Perfect Matching Polytope for general graphs and its Separation Oracle

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Introduction

The paper deals with the problem of finding linear polytopes for the convex hull of matching and perfect matching characteristic vectors for non-bipartite graphs. It gives a proof of Edmonds Theorem and then moves onto giving a proof of the matching problem. This is followed by providing a strongly polynomial time algorithm to determine whether a given characteristic vector satisfies the constraints of perfect matching. Lastly, the paper briefly delves on the problem of edge cover by relating it to the matching problem.

Definitions

Let $G(V, E)$ be a general graph.

1. **Characteristic Vector**
   For a set $E' \subseteq E$, the characteristic vector $\chi^{E'} \in \mathbb{R}^{|E|}$ is a vector such that $\chi^{E'}(e_i) = 1$ if the edge $e_i \in E'$, 0 otherwise. That is, the characteristic vector is a bit-wise encoding of a subset of edges of a graph in a vector form.
2. **Perfect Matching Polytope**
   A perfect matching polytope, wrt. a graph G, $P_{perfect\_matching}(G)$ is a convex combination of the perfect matching characteristic vectors of the given Graph.
   $P_{perfect\_matching}(G) = convexhull(\chi^M | M \text{ is a perfect matching in } G)$

Notation

The following notation will be used in this presentation:

- $\delta(U)$, where $U \subseteq V$, is the set of edges which have one endpoint in U and the other in $V - U$.
- $E(U)$, where $U \subseteq V$, is the set of edges which have both of their endpoints in U.
- $x(E')$, where x is a point in $\mathbb{R}^{|E|}$ and $E' \subseteq E$, is the sum of $x(e)$ for all $e \in E'$
- $\alpha_G(u, v)$, where $u, v \in V$, is defined to be the value of a minimum u, v cut in G.
- $c(\delta(U))$, where $U \subseteq V$, is the value of the cut formed by the cut-set $\delta(U)$.

Definition: A Tree $T = (V(G), E_T)$ is a Gomory-Hu tree if for all $st \in E_T$, $\delta(W)$ is a minimum s, t cut in G, where W is one component of $T - st$.

Theorem: Let T be a Gomory-Hu tree for a graph $G = (V, E)$. Then, for all $u, v \in V$, let st be the edge on the unique path in T from u to v such that $\alpha_G(s, t)$ is minimized. Then,

$$\alpha_G(u, v) = \alpha_G(s, t)$$

and the cut $\delta(W)$ induced by $T - st$ is a u, v minimum cut in G. Thus $\alpha_G(s, t) = \alpha_T(s, t)$ for each $s, t \in V$ where the capacity of an edge st in T is equal to $\alpha_G(s, t)$.

This theorem can be proved using the triangle inequality:

$$\alpha_G(a, b) \geq min(\alpha_G(a, c), \alpha_G(c, b))$$

Gomory-Hu Trees

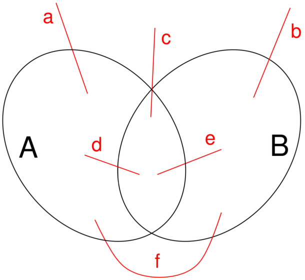
Two important properties of the cut function $[c(\delta(U))]$ **assuming non-negative edge weights**:

- **Property 1**:

$$c(\delta(A)) + c(\delta(B)) \geq c(\delta(A \cup B)) + c(\delta(A \cap B))$$

- **Property 2**:

$$c(\delta(A)) + c(\delta(B)) \geq c(\delta(A - B)) + c(\delta(B - A))$$

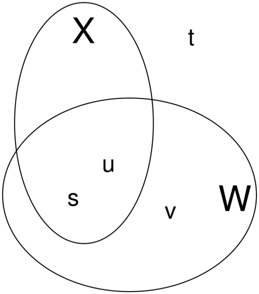


Key Lemma

Let $\delta(W)$ be an s, t minimum cut in a graph G with respect to a capacity/cut function c. Then for any $u, v \in W, u \neq v$, there is a u, v minimum cut $\delta(X)$ where $X \subseteq W$.

