CS602 Applied Algorithms **Jan 2022**

Assignment 1 Solutions

Total Marks: 30 *Deadline: February 13, 23:59*

Que 1 [10 marks]. Let $p_1, p_2, \ldots, p_k$ be points in $\mathbb{R}^n$. Let $q \in \mathbb{R}^n$ be another point which is **not** in the convex hull of $\{p_1, p_2, \ldots, p_k\}$. Prove that there is a separating hyperplane H such that the point q is on one side of H and the points $p_1, p_2, \ldots, p_k$ are on the other side of H. In other words,

$$\exists a \in \mathbb{R}^n, b \in \mathbb{R} \text{ s.t. } a^T q > b \text{ and } a^T p_i \leq b \text{ for each } 1 \leq i \leq k.$$

Ans 1. From Farkas' lemma: q is not in the convex hull of $\{p_1, p_2, \ldots, p_k\}$ means that there **does not** exist $\lambda_1, \lambda_2, \ldots, \lambda_k$ satisfying

$$\begin{aligned} \lambda_i &\geq 0 \qquad \text{for } 1 \leq i \leq k, \\ \lambda_1 + \lambda_2 + \cdots + \lambda_k &= 1 \\ q &= \sum_{i=1}^{k} \lambda_i p_i \end{aligned}$$

Let's write the third constraint in the expanded form. Suppose $p_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,n})$ for each i and $q = (q_1, q_2, \ldots, q_n)$. Then we can write there does not exist $\lambda_1, \lambda_2, \ldots, \lambda_k$ satisfying

$$\begin{aligned} \lambda_i &\geq 0 \qquad \text{for } 1 \leq i \leq k, \\ \lambda_1 + \lambda_2 + \cdots + \lambda_k &= 1 \\ q_j &= \sum_{i=1}^{k} \lambda_i p_{i,j} \qquad \text{for } 1 \leq j \leq n. \end{aligned}$$

Think of the above as a system of constraints in variables $\lambda_1, \lambda_2, \ldots, \lambda_k$. We are saying that the above system is infeasible. Hence, we can use Farkas' lemma. One of the form of Farkas' lemma described in the class states:

$$Ax = b, x \geq 0 \text{ is infeasible } \implies \exists y \text{ such that } y^T A \geq 0, y^T b < 0.$$

To apply Farkas' lemma on the above system, we can take A and b to be as follows:

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ p_{1,1} & p_{2,1} & \cdots & p_{k,1} \\ p_{1,2} & p_{2,2} & \cdots & p_{k,2} \\ & & \vdots & \\ p_{1,n} & p_{2,n} & \cdots & p_{k,n} \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

We conclude that there exists $y_0, y_1, \ldots, y_n$ such that

$$y_0 + y_1 p_{i,1} + y_2 p_{i,2} + \cdots + y_n p_{i,n} \geq 0 \text{ for each } 1 \leq i \leq n$$

and

$$y_0 + y_1 q_1 + y_2 q_2 + \cdots + y_n q_n < 0.$$

Let's define $y = (y_1, y_2, \ldots, y_n)$. We can rewrite the above as

$$y_0 + y^T p_i \geq 0 \text{ for each } 1 \leq i \leq n$$

and

$$y_0 + y^T q < 0.$$

Let's define $a = -y$ and $c = y_0$. We get that

$$a^T p_i \leq c \text{ for each } 1 \leq i \leq n \text{ and } a^T q > c.$$

This is what we desired to show.

A straightforward argument: I didn't realize that there is a much simpler argument using a result proved in the class. Some students pointed out this. We proved in the class (Lecture 4) that any convex hull of finitely many points can be described by a system of linear constraints. Hence, there exists a system of constraints $Cx \leq d$ such that a point α is in the convex hull of $p_1, p_2, \ldots, p_k$ if and only if $C\alpha \leq d$. If q is not in this convex hull then one of the inequalities in $Cx \leq d$ must not be satisfied by q, say $a^T x \leq b$. Hence, $a^T q > b$. But, these inequalities are satisfied by each p_i, since they are inside the convex hull. Hence $a^T p_i \leq b$ for each $1 \leq i \leq k$.

