Zero Sum games and LP Duality
A CS 602 presentation

Derivation of Strong LP Duality from Minimax Theorem of Zero Sum Games

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Table of Contents

1. Preliminaries
2. Strong LP Duality proves Minimax Theorem for Zero Sum Games
3. Proof given by Dantzig
4. Proof Sequence
5. Strong LP Duality from Minimax Theorem of Zero Sum Games
6. Appendix
7. References

Table of Contents

1. Preliminaries
2. Strong LP Duality proves Minimax Theorem for Zero Sum Games
3. Proof given by Dantzig
4. Proof Sequence
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Notations used

- m and n are positive integers, $[n] = \{1,...n\}$.
- All vectors are column vectors unless specified otherwise. The jth component of a vector x is written as x_j.
- All matrices have real entries.
- The transpose of a matrix A is written A^T.
- The all-zero and the all-one vector are written as $\mathbf{0} = (0, \ldots, 0)^T$ and $\mathbf{1} = (1, \ldots, 1)^T$, their dimension depending on the context, and the all-zero matrix as just 0.
- Inequalities between vectors or matrices such as $x \geq \mathbf{0}$ represent inequality between all respective components.

LP and Dual LP

- A linear program (LP) in *inequality* form, given by an $m \times n$ matrix A and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ for a vector $x \in \mathbb{R}^n$ is:

$$\max_x c^T x, \text{ subject to } Ax \leq b, x \geq \mathbf{0} \tag{1}$$

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- The *dual* of the above *primal* LP for a vector $y \in \mathbb{R}^m$ is given by:

$$\min_y b^T y, \text{ subject to } A^T y \geq c, y \geq \mathbf{0} \tag{2}$$

LP duality

- **Weak LP duality**
  It states that if both *primal LP* (1) and the *dual LP* (2) have feasible solutions x and y, respectively, then their objective function values are mutual bounds, that is,

$$c^T x \leq b^T y \tag{3}$$

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- **Strong LP duality**
  If the *primal LP* (1) and the *dual LP* (2) are feasible, then there exist feasible x and y with $c^T x = b^T y$, which are therefore optimal solutions.

Definition - Zero Sum Game

- A **zero sum game** given by an $m \times n$ matrix A, is played between a row player who secretly chooses a row i of A and a column player who secretly chooses a column j.
- Then both players reveal their choices, after which the row player receives the *payoff* a_{ij} from the column player (and since it is a zero-sum game is like a *cost* to the column player, or *payoff* for column player is $-a_{ij}$).
- Such games are called *zero-sum games* since whatever one player gains is what the other player loses.
- A common example is *Rock-Paper-Scissors*.

Definition - Strategy

- A *strategy* refers to a player's plan specifying which choices it will make in every possible situation, leading to an eventual outcome.
- The rows and columns are called the players' **pure strategies**.
- The players can *randomize* their strategies by choosing actions according to a probability distribution called a *mixed strategy*. The row player is then assumed to maximize their *expected payoff* and the column player to minimize their *expected cost*.

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- We denote the set of mixed strategies of the row player by:

$$Y = \{y \in \mathbb{R}^m | y \geq \mathbf{0}, y^T\mathbf{1} = 1\}$$

  and the column player by:

$$X = \{x \in \mathbb{R}^n | x \geq \mathbf{0}, x^T\mathbf{1} = 1\}$$

- With the mixed strategies of the row and column player, the expected payoff of the row player and the expected cost to the column player is $y^T Ax$.

Von Neumann's Minimax theorem

- A is the matrix associated with some *zero-sum game*.
- Suppose column player plays first with mixed strategy x and then row player plays with mixed strategy y, then expected payoff for row player is given by:

$$\min_x \max_y y^T Ax \tag{4}$$

  In other words choose $x \in X$ such that the maximum over $y \in Y$ of $y^T Ax$ is minimized.

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$$\max_y \min_x y^T Ax \tag{5}$$
- The **minimax** theorem due Von Neumann states that optimum value of (4) and (5) are equal to some unique value ν called the **value** of the game.

