Lower Bounds on LP extensions

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Overview

Abstract

This text mainly focuses on providing lower-bound results for some NP-hard problems. In order to do so, they provide a lower bound in complexity in terms of another NP-hard problem. That being said, this text only focuses on methods to do so and not on any real example. We end this presentation by claiming a result that is derived using this method.

Control flow of ideas

- Lower bounding LP formulations using properties of the constraint matrix
- Upper bounding NP-hard(communication complexity) problem in the same terms as constraint matrix.
- Using lower bound results derived for communication complexity and applying them to lower bound LP- formulations

Note 1.1

We define terms like extension complexity and also mention theorems in order that is useful for comprehensive understanding rather than the order

Extended formulation

Def 0.1 Projecting Polytopes by eliminating extra variables

Projection of a polytope P of d dimensions to its first k dimensions is

$$\pi_k(P) = \{x \in \mathbb{R}^k | \exists y \in \mathbb{R}^{d-k}, (x, y) \in P\}$$

Def 0.2 Extended Formulation

An extended formulation of a polytope $P \subset \mathbb{R}^n$ define by $Ax \leq b$ is a polytope $P^1 \subset \mathbb{R}^{n+r}$ defined by a system

$$Cx + Dy \leq d, x \in \mathbb{R}^n, y \in \mathbb{R}^r$$

such that $P = \pi_n(P^1)$.

Intuition: Polyhedra usually have fewer high dimensional facets than lower dimensional facets.

Extended formulation: an example

For any integer n, the permutahedron P_n is defined as the convex hull of the set of all permutations of the numbers $[n] = \{1, 2, 3, ..., n\}$.

Claim 0.1 Low Dimensional Polytope for Permutahedron

$$P_n = \{x \in \mathbb{R}^n : \sum_{1}^{n} x_i = n(n+1)/2 \text{ and } \sum_{i \in S} x_i \geq |S|(|S|+1)/2, \forall S \subset [n]\}$$

Proof of Claim 0.1

Page 12 of Goemans's notes.

This low-dimensional polytope with n variables gives us $2^n - 2$ facets defining P_n.

Extended formulation: an example

Instead, lets try and switch to a higher dimensional polytope by introducing extra variables $y_{i,j}$ for all pairs i, j. $y_{i,j}$ intuitively represents the indicator function $\chi(\pi(i) == j)$ where π is a permutation, not the projection polytope.

Claim 0.2 Higher Dimensional Polytope for Permutahedron

$P_n = \pi_n(P^1)$ where P^1 is a $n^2 + n$ dimensional polytope defined as

$\mathrm{P}^1 = \{(x_1, ..., x_n, y_{1,1}, ..., y_{n,n}) \in \mathbb{R}^{n^2+n}$:
$\sum_j y_{i,j} \leq 1, \forall i \in [n]$
$\sum_i y_{i,j} \leq 1, \forall j \in [n]$
$\mathrm{y}_{i,j} \geq 0, \forall i, j$
$\mathrm{x}_i = \sum_1^n j y_{i,j}, \forall i \in [n]\}$

Note that P_1 has only $n^2 + 3n$ facets defining it, so we can efficiently solve LP problems on P^1.

Extended formulation: an example

Proof of Claim 0.2

Prove using the Birkoff-von Neumann theorem. For details check out section 3 of Sitan Chen's notes.

Def 0.3 extension complexity

The extension complexity $xc(P)$ of a polytope P is the minimum size of an extended formulation P_n^1 of P.

It is easy to see that for the permutahedron, $xc(P_n) \leq n^2 + 3n$
Fun Fact: The lower bound on extension complexity for a permutahedron is $O(nlogn)$ and is in fact tight. (Reference: Goemans's paper)

Yannakakis rank

Def 1.1 Yannakakis Rank

Denoted as $rank_+(M)$. M has $rank_+(M) \leq r$ if it can be written as $M = AB$ where A is a non-negative $m \times r$ matrix and B is a non-negative $r \times n$ matrix.

Cor 1.1 Yannakakis Rank

$rank_+(M) \leq r$ iff $M = \sum_{i=1}^{r} M_i$ where each M_i is a non negative matrix with $rank(M_i) = 1$.

Def 1.2 Slack Matrix

If P is a polytope bounded by V vertices $(x_1, x_2, \ldots x_v)$ and P is defined by the set of inequalities $Ax \leq b$, then

$S_{i,j} = b_i - A_i \cdot x_j$

S is a $v \times f$ matrix. Clearly, $S \geq 0$.

Theorem 1.1(Yannakakis Theorem)

$xc(P) = rank_{+}(S_P)$

Proof: Using farkas lemma, if

$$Ax \leq b \implies A_i^l x \leq c_i$$

then we have

$$\exists y \ A_i^T = A^T y \text{ and } c_i^T \geq b^T y$$

Consider the following lemma which is the core of proof:

lemma 1.1

$rank_{+}(S_P) \leq m$ where S_P is a mXn matrix. In other terms $rank_{+}(S_P)$ is bounded by number of facets in the system.

Proof of this lemma is trivial as we can split the matrix m_i where

m_i=$[000...M_i^T 0]^T$ as this is at-least a rank one matrix.