Que 2 [10 marks]. For a **bipartite graph** $G(V, E)$ prove that the following system describes the vertex cover polytope.

$$\begin{aligned} 0 \leq x_u \leq 1 & \qquad \text{for each vertex } u \in V, \\ x_u + x_v \geq 1 & \qquad \text{for each edge } (u, v) \in E. \end{aligned}$$

Recall that a subset S of vertices is called a vertex cover if every edge in the graph has at least one endpoint in S. And the vertex cover polytope means the convex hull of points $\{\chi^S : S \text{ is a vertex cover in } G\}$, where $\chi^S \in \{0,1\}^V$ is the point with 1 for the vertices in S and 0 for the vertices outside S.

Hint: You can take a non-integral feasible point and show that it cannot be a corner of the polytope defined by the given system. You will conclude that every corner is integral. It is easy to see that every integral feasible solution is a vertex cover.

Ans 2. As suggested in the hint, we will take a non-integral feasible point and show that it is a mid-point of two other feasible points. That will mean it cannot be a corner. Thus, every corner is integral. It is easy to see that every integral feasible solution is a vertex cover. Hence, every corner is a vertex cover and thus, the system describes the vertex cover polytope.

Let's take a non-integral feasible point $\alpha \in \mathbb{R}^V$. We are assuming that α has at least one fractional coordinate, i.e., strictly between 0 and 1. Let F be the set of fractional vertices, i.e., $F = \{u : 0 < \alpha_u < 1\}$. We choose a number $\epsilon > 0$ that is small enough. To be precise, let ϵ to be a nonzero number such that $\epsilon \leq \alpha_u$ and $\epsilon \leq 1 - \alpha_u$ for each $u \in F$.

Recall that our graph is a bipartite graph. Let L and R be the two sets of vertices (left and right side). Let $F_L := F \cap L$ be the fractional vertices on the left side and let $F_R := F \cap R$ be the fractional vertices on the right side. We will define two new points $\beta \in \mathbb{R}^V$ and $\gamma \in \mathbb{R}^V$ by modifying the fractional coordinates of α. To elaborate, we define

$$\beta_u = \begin{cases} \alpha_u + \epsilon & \text{if } u \in F_L \\ \alpha_u - \epsilon & \text{if } u \in F_R \\ \alpha_u & \text{otherwise.} \end{cases}$$

That is, we are adding ϵ on the left side fractional vertices and subtracting ϵ on the right side fractional vertices. We will define γ by doing just the opposite.

$$\gamma_u = \begin{cases} \alpha_u - \epsilon & \text{if } u \in F_L \\ \alpha_u + \epsilon & \text{if } u \in F_R \\ \alpha_u & \text{otherwise.} \end{cases}$$

Claim 1. *β and γ are feasible points for the given constraints.*

Proof. We will just argue for β. Similar arguments will work for γ. Let's go over the constraints one by one.

$\underline{\beta_u \geq 0}$: If $\beta_u = \alpha_u + \epsilon$, then clearly $\beta_u \geq 0$. If $\beta_u = \alpha_u - \epsilon$, then we know that $\epsilon \leq \alpha_u$, and hence $\beta_u \geq 0$. Lastly, if $\beta_u = \alpha_u$ then obviously $\beta_u \geq 0$.

$\underline{\beta_u \leq 1}$: If $\beta_u = \alpha_u - \epsilon$, then $\beta_u \leq \alpha_u \leq 1$. If $\beta_u = \alpha_u + \epsilon$, then using the fact that $\epsilon \leq 1 - \alpha_u$, we get $\beta_u \leq 1$. Lastly, if $\beta_u = \alpha_u$ then obviously $\beta_u \leq 1$.

$\underline{\beta_u + \beta_v \geq 1}$: We are taking $u \in L$ and $v \in R$. If u and v both are fractional vertices then

$$\beta_u + \beta_v = \alpha_u + \epsilon + \alpha_v - \epsilon = \alpha_u + \alpha_v \geq 1.$$

If both u and v are not fractional vertices then

$$\beta_u + \beta_v = \alpha_u + \alpha_v \geq 1.$$

If u is a fractional vertex but v is not then

$$\beta_u + \beta_v = \alpha_u + \epsilon + \alpha_v \geq \alpha_u + \alpha_v \geq 1.$$

If v is a fractional vertex but u is not then α_u must be 1. Because otherwise $\alpha_u = 0$ and $\alpha_v < 1$ which gives $\alpha_u + \alpha_v < 1$, which violates feasibility. Then we can write

$$\beta_u + \beta_v = \alpha_u + \alpha_v - \epsilon \geq 1 + \alpha_v - \epsilon \geq 1.$$

The last inequality holds because by our choice $\epsilon \leq \alpha_v$. To conclude β satisfies all the constraints. ☐

Observation 2. $\alpha = (\beta + \gamma)/2$

The above observation is straightforward to verify. Hence, we have shown α to be a mid-point of two other feasible points. This concludes the argument as explained in the beginning of the answer.