Table of Contents

1. Preliminaries
2. Strong LP Duality proves Minimax Theorem for Zero Sum Games
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6. Appendix
7. References

Reconstructing mini-max and maxi-min as LPs

We rewrite the optimization problem $\max_y y^T Ax$, where $y \in Y$ for a given x as

$$\min_v v, \;\; Ax \leq \mathbf{1}v$$

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Then, (4) corresponds to minimizing this over $x \in X$

$$\min_{x,v} v \text{ subject to } Ax \leq \mathbf{1}v, \;\; x \in X \tag{6}$$

and (5) corresponds, similarly to

$$\max_{y,u} u \text{ subject to } A^T y \geq \mathbf{1}u, \;\; y \in Y \tag{7}$$

Strong LP Duality proves Minimax Theorem for Zero Sum Games

- We modify LP(6) to:

$$\max_{x,v} -v \text{ subject to } Ax - \mathbf{1}v \leq \mathbf{0},\ -\mathbf{1}^T x = -1 \tag{8}$$

and LP(7) to:

$$\min_{y,u} -u \text{ subject to } A^T y - \mathbf{1}u \geq \mathbf{0},\ -\mathbf{1}^T y = -1 \tag{9}$$

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Now, (8) and (9) form a primal-dual pair in general LP form. Since both LPs are feasible, the strong LP duality theorem (which also holds for LPs in general) implies that their optimal values are equal ($-v = -u$), which proves **Minimax theorem for zero sum games**.

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- Our aim now is to prove the converse. (Note that the min-max and max-min values in (6) and (7) exist without having to assume LP-duality)

Table of Contents

1. Preliminaries
2. Strong LP Duality proves Minimax Theorem for Zero Sum Games
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6. Appendix
7. References

Dantzig's game (a fake proof!)

This portion assumes minimax theorem.

Dantzig's Theorem

Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. Consider the zero sum game with the payoff matrix $B \in \mathbb{R}^{(m+n+1)\times(m+n+1)}$, defined as:

$$B = \begin{bmatrix} 0 & A & -b \\ -A^T & 0 & c \\ b^T & -c^T & 0 \end{bmatrix} \tag{10}$$

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Then B has game value 0, with a min-max strategy $z = (y, x, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ which is also a max-min strategy, with $Bz \leq \mathbf{0}$. If $t > 0$ then $\frac{1}{t}x$ is a *optimal* solution to the primal LP (1) and $\frac{1}{t}y$ is a *optimal* solution to the dual LP (2). If $(Bz)_{(m+n+1)} < 0$, then $t = 0$ and both the primal and dual are *infeasible*.

Proof by Dantzig

- $B = -B^T$
  $\therefore$ The value of the game is 0 (refer Appendix B).
- $Bz \leq \mathbf{0}$ (using the definition of min-max).
  In other words $Ax - bt \leq \mathbf{0}$, $-A^T y + ct \leq \mathbf{0}$ and $b^T y - c^T x \leq 0$.

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- If $t > 0$, then $\frac{1}{t}x$ and $\frac{1}{t}y$ are primal and dual feasible solutions respectively with $c^T x \geq b^T y$, but by weak duality $c^T x \leq b^T y$, therefore $c^T x = b^T y$ and we have $\frac{1}{t}x$ and $\frac{1}{t}y$ as respective optimal solutions.

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- If $(Bz)_{(m+n+1)} < 0$, i.e., $c^T x < b^T x$, weak duality is violated if $t > 0$, so $t = 0$. Moreover, $Ax \leq \mathbf{0}$ and $A^T y \geq \mathbf{0}$ and, $b^T y \leq \mathbf{0}$ or $c^T x \geq \mathbf{0}$. This leads to infeasiblity of atleast one of the LPs (1) or (2).

Loophole

- This seems to prove strong LP duality from minimax theorem by the construction of a game that gives us either the optimal solutions or proves infeasibility of atleast one of the LPs.

[1] Proof is omitted

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- This seems to prove strong LP duality from minimax theorem by the construction of a game that gives us either the optimal solutions or proves infeasibility of atleast one of the LPs.
- However, this does not cover the case when $(Bz)_{(m+n+1)} = 0$ and $t = 0$.
- If we assume $t - (Bz)_{(m+n+1)} > 0$, then the above proof by Dantzig works!!!.

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- However, this does not cover the case when $(Bz)_{(m+n+1)} = 0$ and $t = 0$.
- If we assume $t - (Bz)_{(m+n+1)} > 0$, then the above proof by Dantzig works!!!.
- However, this assumption turned out to be equivalent[1] to assuming *Farkas' lemma*! This defeats the point of proving LP Duality as we already know the proof of strong LP duality from Farkas' Lemma.

