Minimum Bounded Degree Spanning Tree

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Minimum Spanning Tree (MST)

Given graph \( G = (V, E) \) and cost function \( c : E \rightarrow \mathbb{R} \), find a spanning tree with minimum total edge cost.

ILP for MST:
\[
\begin{align*}
x_e \in \{0, 1\} & \quad \forall e \in E \\
\min \sum_{e} c_e x_e \\
\text{s.t. } \sum_{e} x_e \geq 1 & \quad \forall S \subset V, S \neq \emptyset
\end{align*}
\]
Now we relax this ILP allowing
$0 \leq x \leq 1$, but this LP fails.

\[ \text{eg: } \begin{array}{c}
1 & 10 & 2 \\
10 & 10 & \Rightarrow \\
1 & 10 & 3 \\
\end{array} \Rightarrow \begin{cases}
\text{actual answer is 20, relaxed LP will give 15}
\text{taking all } x = 0.5
\end{cases} \]

We consider an alternate formulation:
\[ \min \sum_{e \in E} x_e \]  
\[ \text{S.t. } \sum_{e \in E(e)} x_e = |V| - 1 \]  
\[ \sum_{e \in E(s)} x_e \leq |S| - 1 \quad \forall S \subset V, S \neq \emptyset \]  
\[ x_e \geq 0 \quad \forall e \in E. \]  
\{ E(s) \text{ is the set of edge with both endpoints in } S \}. \]
Q) Can this LP be solved in polynomial time?
A) Yes. We show that there exist polynomial time separation oracles.
- Given a fractional solution $x$, we find a set $S \subseteq V$ s.t.
  $x(E(s)) > |S| - 1$.

$\left\{ x(F) = \sum_{e \in F} x_e \right\}$
Create a new graph by adding source $s$ and sink $t$, and edges $(s,v), (v,t) \quad \forall v \in V$, s.t.

\[ c(s,v) = \frac{x(\delta(v))}{2} \]

\[ c(v,t) = 1 \]

\[ c(e) = \frac{xe}{2} \]

Now consider $S$ be a $s-t$ cut.
the value of the cut is:

\[ |S| \times 1 \ (\text{N} \times t \text{ edge with } v \in S) \]

\[ + \sum_{e \in \mathcal{E}(S)} x_e \ (\text{cut edge}) \]

\[ + \sum_{v \in |S|} \frac{x(\mathcal{S}(v))}{2} \]

\[ = |S| + x(\mathcal{E}(\mathcal{N})) - x(\mathcal{E}(\mathcal{S})) \]
\[ = |S| + |V| - 1 - x(E(s)) \rightarrow 4 \]

If LP constraint is not violated \( \Rightarrow \)

\[ x(E(s)) \leq |S| - 1 \]

\[ 0 \leq |S| - 1 - x(E(s)) \rightarrow 5 \]

5 and 4 \( \Rightarrow \) Constraint violated only if any cut has value \( < |V| \)
\[ \therefore \text{It is sufficient to ensure}\]
\[\text{min cut } \geq |V| \text{ which can be solved in polynomial time.}\]

\textbf{Note:} \ S \text{ can be null when finding min-cut which is not allowed. So we make a small modification in this case by putting } C(S,V) = \infty \text{ for some edge.}
Uncrossing Technique:

We may have exponential tight constraints. We see how we can characterize any extreme point solution using a set of non-crossing constraints. We define the characteristic vector of a set of edges $F$ to be:
\[ X(F)_e = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise} \end{cases} \]

**Proposition 1:** for \( S, T \subseteq V \)

\[ X(E(S)) + X(E(T)) \leq X(E(S \cup T)) + X(E(S \cap T)) \]

and equality iff \( E(S \setminus T, T \setminus S) = \emptyset \).

**Proof:** There are 6 types of edges as shown and except for \( 6 \) every other edge is counted same number of times.
Let $\mathcal{F} = \{ S | x(E(s)) = |S| - 1 \}$ be the tight constraints for an extremal point $x$.