Que 3 [10 marks]. Prove that the optimal values of the below two LPs are equal. You can use the duality result from the class that we proved for a particular form of LP. Here $A \in \mathbb{R}^{k\times n}$, $b \in \mathbb{R}^k$, $C \in \mathbb{R}^{\ell\times n}$, $d \in \mathbb{R}^\ell$.

$$\begin{aligned} \min w^T x \text{ subject to}& \\ x &\geq 0 \\ Ax &\leq b \\ Cx &= d \end{aligned}$$

$$\begin{aligned} \max b^T y + d^T z \text{ subject to}& \\ y &\leq 0 \\ z &\in \mathbb{R}^\ell \\ A^T y + C^T z &\leq w \end{aligned}$$

Ans 3. Let us call the given LPs as LP1 and LP2. The idea is to first bring LP2 in a standard form, then use strong duality theorem to get a dual LP with the same optimal value. Finally, we will construct an equivalent LP to the dual LP, which will turn out to be the same as LP1.

The above idea can be implemented in many different ways. We could also start with LP1 and then reach LP2. We are just presenting one way here.

Let us first write the LP strong duality result that we proved in the class.

$$\left(\max g^T u \text{ subject to } Eu \leq f\right) = \left(\min f^T v \text{ subject to } v \geq 0, E^T v = g\right).$$

To apply this result on the given LPs, let us take $u = (y, z)$, that is, u represents $k + \ell$ variables. Further, let us define

$$g = (b, d) \qquad \rightarrow k + \ell \text{ dim vector}$$

$$E = \begin{pmatrix} I & 0 \\ A^T & C^T \end{pmatrix} \qquad \rightarrow (k+n) \times (k+\ell) \text{ dim matrix}$$

$$f = \begin{pmatrix} 0 \\ w \end{pmatrix} \qquad \rightarrow k + n \text{ dim vector}$$

Here I is the $k \times k$ identity matrix. Observe that with the above substitutions $g^T u = b^T y + d^T z$. Moreover, $Eu = \begin{pmatrix} y \\ A^T y + C^T z \end{pmatrix}$. And thus, the constraint $Eu \leq f$ is same as $y \leq 0$ and $A^T y + C^T z \leq w$. With this, we can say that OPT(LP2) is same as

$$\max g^T u \text{ subject to } Eu \leq f.$$

But, from strong duality mentioned above, this is same as

$$\min f^T v \text{ subject to } v \geq 0, E^T v = g \tag{1}$$

Since v has $k + n$ variables, let's view it as $v = (v_1, v_2)$ where v_1 has k and v_2 has n variables. Now, observe that

$$f^T v = w^T v_2$$

$v \geq 0$ means

$$v_1 \geq 0, v_2 \geq 0$$

$E^T v = g$ means

$$I v_1 + A v_2 = b \text{ and } 0\, v_1 + C v_2 = d.$$

Thus, the LP in (1) is equivalent to

$$\begin{aligned} \min w^T v_2 & \qquad \text{subject to} \\ v_1 \geq 0, v_2 \geq 0 & \\ v_1 + A v_2 = b & \\ C v_2 = d & \end{aligned}$$

The variables in v_1 don't participate in the objective function. Let's eliminate v_1 via Fourier Motzkin elimination. Together $v_1 \geq 0$ and $v_1 + A v_2 = b$ implies $A v_2 \leq b$. Thus, the optimal value of the above LP is same as the following LP

$$\begin{aligned} \min w^T v_2 & \qquad \text{subject to} \\ v_2 \geq 0 & \\ A v_2 \leq b & \\ C v_2 = d & \end{aligned}$$

Replace variables v_2 with x and observe that this is exactly the same as LP1 given in the question.