[1] Proof is omitted

Table of Contents

1. Preliminaries
2. Strong LP Duality proves Minimax Theorem for Zero Sum Games
3. Proof given by Dantzig
4. Proof Sequence
5. Strong LP Duality from Minimax Theorem of Zero Sum Games
6. Appendix
7. References

Proof Sequence

The objective at hand is to derive **strong LP duality** from the **minimax theorem of zero sum games**. We will proceed in the subsequent order of demonstrations[1]:

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4. Tucker's Theorem **proves** Farkas' Lemma

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2. Ville's theorem **proves** Gordan's Theorem.
3. Gordan's Theorem **proves** Tucker's Theorem
4. Tucker's Theorem **proves** Farkas' Lemma
5. Farkas' Lemma **proves** strong LP duality

Therefore, we have **minimax theorem of zero sum games** proving **strong LP duality**.

Table of Contents

1 Preliminaries

2 Strong LP Duality proves Minimax Theorem for Zero Sum Games

3 Proof given by Dantzig

4 Proof Sequence

5 Strong LP Duality from Minimax Theorem of Zero Sum Games

6 Appendix

7 References

Let $A \in \mathbb{R}^{m\times n}$. The following Theorem of Gordan (11) proves the Theorem of Ville (12) and vice versa and (12) proves the minimax theorem and vice versa.

$$\nexists x \in \mathbb{R}^n : Ax = \mathbf{0},\ x \geq \mathbf{0},\ x \neq \mathbf{0} \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A > \mathbf{0}^T \tag{11}$$

$$\nexists x \in \mathbb{R}^n : Ax \leq \mathbf{0},\ x \geq \mathbf{0},\ x \neq \mathbf{0} \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A > \mathbf{0}^T,\ y \geq \mathbf{0} \tag{12}$$

Proof of Ville's theorem from Gordan's Theorem

- Assume (11). We prove (12).
- (12)'s ($\Leftarrow$) direction is trivial (multiply y^T to both sides of the inequality $Ax \leq \mathbf{0}$ to get $y^T Ax > 0$ which contradicts $y^T Ax \leq y^T \mathbf{0} = 0$).

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- We now prove the other direction.
- Assume $\nexists x \geq \mathbf{0}, x \neq \mathbf{0}, Ax \leq \mathbf{0}$, then $\nexists x, s \geq \mathbf{0}, (x, s) \neq (\mathbf{0}, \mathbf{0}), Ax + s = \mathbf{0}$. Now consider the matrix $B = (A\ I)$ where I is the $m \times m$ identity and the vector $z = (x, s)^T$.
- Equivalently, $\nexists z \geq \mathbf{0}, Bz = 0, z \neq \mathbf{0}$, then by (11), $\exists y', y'^T B > \mathbf{0}^T$ and thus $y = y'$, gives us the y in (12) (note $y' > \mathbf{0}$ as $y'^T I > \mathbf{0}^T$).
- Hence proved $\square$

Proof of Gordan's theorem from Ville's Theorem

- Assume (12). We prove (11).
- Similar to before, the ($\Leftarrow$) is trivial, so we prove the ($\Rightarrow$) direction.
- Assume $\nexists x, Ax = \mathbf{0}, x \geq \mathbf{0}, x \neq \mathbf{0}$. Then $\nexists x, Ax \leq \mathbf{0}, -Ax \leq \mathbf{0}, x \geq \mathbf{0}, x \neq \mathbf{0}$.
- Then by (12), $\exists y^+, y^- \geq \mathbf{0}$ such that ${y^+}^T A > \mathbf{0}^T, -{y^-}^T A > \mathbf{0}^T$. Therefore, ${y^+}^T A - {y^-}^T A > \mathbf{0}^T$ and $y^+ - y^-$ gives us our y.
- Hence proved $\square$

Proof of Ville's theorem from Minimax Theorem

1. Now we prove the theorem of Ville from minimax theorem.
2. Assume minimax theorem holds on $A \in \mathbb{R}^{m \times n}$. We once again prove the ($\Rightarrow$) side since the other side ($\Leftarrow$) is trivial.
3. Assume $\nexists x \in \mathbb{R}^n : Ax \leq \mathbf{0},\ x \geq \mathbf{0}, x \neq \mathbf{0}$, then by the formulation of the min-max strategy in (6) we see that the value of the game, which is $\min_{x,v} v, Ax \leq \mathbf{1}v$, must be positive. Otherwise, $\exists x \in X$ due to minimax theorem such that $Ax \leq \mathbf{0}$ which contradicts the assumption.

