

## Homework (No submission)

## Linear Programming Algorithms

## Lecture 22 (Oct 27)

Simplex algorithm.

1. Consider the following linear program.

$$\begin{aligned}
 &\max 4x_1 + 3x_2 \text{ subject to} \\
 &\quad x_1 \geq 0 \\
 &\quad x_2 \geq 0 \\
 &\quad 2x_1 + x_2 \leq 6 \\
 &\quad x_1 + 2x_2 \leq 6 \\
 &\quad 2x_1 + 2x_2 \leq 7
 \end{aligned}$$

First write the LP in the standard form  $Ax = b, x \geq 0$ . Start with the basic feasible solution that has  $x_1 = 0, x_2 = 0$ . Keep repeating the following iterations of the simplex algorithm.

- Choose one of the non-basic variables to increase. It should be among those whose coefficient in the objective function is positive. If there is no such variable then output the current solution. If there is such a variable then increase it till the maximum possible value while maintaining feasibility.
- This gives you a new basic feasible solution and a new set of basic and non-basic variables.
- Express the objective function in terms of the non-basic variables.

2. Consider an LP where  $A$  is a  $k \times n$  matrix.

$$\max w^T x \text{ subject to } Ax = b, x \geq 0.$$

Suppose we have a basic feasible solution  $\alpha$ , where the basic variables are  $\alpha_1, \alpha_2, \dots, \alpha_k$ . That means  $\alpha_{k+1} = \alpha_{k+2} = \dots = \alpha_n = 0$ .

- How will you find the values of  $\alpha_1, \alpha_2, \dots, \alpha_k$  using  $A$  and  $b$ . Note that non-degeneracy assumption means every  $k \times k$  submatrix of  $A$  is invertible.
- How will you express the objective function in  $w^T x$ , as a function of  $x_{k+1}, x_{k+2}, \dots, x_n$ .
- Prove that  $\alpha$  is an optimal solution if and only if the objective function, when expressed as a function of  $x_{k+1}, x_{k+2}, \dots, x_n$ , has no positive coefficient.

3. Consider the bipartite maximum matching LP.

$$\begin{aligned}
 &\max \sum x_e \text{ subject to} \\
 &\quad x_e \geq 0 \text{ for } e \in E \\
 &\quad \sum_{e \in \delta(v)} x_e \leq 1 \text{ for } v \in V
 \end{aligned}$$

If we run the simplex algorithm on this LP, what would the successive basic feasible solutions look like? Does it look like the augmenting path algorithm?

Finding the initial feasible solution, convert LP.

Lecture 23, 24 (Oct 31, Nov 3)

Ellipsoid algorithm.

4. Prove Farkas' Lemma from Separating Hyperplane theorem.
5. Prove that for any  $0 \leq \alpha < 1/2$ , the following ellipsoid

$$\{x \in \mathbb{R}^n : (x_1 - \alpha)^2 + x_2^2(1 - 2\alpha) + \cdots + x_n^2(1 - 2\alpha) \leq (1 - \alpha)^2\}$$

contains the half ellipsoid given by

$$\{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1, x_1 \geq 0\}.$$

6. Minimize  $\frac{(1-\alpha)^n}{(1-2\alpha)^{(n-1)/2}}$  for  $0 \leq \alpha < 1/2$ .
7. Suppose we are given an ellipse

$$E_1 = \{(x_1, x_2) : (27x_1 - 36x_2 - 27)^2 + (64x_1 + 48x_2 - 64)^2 \leq 900\}.$$

Find the smallest ellipse  $E_2$  that contains the intersection of  $E_1$  and the half-space  $H_1 = \{(x_1, x_2) : x_1 \geq 1\}$ . You also need to explain how you found it.

Hint: You may want to follow these steps: (1) find a transformation  $T$  that transforms  $E_1$  into a circle  $C$  of radius 1 centered at origin. Apply the same transform on  $H_1$  to get  $H'_1$ . (2) Do a rotation  $R$  so that  $H'_1$  becomes  $H$ . (3) First apply the inverse of  $R$  on  $E$  and then apply inverse of  $T$ . The resulting ellipse is the desired one.

8. Consider an LP  $Ax = b, x \geq 0$  with  $n$  variables and  $m$  equality constraints. Let the entries in  $A$  and  $b$  be integers with at most  $\ell$  bits. Assume that this LP has a basic feasible solution (i.e., solution with  $n - m$  zero coordinates). Show that any coordinate in a basic feasible solution is at most  $2^{f(n, \ell)}$  for some polynomial function  $f(n, \ell)$ .
9. Recall the Steiner forest LP we wrote. Let the given pair of terminals be  $\{(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)\}$ .

$$\begin{aligned} \min \sum_{e \in E} w_e x_e \text{ subject to} \\ x_e \geq 0 \\ \sum_{e \in \delta(S)} x_e \geq 1 \quad \text{for } S \subseteq V \text{ which separates some terminal pair } (s_i, t_i) \end{aligned}$$

Design a polynomial time separation oracle for this LP.

10. To check if a matrix is positive semidefinite, one can do a two-sided Gaussian elimination, where one does the same row and column transformations on a symmetric matrix and obtains a diagonal matrix at the end. See the example.

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} &\xrightarrow{\text{row}_2 \leftarrow \text{row}_2 - 2 \times \text{row}_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{\text{col}_2 \leftarrow \text{col}_2 - 2 \times \text{col}_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 1 \end{pmatrix} &\xrightarrow{\text{row}_3 \leftarrow \text{row}_3 + (2/3) \times \text{row}_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 7/3 \end{pmatrix} \xrightarrow{\text{col}_3 \leftarrow \text{col}_3 + (2/3) \times \text{col}_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7/3 \end{pmatrix} \end{aligned}$$

Show that a symmetric matrix is PSD if and only if the diagonal matrix at the end of two-sided Gaussian elimination has all non-negative entries. Essentially, you need to show that PSD property is preserved when we do same row and column transformations.

11. Recall the below semidefinite program for the MAXCUT problem. Design a polynomial time separation oracle for this SDP. First solving the above problem will be helpful.

$$\max \sum_{i < j} w_{i,j} x_{i,j} \text{ subject to } \begin{pmatrix} 1 & 1 - 2x_{1,2} & \cdots & 1 - 2x_{1,n} \\ 1 - 2x_{1,2} & 1 & \cdots & 1 - 2x_{2,n} \\ & \vdots & & \\ 1 - 2x_{1,n} & 1 - 2x_{2,n} & \cdots & 1 \end{pmatrix} \succeq 0.$$

Lecture 25, 26 (Nov 7, 10) Interior point method

12. Consider the following linear program.

$$\begin{aligned} \min & -4x_1 + 3x_2 \text{ subject to} \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 + x_2 \leq 3 \end{aligned}$$

We want to convert this an unconstrained optimization problem with parameter  $\eta$  and a barrier function  $B(x_1, x_2)$ .

$$\min \eta \times (-4x_1 + 3x_2) + B(x_1, x_2).$$

What will be the barrier function corresponding to the above constraints. Let  $x_\eta^* \in \mathbb{R}^2$  be the optimal point for the above function with parameter  $\eta$ . Plot  $x_\eta^*$  as a function of  $\eta$ , from  $\eta = 0$  to  $\eta = \infty$ . For example, you can try <https://www.wolframalpha.com/> to plot this.