CS602 Applied Algorithms **Jul 2023**

Assignment 1 Solutions

Total Marks: 40 *Deadline: Sep 12, 14:00*

Note: The assignment needs to be done individually. Any kind of discussion with other students is not allowed. If you feel the need to discuss anything, you can reach out to the instructor. You can directly use any result proved in the class. You can use other sources (book/internet), but you should give a reference and specify for which part it has been used.

Que 1 (5+5 marks). Red-blue s-t connectivity: In this problem, we are given an undirected graph G, with each edge colored either red or blue. We are also given a source vertex s and a destination vertex t. The goal is to find an alternating red-blue path between s and t. That is, a path that starts on s with a red edge, alternates between red and blue edges, and ends at t with a blue edge.

We try to reduce this problem to the matching problem as follows. Naturally, first we can delete any blue edges incident on s and any red edges incident on t. We will construct another graph H based on the given graph.

- For every vertex v in G other than s and t, create two vertices in H, v_r and v_b.
- Create two more vertices in H, s_r and t_b.
- For any edge (u, v) in G: if it is red then create an edge (u_r, v_r) in H and if it is blue then create an edge (u_b, v_b) in H.
- Create an edge (u_r, u_b) for every vertex u other than s and t.

Prove or disprove using a counter-example the following: graph G has an alternating red-blue path between s and t **if and only if** the new graph has a perfect matching.

Ans 1. ($\implies$). Suppose G has an alternating red-blue path P between s and t, which has vertices $(s, v^1, v^2, \ldots, v^k, t)$. Here red edges are (s, v^1) and (v^i, v^{i+1}) for any even i. And blue edges are (v^i, v^{i+1}) for any odd i and (v^k, t) (k is odd). The graph H has a perfect matching with following edges.

- $(s_r, v_r^1), (v_r^2, v_r^3), (v_r^4, v_r^5), \ldots, (v_r^{k-1}, v_r^k)$.
- $(v_b^1, v_b^2), (v_b^3, v_b^4), \ldots, (v_b^{k-2}, v_b^{k-1}), (v_b^k, t_b)$.
- $\{(u_r, u_b) : u \text{ is a vertex that does not lie on } P\}$.

It can be seen that all vertices in H are matched.

($\impliedby$). Suppose H has a perfect matching M. Let $(v^1, v^2, \ldots, v^k)$ be the maximal sequence of distinct vertices in $G - \{s, t\}$ such that

- v_r^1 is the vertex matched with s_r (s_r is only connected to u_r vertices).
- for odd $1 \le i \le k-1$, v_b^{i+1} is the vertex matched with v_b^i.
- for even $1 \le i \le k-1$, v_r^{i+1} is the vertex matched with v_r^i.

The sequence cannot have repeated vertices because every vertex is matched exactly once in a perfect matching. As $(v^1, v^2, \ldots, v^k)$ is chosen to be the maximal sequence of vertices in $G - \{s, t\}$, it must be that k is odd and v_b^k is matched with t_b. From the construction of H, we conclude that following edges exists in G with the mentioned colors:

- (s, v^1) is red,

- for odd $1 \leq i \leq k-1$, (v^i, v^{i+1}) is blue,
- for even $1 \leq i \leq k-1$, (v^i, v^{i+1}) is red.
- (v^k, t) is blue.

Since $v^1, v^2, \ldots, v^k$ are distinct vertices, we get the desired red-blue path in G.

Que 2 (10 marks). Suppose S is a convex set and we are maximizing a linear function $w^T x$ over it. If a point $x^* \in S$ locally maximizes the function, then prove that it maximizes the function over all S. Locally maximizes means the following: there exists an $\epsilon > 0$ such that for all points $y \in S$ within distance ϵ from x^*, we have $w^T x^* \geq w^T y$. You will need to prove such an inequality for all points y in S.