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4. Then, by minimax theorem there is an optimal $y \in Y$ and $u = v$ such that $y^T A \geq \mathbf{1}^T u > \mathbf{0}^T$.
   $\therefore y^T A > \mathbf{0}^T$.
5. Hence proved $\square$

Proof of Minimax theorem form Ville's Theorem

1. To prove minimax theorem from the theorem of Ville.
2. Assume (12).
3. Consider the game on A. It has a max-min payoff u and according to (7) a max-min strategy $y \in Y$. Let $A' = A - \mathbf{1}u\mathbf{1}^T$.
4. $y^T A' = y^T A - u\mathbf{1}^T \geq \mathbf{0}^T$ (by (7)).
5. $\exists x \geq \mathbf{0}, x \neq \mathbf{0}$ such that $A'x \leq \mathbf{0}$.

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5. $\exists x \geq \mathbf{0}, x \neq \mathbf{0}$ such that $A'x \leq \mathbf{0}$.
6. If not, by Ville's theorem, $\exists z, z^T A' > \mathbf{0}^T, z \geq \mathbf{0}$ and we can (because $z \neq 0$) scale z so that it belongs to Y. $z^T A' > \mathbf{0}^T \Rightarrow z^T A' \geq \varepsilon \mathbf{1}^T$ for some ε or $z^T A \geq (u + \varepsilon)\mathbf{1}^T$ which means u is not optimal $\#$.

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7. Therefore $A'x \leq \mathbf{0}$ for some $x \in X$ (by scaling). So $Ax \leq \mathbf{1}u$, or the min-max value is atmost u. But min-max is trivially greater than or equal to max-min. Thus, the min-max value is u, proving minimax theorem.
8. Hence proved $\square$

The Lemma of Tucker

The following is the lemma of Tucker for $A \in \mathbb{R}^{m \times n}$:

$$\exists y \in \mathbb{R}^m, x \in \mathbb{R}^n : y^T A \geq \mathbf{0}^T,\ x \geq \mathbf{0},\ Ax = \mathbf{0}, x_n + (A^T y)_n > 0 \tag{13}$$

and it has an equivalent inequality form

$$\exists y \in \mathbb{R}^m, x \in \mathbb{R}^n : y^T A \geq \mathbf{0}^T,\ y \geq \mathbf{0},\ x \geq \mathbf{0},\ Ax \leq \mathbf{0}, x_n + (A^T y)_n > 0 \tag{14}$$

It has a form for skew symmetric matrices $B \in \mathbb{R}^{k \times k}$ (provable from the lemma itself),

$$\exists z \in \mathbb{R}^k : z \geq \mathbf{0},\ Bz \leq \mathbf{0},\ z_k - (Bz)_k > 0 \tag{15}$$

This can be proved[2] using induction on n, the number of columns of A.

[2] We omit the proof in this presentation

The Theorem of Tucker

1. The last column of A in Tucker's lemma plays a special role which can be taken by any other column. We have the stronger version, the theorem of Tucker, provable from his lemma:

$$\exists y \in \mathbb{R}^m, x \in \mathbb{R}^n : y^T A \geq \mathbf{0}^T,\ x \geq \mathbf{0},\ Ax = \mathbf{0}, x + A^T y > 0 \quad (16)$$

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2. *Proof,*
Choose $x^{(i)}$, $y^{(i)}$ in (13) such that $x_i^{(i)} + (A^T y^{(i)})_i > 0$ (these come from Tucker's Lemma).
Then $x = \sum_{i=1}^n x^{(i)}$ and $y = \sum_{i=1}^n y^{(i)}$ satisfy (16).
3. We now move to the climax, an unexpected proof ... (*or was it?*)

Proof of Tucker's theorem from Gordan's theorem

Observation

If $Ax = \mathbf{0}$ and $x \geq \mathbf{0}$, then $\forall y$ such that $y^T A \geq \mathbf{0}^T$: if $x_j > 0$ then $(y^T A)_j = 0$ because otherwise, $y^T Ax = 0 = \sum_{j \in [n]} (y^T A)_j x_j > 0$

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1. Hence, (for a given A) for any x satisfying the conditions of (16), the set:
$$S = \text{supp}(x) = \{\, j \in [n] \,\big|\, x_j > 0\}$$
is unique.
2. The main idea is that the nonnegativity constraints for the variables x_j, $j \in S$ can be dropped and these variables therefore be eliminated, which allows applying Gordan's Theorem to the remaining variables.

