602 Applied Algorithms	**Jul-Nov 2023**

Midsem exam

Total Marks: 75 — *Time: 210 minutes*

Instructions.

- Please write your answers concisely.
- You can directly use anything proved in the class.

Que 1 [3+4+5 marks]. Consider the following LP.

$$\min \sum_{i=1}^{n} i \cdot x_i \text{ subject to}$$
$$x_i \geq 0 \quad \text{for each } 1 \leq i \leq n$$
$$x_i \leq 1 \quad \text{for each } 1 \leq i \leq n$$
$$\sum_{i=1}^{n} x_i = k$$

- Give an optimal solution and the optimal value for this LP (no need to argue that the solution is optimal, as the third part will guarantee that).

  Optimal solution $x_1 = 1, x_2 = 1, \ldots, x_k = 1, x_{k+1} = 0, \ldots, x_n = 0$.

  Optimal value $= \sum_{i=1}^{k} i = k(k+1)/2$.

- Write the dual LP for the above LP.

$$\max kz - \sum_{i=1}^{n} y_i \text{ subject to}$$
$$y_i \geq 0 \text{ for each } 1 \leq i \leq n$$
$$z - y_i \leq i \text{ for each } 1 \leq i \leq n$$

- Give a dual feasible solution such that the dual objective value is equal to the primal objective value from the first part.

  Dual feasible solution: $y_1 = k, y_2 = k-1, \ldots, y_k = 1, y_{k+1} = 0, y_{k+2} = 0, \ldots, y_n = 0$, $z = k+1$.

  Dual objective value: $k(k+1) - (k + (k-1) + \cdots + 1) = k(k+1)/2$.

Que 2 [5 + 5 marks] (Clustering). Suppose we are given n objects and their pairwise dissimilarities $\{d_{i,j} : 1 \leq i < j \leq n\}$. Dissimilarities are positive real numbers. We want to partition the objects into **two** clusters so as to maximize the total dissimilarity of pairs lying in different clusters. That is, maximize the following objective function

$$\sum_{\substack{i,j \\ \text{which are in} \\ \text{different clusters}}} d_{i,j}.$$

- Write an integer linear program (ILP) for this problem.

  Hint: You may take variables $y_i \in \{0,1\}$ to denote the cluster in which the ith object lies. You may take variables $x_{i,j} \in \{0,1\}$ to denote which pairs are in different clusters.

$$\begin{aligned} \max \sum_{i,j} d_{i,j} x_{i,j} & \text{ subject to} \\ y_i \in \{0,1\} & \text{ for each } 1 \le i \le n \\ x_{i,j} \in \{0,1\} & \text{ for each } 1 \le i < j \le n \\ x_{i,j} \le y_i + y_j & \text{ for each } 1 \le i < j \le n \\ x_{i,j} \le 2 - y_i - y_j & \text{ for each } 1 \le i < j \le n \end{aligned}$$

- Relax the integer condition to make it a linear program. Show an example, where the LP optimal is not equal to the maximum possible total dissimilarity.

  Hint: you may show that there is an feasible (fractional) solution for the LP that achieves objective value $\sum_{i,j} d_{i,j}$.

$$\begin{aligned} \max \sum_{i,j} d_{i,j} x_{i,j} & \text{ subject to} \\ 0 \le y_i \le 1 & \text{ for each } 1 \le i \le n \\ 0 \le x_{i,j} \le 1 & \text{ for each } 1 \le i < j \le n \\ x_{i,j} \le y_i + y_j & \text{ for each } 1 \le i < j \le n \\ x_{i,j} \le 2 - y_i - y_j & \text{ for each } 1 \le i < j \le n \end{aligned}$$

  Consider for example, $n = 3$. Suppose $d_{1,2} = 5, d_{2,3} = 5, d_{1,3} = 5$. Maximum possible total dissimilarity is 5+5=10. This is because some 2 objects have to be in the same cluster.

  A dual feasible solution: $y_i = 1/2$ for $1 \le i \le n$. And $x_{i,j} = 1$ for $1 \le i < j \le n$. Dual objective value= 5+5+5 = 15.

Que 3 [5 marks]. Let S be a set points in $\mathbb{R}^n$ and let $f(x) = w^T x$ be a linear function. Let conv (S) denote the convex hull of the set S (i.e., smallest convex set containing S). Then prove that

$$\max f(x) \text{ over } S = \max f(x) \text{ over } \operatorname{conv}(S).$$

Assuming S is finite. Let $S = \{s_1, s_2, \dots, s_k\}$. Let $c^* = \max\{w^T s_1, w^T s_2, \dots, w^T s_k\}$. Since S is contained in conv (S), we have

$$c^* \le \max f(x) \text{ over } \operatorname{conv}(S).$$

We need to prove the other inequality. Any point in $\alpha \in \operatorname{conv}(S)$ can be written as

$$\alpha = \lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_k s_k,$$

where $\lambda_i \ge 0$ and $\sum_i \lambda_i = 1$. We get

$$\begin{aligned} w^T \alpha &= \lambda_1 w^T s_1 + \lambda_2 w^T s_2 + \cdots + \lambda_k w^T s_k \\ &\le \lambda_1 c^* + \lambda_2 c^* + \cdots + \lambda_k c^*. \\ &= c^*. \end{aligned}$$

Hence, $\max f(x)$ over $\text{conv}(S) \le c^*$.