Gomory-Hu Trees (Key Lemma Proof)

To prove this, we will look at two cases.
Case 1: $t \notin X$



From Property 1:

$$c(\delta(X)) + c(\delta(W)) \geq c(\delta(X \cup W)) + c(\delta(X \cap W))$$

But, since $\delta(X)$ and $\delta(W)$ are min-cuts, the equality holds.

Case 2: $t \in X$
Using a figure similar to case 1, and using Property 2:

$$c(\delta(X)) + c(\delta(W)) \geq c(\delta(X - W)) + c(\delta(W - X))$$

But, since $\delta(X)$ and $\delta(W)$ are min-cuts, the equality holds.

Gomory-Hu Trees (Generalized)

Definition: Let $G = (V, E)$, $R \subseteq V$. Then a generalized Gomory-Hu tree for R in G is a pair consisting of $T = (R, E_T)$ and a partition $(C_r | r \in R)$ of V associated with each $r \in R$ such that:

1. For all $r \in R$, $r \in C_r$
2. For all $st \in E_T$, $T - st$ induces a minimum cut in G between s and t defined by:

$$\delta(U) = \bigcup_{r \in X} C_r$$

where X is the vertex set of a component of $T - st$.

Gomory-Hu Trees (Algorithm)

Using the definition of generalized Gomory-Hu Trees, we can recursively build a Gomory-Hu tree for a graph G.

Algorithm 1 GomoryHuAlg(G, R)

if $|R| = 1$ **then**
  return $T = (\{r\}, \varnothing), C_r = V$
else
  Let $r_1, r_2 \in R$, and let $\delta(W)$ be an r_1, r_2 minimum cut

  ⟨⟨*Create two subinstances of the problem*⟩⟩
  $G_1 = G$ with $V \setminus W$ shrunk to a single vertex, v_1; $R_1 = R \cap W$
  $G_2 = G$ with W shrunk to a single vertex, v_2; $R_2 = R \setminus W$

  ⟨⟨*Now we recurse*⟩⟩
  $T_1, (C_r^1 \mid r \in R_1) = \text{GomoryHuAlg}(G_1, R_1)$
  $T_2, (C_r^2 \mid r \in R_2) = \text{GomoryHuAlg}(G_2, R_2)$

  ⟨⟨*Note that r', r'' are not necessarily r_1, r_2!*⟩⟩
  Let r' be the vertex such that $v_1 \in C_{r'}^1$
  Let r'' be the vertex such that $v_2 \in C_{r''}^2$

  ⟨⟨*See figure 4*⟩⟩
  $T = (R_1 \cup R_2, E_{T_1} \cup E_{T_2} \cup \{rr'\})$
  $(C_r \mid r \in R) = \text{ComputePartitions}(R_1, R_2, C_r^1, C_r^2, r', r'')$
  return T, C_r
end if

Algorithm 2 ComputePartitions($R_1, R_2, C_r^1, c_r^2, r', r''$)

⟨⟨*We use the returned partitions, except we remove v_1 and v_2 from $C_{r'}$ and $C_{r''}$, respectively*⟩⟩
For $r \in R_1$, $r \neq r'$, $C_r = C_r^1$
For $r \in R_1$, $r \neq r''$, $C_r = C_r^2$
$C_{r'} = C_{r'}^1 - \{v_1\}, C_{r''} = C_{r''}^2 - \{v_2\}$
return $(C_r \mid r \in R)$