Proof of Tucker's theorem from Gordan's theorem

1. Let $A = [A_1 \dots A_n]$. For any $S \subseteq [n]$ and $J = [n] - S$, we write $A = [A_J \, A_S]$ and $x = [x_J \, x_S]$ for $x \geq \mathbf{0}$.

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2. $Ax = 0, x \geq \mathbf{0}$ and $Ax' = 0, x' \geq \mathbf{0} \Rightarrow A(x + x') = 0$ and $x + x' \geq \mathbf{0}$.
3. $\mathrm{supp}(x + x') = \mathrm{supp}(x) \cup \mathrm{supp}(x')$.

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6. We now show that

$$\exists y \in \mathbb{R}^m, x = (\mathbf{0}\ x_S) : y^T A_J > \mathbf{0}^T, y^T A_S = \mathbf{0}^T, Ax = 0, x_s > \mathbf{0} \tag{17}$$

which implies Tucker's theorem (we will use Gordan's theorem).

Proof ctd[1]

1. Consider some $\tilde{x} \geq \mathbf{0}, A\tilde{x} = \mathbf{0}$ with maximal support (S), i.e., $x_S > \mathbf{0}$. If $S = [n]$ we are done, as then $\tilde{x} > 0$, giving (16).
2. Let k be the rank of A_S. If $k = m$, we claim that $S = [n]$.
3. If $k = m$, then A_S is full rank, hence for any $j \in J = [n] - S$, $A_j = A_S \hat{x_S}$, for some $\hat{x_S}$.
4. Consider x' which is 1 at the jth position and $x'_S = \alpha(\tilde{x_S}) - \hat{x_S}$ for a sufficiently large α so that $x'_S > \mathbf{0}$. Then $Ax' = 0, x' \geq \mathbf{0}$ and $\text{supp}(x') = S \cup j$ contradicting the maximality of S if $S \neq [n]$.

Proof ctd^2

1. Hence, let $k < m$. In order to apply Gordan's theorem, eliminate x_S from $Ax = A_J x_J + A_S x_S = \mathbf{0}$ by replacing it with an equivalent system $CAx = \mathbf{0}$ for a suitable invertible matrix $C \in \mathbb{R}^{m \times m}$.
2. Note that any solution of $A_J x_J + A_S x_S = \mathbf{0}$ is a solution of $CA_J x_J + CA_S x_S = \mathbf{0}$ and since C is invertible, vice versa.

Proof ctd[3]

1. WLOG let the last k rows of A_S be linearly independent and form the matrix F. Let the ith row of A_S be a_{iS}. Then $\forall i \in \{1, ...m-k\}$, $a_{iS} = z^{(i)} F$ for some $z^{(i)} \in \mathbb{R}^{(1 \times k)}$.
2. Then we construct C as:

$$C = \begin{bmatrix} 1 & \dots & 0 & & -z^1 & \\ & \ddots & & & \vdots & \\ 0 & \dots & 1 & & -z^{(m-k)} & \\ 0 & \dots & 0 & 1 & \dots & 0 \\ & \vdots & & & \ddots & \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \tag{18}$$

C is upper triangular and hence invertible $(\det(C) = 1)$

Proof ctd[4]

We have:

$$CA_J = \begin{bmatrix} D \\ E \end{bmatrix}, CA_S = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

Where $D \in \mathbb{R}^{(m-k)\times|J|}$, $E \in \mathbb{R}^{k\times|J|}$ and $F \in \mathbb{R}^{k\times|S|}$ is as defined earlier.