Ans 2. Consider any point y in S. We consider a point p on the line joining x^* and y, which is close to x^*:

$$p = x^* + \mu(y - x^*) = (1-\mu)x^* + \mu y.$$

Here, $\mu \in \mathbb{R}$ is chosen to be a small enough positive number so that the distance between p and x^* is less than ϵ. From convexity of S, we know that p is also in S. By the local maximizing property we have $w^T x^* \geq w^T p$. We conclude

$$w^T x^* \geq w^T p = (1-\mu) w^T x^* + \mu w^T y.$$

This implies that

$$w^T x^* \geq w^T y.$$

Que 3 (10 marks). Use Fourier Motzkin procedure to compute the linear inequalities in variables x_1, x_2, x_3, which describe the cone $\{\lambda_1(1,2,3) + \lambda_2(2,3,1) + \lambda_3(3,1,2) : \lambda_1, \lambda_2, \lambda_3 \geq 0\} \subset \mathbb{R}^3$. Don't just write the final answer. You need to show the steps of Fourier Motzkin procedure.

Ans 3.

$$(x_1, x_2, x_3) = \lambda_1(1,2,3) + \lambda_2(2,3,1) + \lambda_3(3,1,2) : \lambda_1, \lambda_2, \lambda_3 \geq 0.$$

Writing another way

$$\begin{aligned} x_1 &= \lambda_1 + 2\lambda_2 + 3\lambda_3 \\ x_2 &= 2\lambda_1 + 3\lambda_2 + 1\lambda_3 \\ x_3 &= 3\lambda_1 + 1\lambda_2 + 2\lambda_3 \\ 0 &\leq \lambda_1, \lambda_2, \lambda_3. \end{aligned}$$

Using the first equation $\lambda_1 = x_1 - 2\lambda_2 - 3\lambda_3$ to eliminate λ_1.

$$\begin{aligned} x_2 &= 2(x_1 - 2\lambda_2 - 3\lambda_3) + 3\lambda_2 + 1\lambda_3 \\ x_3 &= 3(x_1 - 2\lambda_2 - 3\lambda_3) + 1\lambda_2 + 2\lambda_3 \\ 0 &\leq x_1 - 2\lambda_2 - 3\lambda_3 \\ 0 &\leq \lambda_2, \lambda_3. \end{aligned}$$

Equivalently,

$$\begin{aligned} x_2 &= 2x_1 - \lambda_2 - 5\lambda_3 \\ x_3 &= 3x_1 - 5\lambda_2 - 7\lambda_3 \\ 0 &\leq x_1 - 2\lambda_2 - 3\lambda_3 \\ 0 &\leq \lambda_2, \lambda_3. \end{aligned}$$

Using the first equation $\lambda_2 = 2x_1 - x_2 - 5\lambda_3$ to eliminate λ_2.

$$\begin{aligned}
x_3 &= 3x_1 - 5(2x_1 - x_2 - 5\lambda_3) - 7\lambda_3\\
0 &\le x_1 - 2(2x_1 - x_2 - 5\lambda_3) - 3\lambda_3\\
0 &\le 2x_1 - x_2 - 5\lambda_3\\
0 &\le \lambda_3.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
x_3 &= -7x_1 + 5x_2 + 18\lambda_3\\
0 &\le -3x_1 + 2x_2 + 7\lambda_3\\
0 &\le 2x_1 - x_2 - 5\lambda_3\\
0 &\le \lambda_3.
\end{aligned}$$

Using the first equation $\lambda_3 = (7/18)x_1 - (5/18)x_2 + (1/18)x_3$ to eliminate λ_3.

$$\begin{aligned}
0 &\le -3x_1 + 2x_2 + 7((7/18)x_1 - (5/18)x_2 + (1/18)x_3)\\
0 &\le 2x_1 - x_2 - 5(7/18)x_1 - (5/18)x_2 + (1/18)x_3\\
0 &\le (7/18)x_1 - (5/18)x_2 + (1/18)x_3
\end{aligned}$$

Equivalently,

$$\begin{aligned}
-5x_1 + x_2 + 7x_3 &\ge 0\\
x_1 + 7x_2 - 5x_3 &\ge 0\\
7x_1 - 5x_2 + x_3 &\ge 0
\end{aligned}$$