**Lemma 1**: If $S \in \mathcal{F}$ and $S \not= \emptyset$ then $S \cup \mathcal{T}, S \cup \mathcal{T} \in \mathcal{F}$

**Proof**: \[ |S| - 1 + |\mathcal{T}| - 1 = x(E(s)) + x(E(T)) \leq x(E(S \cup \mathcal{T})) + x(E(S \cup \mathcal{T})) \text{ (Prop 1)} \leq |S \cup \mathcal{T}| - 1 + |S \cup \mathcal{T}| - 1 \]

\[ = |S| - 1 + |T| - 1 \]

\[ \therefore \text{ All inequalities are equal.} \]
Laminar Family:

Two sets $S,T$ are intersecting if $S \cap T$ and $T \subseteq S$ are non-empty.

A family of sets is called laminar if all sets are non-intersecting.

Onion-like structure
Lemma 2: If $L$ is maximal laminar subfamily of $F$ then
\[ \text{Span}(F) = \text{Span}(L) \]

Proof: For sake of contradiction, assume $\text{Span}(L) \subset \text{Span}(F)$. \\
\[ \Rightarrow \chi(E(S)) \notin \text{Span}(L) \text{ and } S \notin F \]

Pick $S$ that intersects fewest sets in $L$. Let $T$ be a set that intersects $S$. Then by Lemma 1, $S$ and $T$ are also in $F$. Clearly $S \cap T$
and SUT intersects fewer sets in \( L \) than \( S \), \( \Rightarrow \) SUT, SUT \( \in \) \( \text{Span}(L) \)

but again by lemma 1
\[
X(E(CS)) = X(E(SUT)) + X(E(CST)) - X(E(C))
\]
\[\therefore X(E(s)) \text{ is also in } L, \text{ which is a contradiction.} \]

Proposition 2:- A laminar family over \( V \) without singletons has almost \( |V| - 1 \) distinct elements.
Corollary 1: LP-MST is integral.

Proof: Let $E^* \subseteq E$ s.t. $x_e > 0 \forall e \in E^*$.

Then $|E^*| = |L| \leq |V|-1$.

Also $\sum x_e = |V|-1$

$\forall e \in E^*$

$\Rightarrow x_e = 1 \quad \forall e \in E^*$
Minimum bounded degree spanning tree:

Given graph \( G(V,E) \) and cost \( c:E \rightarrow \mathbb{R} \) and degree bounds \( B_v \) for each \( v \in W, W \subseteq V \), find min spanning tree respecting degree bounds.
Setting degree bound of 1 to 2 vertices and 2 to all other will convert this to the Hamiltonian path problem which is NP-complete. Hence MBDST is NP-complete.
LP-MBDST:\[\]
\[
\min \sum_{e \in E} c_e x_e, \; x_e \geq 0
\]
\[
\text{s.t.} \sum_{e \in E(v)} x_e = |V|-1 \quad \forall e \in E(v) \cup S \neq \emptyset
\]
\[
\sum_{e \in E(s)} x_e = |S|-1 \quad \forall s \in S \neq \emptyset
\]
\[
x(\delta(v)) \leq b_v + \forall v \in V.
\]
A+2 Approximation:

There exist an algorithm in polynomial time whose degree bounds is violated by at most 2 and cost is less than or equal to optimal cost.
Brief explanation:—
find a basic optimal solution $x$, and construct graph with edges $e$ for which $xe > 0$.

1. If any vertex has degree 1, add its edge to final soln, remove this vertex and repeat algorithm for remaining graph.
else if any vertex has degree $\leq 3$ in the resulting graph, remove the constraint for that vertex.

Note: proof for this is in the reference (2)
LP-MBDCT (G, B, W, F)

\[ \min c(x) = \sum_{e \in E} x_e \]

\[ x(E(v)) = |V| - |F(v)| - 1 \]

\[ x(E(s)) = |S| - |F(s)| - 1 \quad \forall s \in I(F) \]

\[ x(\delta(v)) \leq B \]

\[ x_e \geq 0 \quad \forall e \in E \]

H is an F-tree of G
MBDCT Algorithm (G, B, W, F)

1. If F is a spanning tree return $\phi$
   else let $F' = \phi$
2. Find basic optimal solution of LP-MBDCT(G, B, W, F)
   remove every edge $e$ with $x_e^* = 0$
   Let $E^*$ be the support of $x^*$
3. If there exists an edge $e = \{v, w\}$ s.t $x_e^* = 1$
   set $F' = \{e\}, F = F \cup \{e\}, G_1 = G_1 \setminus \{e\}$
   $b_v' = b_v - 1, b_w' = b_w - 1$
4. If there exists a vertex $w \in W$ s.t $\deg_{G_1}(w) \leq b_w + 1$, then set $W = W \setminus \{w\}$
5. Return $\hat{F} \cup$ MBDCT Algorithm ($G_1, B, W, F'$)

Let $x^*$ be a basic optimal solution to LP-MBDCT($G, B, W, F$).