Yannakakis theorem

We now see another lemma that helps us to convert S_P into a more efficient form defined by extension formulation

lemma 1.2

$rank_+(M) \geq rank_+(yM)\ \forall y \geq 0$

To see the proof of this lemma, we just need to see that yM can be split into ym_i as $y \geq 0$ and ym_i has rank atmost $1 (rank(AB) \leq min(rank(A), rank(B))$
Now consider the matrix S_P

$$S_P^{ij} = b_i - A_i V_j$$

Use farkas lemma from the extension matrix $A^1 x^1$ and b^1 to see that

$$\exists y_i\ A_i = y_i A^1 \text{ and } b_i \geq y_i b_i^1\ \forall i$$

Yannakakis theorem

To make this proof more readable, we are assuming that A is padded with zeros to suit the equations.
Now observe that

$$S_P^{ij} = y_i(b^1 - A^1 V_j) + rem_i,\ rem_i \geq 0 \text{ is the remainder term from the inequality of farkas lemma}$$

$$S_P = YS_P^1 + rem,\ rem \geq 0 \text{ is atmost rank one matrix}$$

$$Y = [y_1^T, y_2^T .....] \text{ and } S_P^1 \text{ is the matrix defined using extended equations}$$

Now carefully use lemma 1.2 on S_P and S_P^1 to see that

$$rank_+(S_P) \leq rank_+(S_P^1)$$

Use lemma 1.1 to conclude that

$$rank_+(S_P) \leq xc(P)$$

Fortunately, we only need the inequality moving forward

Communication Complexity

Def 2.1 Communication Complexity Problem

let n_1, n_2 be two positive integers and

$$f\colon\ [n_1]X[n_2] \rightarrow \{0,1\}$$

be a function represented by M_f
The problem asks us to find minimum number of bits needed to convince users a,b that hold x,y indices that $f(x,y) = 1$ in a case that it is.(it is not guaranteed always to a,b that it is)

Def 2.2 $FV(P)$

$FV(P)$ is the function defined by the matrix

$$J^{ij} = \left\{ \begin{array}{ll} 0 & \text{if } S_P^{ij} = 0 \\ 1 & \text{otherwise} \end{array} \right\}$$

Communication Complexity

Theorem 2.1

$ncc(FV(P)) \leq log_2(rank_+(S_P))$

Proof: To see proof of such equations, we try and find a mechanism that can convince a,b. Before defining the mechanism, consider the following lemma

lemma 2.1

let M be a mXn matrix

$$rank(M) = 1 \implies \exists \alpha \in R^{mX1} \exists \beta \in R^{1Xn} \; M = \alpha\beta$$

This result can either be seen as corollary of positive rank in the case of positive matrices or can also be easily proved by vector space results.

Communication Complexity

Assume that we put some constraints on β_i so that both players know the exact value of α_i, β_i for all M_i that sum up to S_P
Proof that we submit to players is index k such that

$$\alpha_k^i \beta_k^j > 0$$

Note that for all $M^{ij} > 0$, there must exist such proof as all the matrices M_i are non . Other side is trivial to observe as well.
Observe that this proof requires only $log_2(rank_+(S_P))$ number of bits proving our upper bound.

Connection to approximate LP problems

Def 3.1 $\rho-$*approximate* extended formulation

A formulation P$^1 is called \rho-$*approximate* extended formulation of P if for all linear objective functions w

$$max\{w^Tx : x \in P\} \leq max\{w^Tx : x \in P^1\} \leq \rho \ max\{w^Tx : x \in P\}$$

Def 3.2 $xc_\rho(P)$

It is defined as the minimum size of such P^1

The following definition is not present in the reading text and is defined for the ease of reader,

Def 3.3 $xc_\rho(P, Q)$

It is defined as the minumum size of $\rho-$*approximate* extended formulation P^1 of P such that

$$P \subseteq P^1 \subseteq Q$$

Connection to approximate LP problems

Def 3.4 $S_{P,Q}$

In a setting $P \subseteq Q$, we define $S_{P,Q}$ as follows,

$$S_{P,Q} = (b_i - A_i V_j)_{i,j}$$

where

$$Q := Ax \leq b \text{ and } \{V_1, V_2 ...\} \text{ are vertices of P.}$$

lemma 3.1

For any $\rho-$*approximate* extended formulation P^1 of P, we have

$$P^1 \subseteq \rho P \text{ and } P \subseteq P^1$$

To see proof of this lemma, observe that any optimal value in ρP is ρ times optimal value in P. Following this, if we have a point outside of ρP in P^1, we construct a separating hyperplane and provide a contradiction. The second part is trivial to see in the same way.

Connection to approximate LP problems

This lemma 3.1 helps us to reformulate $\rho-$*approximate* extensions by searching in (P,ρQ) where P$\subseteq$Q

Theorem 3.1(Braun et al.)

$xc_{\rho}(P) = rank_{+}(S_{P,\rho Q})$

We instead need(for the sake of lower bounds) a weaker Theorem that we can easilty prove.

Theorem 3.2

$\forall P^1 (Q \subseteq P^1) \implies xc_{\rho}(P) \geq rank_{+}(S_{P,\rho Q})$

Proof: We need clever definitons of Q in order to satify the first clause, observe that P satisifes all the properties we need.

Connection to approximate LP problems

lemma 3.2

$Q \subseteq P^1 \implies rank_+(S_{P,P^1}) = rank_+(S_{P,\rho Q})$

Follow the same methodology of proof used to prove Theorem 1.1 to observe this result.
This lemma 3.2 in combination with the fact that

$$xc_\rho(P) \geq rank_+(S_{P,P^1})$$

where P^1 is the extended formulation with least size.
will prove Theorem 3.2

Note 3.1

Whenever we refer to P^1 in this presentation section 3, it is assumed that it is $\rho-$*approximate* extended formulation of P.

Examples

This theorem is proved using Theorem 3.2 we proved and a result from communication complexity lower bound.

Theorem 4.1(Braverman-Moitra)

Obtaining a $n^{1-\epsilon}-$*approximation* of max clique has extension complexity lower bound of $2^{(n^{\epsilon})}$

Proof: Please follow the reference in section 5 for proof.

References

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Thank You