Edmonds Theorem

For bipartite graphs, the linear polytopes for the convex hull of matching and perfect matching comes out to be fairly straightforward. However, using the same results for non-bipartite graphs yield fractional solutions (for example, the K3 graph).
Thus, for non-bipartite graphs, the following system of constraints is proved to be correct, and is known as the Edmonds Theorem:

$$x(e) \geq 0 \ \forall e \in E$$

$$x(\delta(v)) = 1 \ \forall v \in V$$

$$x(\delta(U)) \geq 1 \ \forall U \subseteq V, |U| odd, |U| \geq 3$$

Proof of Edmonds Theorem

We will prove the Edmonds Theorem by contradiction.
Let $Q(G)$ be the polytope decribed by the inequalities of the Edmonds Theorem. It is easy to see that $P_{perfect\_matching}(G) \subseteq Q(G)$. Now, suppose there exists a graph G such that $Q(G) \not\subseteq P_{perfect\_matching}(G)$.

- Among all such graphs, choose the one which minimizes $|E| + |V|$.
- Choose a basic feasible solution (vertex) x of $Q(G)$ such that $x \notin P_{perfect\_matching}(G)$.
- For this solution, $x(e) \in (0,1)\ \forall e \in E$ which implies that $|E| \geq |V|$
- But, since x is not a Perfect Matching solution, $|E| > |V|$ and this implies that there exists some odd set $U \subseteq V$ such that $x(\delta(U)) = 1$ as x is chosen to be a vertex of the polytope.

Proof of Edmonds Theorem

Proof Continued:

- Next, we divide the vertices V, into two sets, U and $\bar{U} = V - U$ and then we create two graphs $G' = G/U$, where U is collapsed to a single vertex u', and $G'' = G/U'$, where U' is collapsed to a single vertex u''.

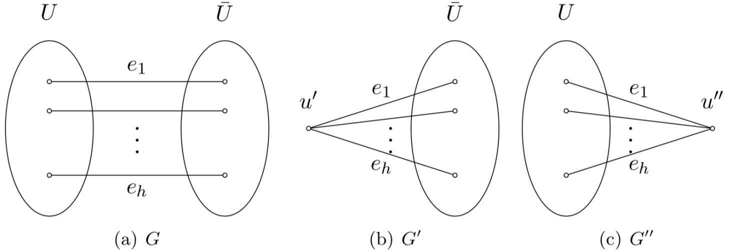


Figure: To show G, G' and G''

Proof Continued:

- The vector x when restricted to G' induces $x' \in Q(G')$ and, similarly, x induces $x'' \in Q(G'')$.
- From these x' and x'', which are perfect matching solutions because $|E| + |V|$ of G' and G'' are smaller than that of G, we show that x is also a perfect matching solution, which is a contradiction.

Separation Oracle

The inequality systems that we saw for perfect matching and matching have an exponential number of inequalities. Therefore, we cannot use them directly to solve the optimization problems of interest, namely, the maximum weight matching problem or the minimum weight perfect matching problem. Thus, we need to come up with a strongly polynomial time algorithm to determine whether a given characteristic vector satisfies Edmonds constraints.
The algorithm involves computing a Gomory-Hu tree for the given graph followed by computing the capacity of the minimum odd-cut induced by the edges of the tree.

Matching Polytope

Just as Edmonds Theorem gives a set of constraints for the perfect matching polytope, the following is the same for the matching polytope:

$$x(e) \geq 0 \; \forall e \in E$$

$$x(\delta(v)) \leq 1 \; \forall v \in V$$

$$x(E[U]) \leq \frac{|U| - 1}{2} \; \forall U \subseteq V, |U| odd$$

Proof of the Matching Polytope

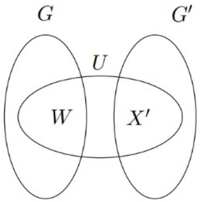
The proof employs constructing a graph $\tilde{G} = (\tilde{V}, \tilde{E})$ from G such that $\tilde{V} = V \cup V^{'}$ and $\tilde{E} = E \cup E^{'} \cup \{(v, v^{'}) | v \in V\}$, where $G^{'} = (V^{'}, E^{'})$ is a copy of G.
Now, its easy to see that χ^{M} satisfies the inequalities for every matching M. So all we need to show is that given a feasible solution x, it can be written as a convex combination of matchings in G.
So, we construct $\tilde{G}$ from G as explained above and define a fractional solution $\tilde{x} : \tilde{E} \rightarrow R^{+}$ as follows:

$$\tilde{x} = \begin{cases} x(e); \text{ if } e \in E \\ x'(e); \text{ if } e \in E' \\ 1 - x\left(\delta(v)\right); \text{ if } e = vv' \end{cases}$$