$F'$-tree $H'$ of $G_1$ whose cost is at most the cost of an optimal solution to LP-MBDCT($G_1, B', W, F'$)

such that $d_{H'}(w) \leq b_w' + 1, \forall w \in W$.

$F$-tree $H = H' \cup \{e\}$ of $G_1$.

$c(H) = c(H') + c_e \leq c(x^*) + c_e \leq c(x^*) + c_e^{\text{rest}}$

The cost of $H$ is at most cost of an optimal solution to LP-MBDCT($G, B, W, F$).
Theorem: One of the following is satisfied:

1. There is an edge e with $x_e^* = 1$
2. There is a vertex $w \in W$ s.t. $\deg_{E^*}(w) = B_w$

Not True $\Rightarrow x_e^* < 1 \quad \forall e$
\[ \deg_{E^*}(v) \geq 3, \forall v \in W \quad \text{and} \quad \deg_{E^*}(v) = 3 \Rightarrow B_v = 1 \]

Claim: If active vertex $v$ has only one excess token then:
$\Rightarrow v \notin T, \deg(v) = 1$
OR
$v \in T, \deg(v) = 3, B_v = 1$
$\deg(v) = 2, B_v = 1$

If $B_v \geq 2$, some $x_e^*$ can be 1
If $S$ is a leaf

- $S$ has 4 or more active vertices (only 3 vertices needed)
- If someone has 2 excess tokens, then satisfied
  Suppose $u, v, w$ has exactly excess token
  \[ |E^*(S)| \leq 3 \] (3 vertices)
  \[ x^*(E^*(S)) \leq 1 \] (S cannot be tight now or constraint)
  \[ \Rightarrow |E^*(S)| \geq 2 \] (no 1-edge)

- 2 active vertices
  This case is not possible since
  \[ x^*(E^*(S)) \geq 1 \]

\[ |E^*(S)| = 2 \]

**Claim**: \( uvw \in T \) and \( B_u = B_v = B_w = 1 \), deg = 3

\[ x^*(E(A)) \leq a - 1 \]
\[ x^*(E(B)) \leq b - 1 \]
\[ x^*(E(C)) \leq c - 1 \]

1) \( x^*(E(A)) + x^*(E(B)) + x^*(E(C)) \leq a + b + c - 3 \)
2) \( x^*(E(V)) = |V| - 1 \)

\[ \Rightarrow x^*(u, v) + x^*(v, w) + x^*(w, u) \geq 2 \]

\[ \Rightarrow \sum_{z \in \{u, v, w\}} x^*(S(z)) \geq 4 \]

\[ \text{but} \quad \sum_{z \in \{u, v, w\}} x^*(S(z)) = B_u + B_v + B_w = 3 \]
S has at least 1 child

S has 2 or more children. By induction, each child has 2 excess tokens. S can get 4

S has only 1 child R
S/R has at least one active vertex

If only one active vertex v
v has only 1 excess token

\[ x^*(E(s)) = x^*(E^*(R)) + x^*(\delta(v, R)) \]

int int

\[ \Rightarrow x^*(\delta(v, R)) \geq 1 \]

\[ \text{deg}_v \neq 1 \text{ since } (x^*_v \leq 1) \]

Claim 1: \[ \Rightarrow \text{deg}_v = 3, B_v = 1, \forall v \in T \]

1 = B_v = x^*(\delta(v)) \geq x^*(\delta(v, R)) \geq 1

equality throughout

\[ \chi_{E(s)} = \chi_{E(R)} + \chi_{\delta(v, R)} = \chi_{E(R)} + \chi_{\delta(v)} \]

contradicts linear independence
References

[2] mbdst.pdf (gatech.edu)
[3] 56413_goemans_michel.dvi (mit.edu)