to the length of the shortest path.

proof: We will construct a dual feasible solution whose dual objective value is equal to the length of the shortest path.

Bipartite Maximum matching Linear Program.