Que 1 (5+5 marks). **Red-blue s-t connectivity:** In this problem, we are given an undirected graph $G$, with each edge colored either red or blue. We are also given a source vertex $s$ and a destination vertex $t$. The goal is to find an alternating red-blue path between $s$ and $t$. That is, a path that starts on $s$ with a red edge, alternates between red and blue edges, and ends at $t$ with a blue edge.

We try to reduce this problem to the matching problem as follows. Naturally, first we can delete any blue edges incident on $s$ and any red edges incident on $t$. We will construct another graph $H$ based on the given graph.

- For every vertex $v$ in $G$ other than $s$ and $t$, create two vertices in $H$, $v_r$ and $v_b$.
- Create two more vertices in $H$, $s_r$ and $t_b$.
- For any edge $(u, v)$ in $G$: if it is red then create an edge $(u_r, v_r)$ in $H$ and if it is blue then create an edge $(u_b, v_b)$ in $H$.
- Create an edge $(u_r, u_b)$ for every vertex $u$ other than $s$ and $t$.

Prove or disprove using a counter-example the following: graph $G$ has an alternating red-blue path between $s$ and $t$ if and only if the new graph has a perfect matching.

Que 2 (10 marks). Suppose $S$ is a convex set and we are maximizing a linear function $w^T x$ over it. If a point $x^* \in S$ locally maximizes the function, then prove that it maximizes the function over all $S$. Locally maximizes means the following: there exists an $\epsilon > 0$ such that for all points $y \in S$ within distance $\epsilon$ from $x^*$, we have $w^T x^* \geq w^T y$. You will need to prove such an inequality for all points $y$ in $S$.

Que 3 (10 marks). Use Fourier Motzkin procedure to compute the linear inequalities in variables $x_1, x_2, x_3$, which describe the cone $\{ \lambda_1(1, 2, 3) + \lambda_2(2, 3, 1) + \lambda_3(3, 1, 2) : \lambda_1, \lambda_2, \lambda_3 \geq 0 \} \subset \mathbb{R}^3$. Don’t just write the final answer. You need to show the steps of Fourier Motzkin procedure.

Que 4 (5+5 marks). We proved the following Farkas’ lemma in the class. For any given $k \times n$ matrix $A$ and $b \in \mathbb{R}^k$, the system

$$Ax = b, x \geq 0$$

has no feasible solution if and only if the system

$$A^T y \geq 0, b^T y = -1$$
has a feasible solution. Use this lemma (or any other way) to prove that for any given numbers $b_1, b_2, b_3, b_4$, the system

\begin{align*}
2x_1 - 3x_2 + x_3 & \leq b_1 \\
-x_1 + x_2 + 2x_3 & \leq b_2 \\
x_1 - x_2 & = b_3 \\
x_2 - 2x_3 & = b_4 \\
x_1, x_2 & \geq 0 \\
x_3 & \in \mathbb{R}
\end{align*}

has no feasible solution if and only if there exists $y_1 \geq 0, y_2 \geq 0, y_3, y_4 \in \mathbb{R}$ such that

\begin{align*}
2y_1 - y_2 + y_3 & \geq 0 \\
-3y_1 + y_2 - y_3 + y_4 & \geq 0 \\
y_1 + 2y_2 - 2y_4 & = 0 \\
b_1y_1 + b_2y_2 + b_3y_3 + b_4y_4 & = -1.
\end{align*}

You need to show both the directions.