Proof ctd[5]

1. Suppose $\exists x_J \in \mathbb{R}^{|J|}$ such that
$$Dx_J = \mathbf{0}, x_J \geq \mathbf{0}, x_J \neq \mathbf{0} \tag{19}$$
2. Because F has rank k, $\exists x_S$ such that $Ex_J = -Fx_S$ or $Ex_J + Fx_S = \mathbf{0}$ hence $CA_Jx_J + CA_Sx_S = \mathbf{0}$ and hence $A_Jx_J + A_Sx_S = \mathbf{0}$.
3. Let $x(\alpha) = (x_J, x_S + \alpha\tilde{x_S})$, we have $Ax(\alpha) = \mathbf{0}$ since $A\tilde{x_S} = \mathbf{0}$ and $x(\alpha) \geq \mathbf{0}$ for arbitrarily large α.
4. But $x(\alpha)$ has a larger support than $\tilde{x_S}$ because $x_J \neq \mathbf{0}$. $\#$.
5. Hence, no such x_J as in (19) exists and for the finishing blow, by **Gordan's theorem** $\exists w \in \mathbb{R}^{m-k}, w^T D > \mathbf{0}^T$ i.e.,
$$(w^T, \mathbf{0}^T)\begin{bmatrix} D \\ E \end{bmatrix} > \mathbf{0}^T, (w^T, \mathbf{0}^T)\begin{bmatrix} 0 \\ F \end{bmatrix} = \mathbf{0}^T$$
With $y = (w^T, \mathbf{0}^T)C$ and $x = \tilde{x}$, this implies (17) and we're done
6. Hence proved $\square$

Farkas' Lemma

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$: The following three are versions of Farkas' lemma (and are provably equivalent)[3]

$$\nexists x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0} \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T,\ y^T b < 0 \tag{20}$$

$$\nexists x \in \mathbb{R}^n : Ax \leq b,\ x \geq \mathbf{0} \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T,\ y \geq \mathbf{0},\ y^T b < 0 \tag{21}$$

$$\nexists x \in \mathbb{R}^n : Ax \leq b, \Leftrightarrow \exists y \in \mathbb{R}^m : y^T A = \mathbf{0}^T,\ y \geq \mathbf{0},\ y^T b < 0 \tag{22}$$

[3] We've shown in class that (21) proves strong LP duality and thus we omit the proof.

Proof of Farkas' Lemma from Tucker's Lemma

To prove Farkas' lemma (20) from Tucker's lemma [2] (we once again prove only the ($\Rightarrow$) side as the other side ($\Leftarrow$) is trivial),

$$(\nexists x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0}\ ) \Rightarrow (\exists y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T,\ y^T b < 0)$$

Rewriting in contrapositive form,

$$(\forall y \in \mathbb{R}^m : \neg(y^T A \geq \mathbf{0}^T) \vee y^T b \geq 0) \Rightarrow (\exists x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0})$$

OR

$$(\forall y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T \Rightarrow y^T b \geq 0) \Rightarrow (\exists x \in \mathbb{R}^n : Ax = b,\ x \geq \mathbf{0}) \quad (23)$$

Proof of Farkas' Lemma from Tucker's Lemma

Proof,

1. Consider the matrix $A' = [A \;\; -b]$ such that it is concatenation of A and $-b$.

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2. By Tucker's lemma we are guaranteed that $\exists x_0, y_0$ with
   $x_0 = ({x'}^T, (x_0)_{n+1})^T$,(where $x' \in \mathbb{R}^n$) such that
   $y_0^T A' \geq \mathbf{0}^T$ (which is equivalent to saying $y_0^T A \geq \mathbf{0}^T \wedge y_0^T b \leq 0$,)
   $x_0 \geq \mathbf{0}$,
   $A' x_0 = \mathbf{0}$ (i.e., $Ax' - b(x_0)_{n+1} = \mathbf{0}$) and
   $(x_0)_{n+1} - y_0^T b > 0$

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   $(x_0)_{n+1} - y_0^T b > 0$
3. Now by the hypothesis of Farkas' lemma (23), i.e.
   $\forall y \in \mathbb{R}^m : y^T A \geq \mathbf{0}^T \Rightarrow y^T b \geq 0$
   and since $y_0 \in \mathbb{R}^m$, and $y_0^T A \geq \mathbf{0}^T$, we have $y_0^T b \geq 0$,
   Also, we have $y_0^T b \leq 0$ and thus $y_0^T b = 0 \Rightarrow (x_0)_{n+1} > 0$
4. This gives $Ax = b$ for $x = x'/(x_0)_{n+1} \geq \mathbf{0}$

Hence proved $\square$

Conclusion

- This brings us to the end of the presentation.