Proof contd

Note that every perfect matching in $\tilde{G}$ induces a matching in G and, thus, if we can prove that $\tilde{x} \in P_{perfect\_matching}(\tilde{G})$ then it follows that $x \in P_{matching}(G)$. Hence, all that remains is to show that

$$\tilde{x}(\tilde{\delta}(U)) \geq 1 \ \forall U \subseteq \tilde{V}, |U| odd$$



Now, if $X^{'} = \phi$ then $\mid W \mid$ is odd, and it follows that

Proof contd

$$
\begin{aligned}
\tilde{x}\left(\tilde{\delta}(U)\right) &= \tilde{x}\left(\tilde{\delta}(W)\right) \\
&= \sum_{v \in W} \tilde{x}\left(\tilde{\delta}(v)\right) - 2\tilde{x}\left(E[W]\right) \\
&= |W| - 2x\left(E[W]\right) \\
&\geq |W| - 2\left(\frac{|W|-1}{2}\right) \\
&\geq 1
\end{aligned}
$$

Thus, for the general case, one can see that $\tilde{x}(\tilde{\delta}(U)) \geq \tilde{x}(\tilde{\delta}(W \backslash X)) + \tilde{x}(\tilde{\delta}(X^{'} \backslash W^{'}))$. Note that one of $\mid W \backslash X \mid$ and $\mid X^{'} \backslash W^{'} \mid$ is always odd and thus we get that $\tilde{x}(\tilde{\delta}(U)) \geq 1$ (using the above result), proving the Edmonds Theorem.

Edge covers & Matchings

Given G = (V, E), an edge cover is the subset $E_0 \subseteq E$ such that each node is covered by some edge in E_0.
An important result in this domain is the Gallai Theorem, which states that the sum of cardinalities of the minimum size edge cover and the maximum matching equals the number of vertices of the graph, i.e
$\boldsymbol{\nu(G) + \rho(G) = \mid V \mid}$.
Note that a matching M covers $2 \mid M \mid$ nodes and for each uncovered node, we choose an edge arbitrarily. This leads to an edge cover of size $\leq\mid V \mid -2 \mid M \mid + \mid M \mid$. All that's left is to show that $\geq$ portion. Note that if v is not incident to an edge of M then since it is covered by E_0 (the minimal edge cover), there is an edge $e_v \in E_0 \setminus M$ that covers v; since M is maximal the other end point of e_v is covered by M.

Proof contd

This implies the fact that $2 \mid M \mid + \mid E_0 \backslash M \mid \geq \mid V \mid$, that is $2 \mid M \mid + \mid E_0 \mid - \mid M \mid \geq \mid V \mid$ and hence $\mid M \mid + \mid E_0 \mid \geq \mid V \mid$. Now consider E_0 as $\nu(G)$ and $\rho(G) \geq \mid M \mid$ as it is the maximal matching, this proves the Gallai Theorem.
Finally, the paper concludes by providing the edge cover polytope, which goes as follows:

$$1 \geq x(e) \geq 0 \; \forall e \in E$$

$$x(\delta(v)) \geq 1 \; \forall v \in V$$

$$x(E[U] \cup \delta(U)) \geq \frac{|U| + 1}{2} \; \forall U \subseteq V, |U| odd$$

QnA

The paper presented the Edmonds Theorem & the matching polytope followed by its separation oracle and the edge cover polytope. That was all we had to present. We would be happy to answer any questions you may have.