Que 4 (5+5 marks). We proved the following Farkas' lemma in the class. For any given $k \times n$ matrix A and $b \in \mathbb{R}^k$, the system

$$Ax = b, x \ge 0$$

has no feasible solution if and only if the system

$$A^T y \ge 0, b^T y = -1$$

has a feasible solution. Use this lemma (or any other way) to prove that for any given numbers b_1, b_2, b_3, b_4, the system

$$\begin{array}{rcl}
2x_1 - 3x_2 + x_3 & \le & b_1\\
-x_1 + x_2 + 2x_3 & \le & b_2\\
x_1 - x_2 & = & b_3\\
x_2 - 2x_3 & = & b_4\\
x_1, x_2 & \ge & 0\\
x_3 \in \mathbb{R} & &
\end{array}$$

has no feasible solution **if and only if** there exists $y_1 \ge 0, y_2 \ge 0, y_3, y_4 \in \mathbb{R}$ such that

$$\begin{array}{rcl}
2y_1 - y_2 + y_3 & \ge & 0\\
-3y_1 + y_2 - y_3 + y_4 & \ge & 0\\
y_1 + 2y_2 - 2y_4 & = & 0\\
b_1y_1 + b_2y_2 + b_3y_3 + b_4y_4 & = & -1.
\end{array}$$

You need to show both the directions.

Ans 4. The first system has a feasible solution if and only if the following system has

$$\begin{aligned}
2x_1 - 3x_2 + x_3 + z_1 &= b_1 \\
-x_1 + x_2 + 2x_3 + z_2 &= b_2 \\
x_1 - x_2 &= b_3 \\
x_2 - 2x_3 &= b_4 \\
x_1, x_2, z_1, z_2 &\geq 0 \\
x_3 &\in \mathbb{R}
\end{aligned}$$

The above system has a feasible solution if and only if the following system has

$$\begin{aligned}
2x_1 - 3x_2 + x_3 - z_3 + z_1 &= b_1 \\
-x_1 + x_2 + 2(x_3 - z_3) + z_2 &= b_2 \\
x_1 - x_2 &= b_3 \\
x_2 - 2(x_3 - z_3) &= b_4 \\
x_1, x_2, z_1, z_2 &\geq 0 \\
x_3, z_3 &\geq 0
\end{aligned}$$

Now, the system is in the form $Ax = b, x \geq 0$. Using Farkas' lemma we can say that this system has no feasible solution if and only if the following system has a feasible solution.

$$\begin{aligned}
2y_1 - y_2 + y_3 &\geq 0 \quad \text{(constraint corresponding to variable } x_1\text{)} \\
-3y_1 + y_2 - y_3 + y_4 &\geq 0 \quad \text{(constraint corresponding to variable } x_2\text{)} \\
y_1 + 2y_2 - 2y_4 &\geq 0 \quad \text{(constraint corresponding to variable } x_3\text{)} \qquad (1) \\
y_1 &\geq 0 \quad \text{(constraint corresponding to variable } z_1\text{)} \\
y_2 &\geq 0 \quad \text{(constraint corresponding to variable } z_2\text{)} \\
-y_1 - 2y_2 + 2y_4 &\geq 0 \quad \text{(constraint corresponding to variable } z_3\text{)} \qquad (2) \\
b_1y_1 + b_2y_2 + b_3y_3 + b_4y_4 &= -1.
\end{aligned}$$

Combining (1) and (2), we get the following system.

$$\begin{aligned}
2y_1 - y_2 + y_3 &\geq 0 \\
-3y_1 + y_2 - y_3 + y_4 &\geq 0 \\
y_1 + 2y_2 - 2y_4 &= 0 \\
y_1, y_2 &\geq 0 \\
y_3, y_4 &\in \mathbb{R} \\
b_1y_1 + b_2y_2 + b_3y_3 + b_4y_4 &= -1.
\end{aligned}$$

This is same as desired.