[4] $\xrightarrow{\star}$ is obvious and we have not shown proof for $\xrightarrow{\star\star}$

Conclusion

- This brings us to the end of the presentation.
- We have shown the following derivations:
  Minimax Theorem $\rightarrow$ Theorem of Ville $\rightarrow$ Gordan's theorem $\rightarrow$ Theorem of Tucker $\xrightarrow{\star}$ Tucker's Lemma $\rightarrow$ Farkas' lemma $\xrightarrow{\star\star}$ Strong LP Duality [4]
- Thus, we have proved that **Strong LP Duality can be derived from Minimax Theorem of Zero Sum Games.** $\square$

[4] $\xrightarrow{\star}$ is obvious and we have not shown proof for $\xrightarrow{\star\star}$

Thank You

Thank you for attending the presentation!!!

Table of Contents

1. Preliminaries
2. Strong LP Duality proves Minimax Theorem for Zero Sum Games
3. Proof given by Dantzig
4. Proof Sequence
5. Strong LP Duality from Minimax Theorem of Zero Sum Games
6. Appendix
7. References

Appendix A

- Since Tucker's lemma (or theorem) proves Farkas' lemma, it isn't surprising that it proves *strict complementary slackness*.
- Recall that for the LP and its dual, the feasible pair x and y respectively is optimal iff $c^T x = y^T Ax = y^T b$ which means $y^T(b - Ax) = (c^T - y^T A)x = 0$.

Appendix A

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- Recall that for the LP and its dual, the feasible pair x and y respectively is optimal iff $c^T x = y^T Ax = y^T b$ which means $y^T(b - Ax) = (c^T - y^T A)x = 0$.
- This means, in each component of say $y^T(b - Ax)$ atleast one of y_j^T and $(b - Ax)_j$ $(j \in [n])$ is 0 and similarly for $(c^T - y^T A)x$. This is **complementary slackness**.

Appendix A

Strict Complementary Slackness

For the LP and dual LP defined earlier, $\exists x, y$, both optimal, that is, they satisfy complementary slackness conditions and

$$y + (b - Ax) > \mathbf{0}, \quad x + (y^T A - c^T) > \mathbf{0} \tag{24}$$

Appendix A

Proof:

Tucker's lemma for skew symmetric matrices (15) shows that for a skew symmetric matrix B, there is some z such that

$$z \geq \mathbf{0}, \;\; Bz \leq \mathbf{0}, \;\; z_k - (Bz)_k > 0$$

This applied to Dantzig's game gives a $z = (x', y', t')$, with $t' > 0$. $x = \frac{1}{t}x'$ and $y = \frac{1}{t}y'$ satisfy (24).

Appendix A

- The proof of strict complementary slackness demonstrates a very good use of Dantzig's game B.
- Geometrically, the LP solutions x and y are then in the relative interior of the set of optimal solutions.
- Unless this set is a singleton, x and y are not unique, but their supports $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are unique, shown similarly to the initial argument in the proof of Tucker's theorem from Gordan's theorem.

Appendix B

A zero sum game formed by a skew-symmetric matrix B has value 0 and the min-max strategy is the same as the max-min strategy. This comes from the fact that:

$$\begin{aligned}
&\min_v v, Bx \leq \mathbf{1}v, x \in X \\
=&\min_v v, -Bx \geq -\mathbf{1}v \\
=&\min_v v, B^T x \geq -\mathbf{1}v \\
=&\max_v -v, B^T x \geq \mathbf{1}(-v) \\
=&\max_u u, B^T y \geq \mathbf{1}u, y \in Y
\end{aligned}$$

Thus, we obtain $x = y$.
Then, the value of the game $u = x^T Bx = -x^T B^T x = -(x^T Bx)^T = -u$.
$\therefore\ u = 0$

Table of Contents

1 Preliminaries

2 Strong LP Duality proves Minimax Theorem for Zero Sum Games

3 Proof given by Dantzig

4 Proof Sequence

5 Strong LP Duality from Minimax Theorem of Zero Sum Games

6 Appendix

7 References

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[2] Tucker, A. W. (1956). Dual systems of homogeneous linear relations. In: H. W. Kuhn and A. W. Tucker (eds.), *Linear Inequalities and Related Systems*, volume 38 of *Annals of Mathematics Studies*, 1–18. Princeton University Press, Princeton, NJ.