CS602 Applied Algorithms **Jan 2022**

Assignment 1

Total Marks: 30 *Deadline: February 13, 23:59*

Note: The assignment needs to be done individually. Any kind of discussion with other students is not allowed. If you feel the need to discuss anything, you can reach out to the instructor.

You can directly use any result proved in the class.

Que 1 [10 marks]. Let $p_1, p_2, \ldots, p_k$ be points in $\mathbb{R}^n$. Let $q \in \mathbb{R}^n$ be another point which is **not** in the convex hull of $\{p_1, p_2, \ldots, p_k\}$. Prove that there is a separating hyperplane H such that the point q is on one side of H and the points $p_1, p_2, \ldots, p_k$ are on the other side of H. In other words,

$$\exists a \in \mathbb{R}^n, b \in \mathbb{R} \text{ s.t. } a^T q > b \text{ and } a^T p_i \leq b \text{ for each } 1 \leq i \leq k.$$

Que 2 [10 marks]. For a **bipartite graph** $G(V, E)$ prove that the following system describes the vertex cover polytope.

$$\begin{aligned} 0 \leq x_u \leq 1 \qquad & \text{for each vertex } u \in V, \\ x_u + x_v \geq 1 \qquad & \text{for each edge } (u, v) \in E. \end{aligned}$$

Recall that a subset S of vertices is called a vertex cover if every edge in the graph has at least one endpoint in S. And the vertex cover polytope means the convex hull of points $\{\chi^S : S \text{ is a vertex cover in } G\}$, where $\chi^S \in \{0,1\}^V$ is the point with 1 for the vertices in S and 0 for the vertices outside S.

Hint: You can take a non-integral feasible point and show that it cannot be a corner of the polytope defined by the given system. You will conclude that every corner is integral. It is easy to see that every integral feasible solution is a vertex cover.

Que 3 [10 marks]. Prove that the optimal values of the below two LPs are equal. You can use the duality result from the class that we proved for a particular form of LP. Here $A \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^k$, $C \in \mathbb{R}^{\ell \times n}$, $d \in \mathbb{R}^\ell$.

$$\begin{aligned} \min w^T x & \text{ subject to} \\ x &\geq 0 \\ Ax &\leq b \\ Cx &= d \end{aligned}$$

$$\begin{aligned} \max b^T y + d^T z & \text{ subject to} \\ y &\leq 0 \\ z &\in \mathbb{R}^\ell \\ A^T y + C^T z &\leq w \end{aligned}$$