When we don't assume S to be finite, we can still write any point $\alpha \in \text{conv}(S)$ as a convex combination of finitely many points. This just follows from the definition of the convex hull. Rest of the proof will follow as it is.

Que 4 [3+8 marks]. Let A be a $k \times n$ matrix and $b \in \mathbb{R}^k$ such that the system $Ax \le b$ is feasible. Then prove that for any given $w \in \mathbb{R}^n$,

The optimal value of $\max w^T x$ subject to $Ax \le b$ is unbounded (i.e., $+\infty$)

if and only if

there exists $z \in \mathbb{R}^n$ such that $Az \le 0$ and $w^T z = 1$.

You can use any results proved in the class, for example, Farkas' Lemma, LP duality. One of the directions is easy, the other is not.

Let's write the dual LP of the given LP $\max w^T x$ subject to $Ax \le b$. It will be

$$\min b^T y \text{ subject to } A^T y = w, y \ge 0.$$

Recall that any feasible solution of the dual gives an upper bound on the primal optimal value (weak duality), i.e.,

$$w^T x = y^T A x \le y^T b = b^T y.$$

Since primal optimal is unbounded, it must be that the dual LP is not feasible. Also, the direction holds, if the primal optimal is bounded (we know primal is feasible), then the dual LP is feasible (strong duality). Now, recall that Farkas' lemma states that the system $A^T y = w, y \ge 0$ is infeasible if and only if there exists z such that

$$z^T A^T \ge 0, z^T w = -1.$$

This is equivalent to

$$Az \le 0, w^T z = 1.$$

We had not proved this form of Farkas' lemma, but it is ok to use it directly.

A proof of the above form of Farkas' lemma. Not expected in the solution.

We proved the below in the class.

$$Ax \le b \text{ is not feasible} \iff y \ge 0, y^T A = 0, y^T b = -1 \text{ is feasible.}$$

We can write $A^T y = w, y \ge 0$ in an equivalent way as

$$A^T y \le w, A^T y \ge w, y \ge 0.$$

Let us define

$$B = \begin{pmatrix} A^T \\ -A^T \\ -I \end{pmatrix} \text{ and } c = \begin{pmatrix} w \\ -w \\ 0 \end{pmatrix}$$

Then the above is equivalent to $By \le c$. From the form of Farkas' lemma we proved, we get $By \le c$ is infeasible if and only if the following is feasible

$$z \ge 0, z^T B = 0, z^T c = -1.$$

Let $z = (z_1, z_2, z_3)$ where z_1, z_2, z_3 are in appropriate dimensions. Above is equivalent to

$$z_1, z_2, z_3 \ge 0, z_1^T A^T - z_2^T A^T - z_3 = 0, z_1^T w - z_2^T w = -1.$$

Let $z' = z_1 - z_2$. Then the above is equivalent to

$$Az' = z_3, z_3 \ge 0, z'^T w = -1.$$

This is equivalent to

$$Az' \ge 0, z'^T w = -1.$$

Que 5 (5+3+5). Let Δ_n be the n-dimensional probability simplex, i.e.,

$$\Delta_n = \{\lambda \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for each } 1 \leq i \leq n\}.$$

Let R be a $m \times n$ real matrix. Let $R_{i,j}$ be the (i,j)-th entry of R. Define

$$h(y) := \min_{x \in \Delta_n} y^T R x$$

- Prove that
$$h(y) = \min \left\{ \sum_{i=1}^{m} y_i R_{i,1}, \sum_{i=1}^{m} y_i R_{i,2}, \ldots, \sum_{i=1}^{m} y_i R_{i,n} \right\}.$$

  For any $1 \leq j \leq n$, let $\delta^j := (0, 0, \ldots, 1, \ldots, 0, 0)$ where 1 is at jth position. Clearly, $\delta^j \in \Delta_n$. Hence, by definition
$$h(y) \leq y^T R \delta^j = \sum_{i=1}^{m} y_i R_{i,j}.$$

  Let us prove the other direction. Let $\min\{\sum_{i=1}^{m} y_i R_{i,j} : 1 \leq j \leq n\}$ be c^*. We can write
$$\begin{aligned} y^T R x &= \sum_j x_j \sum_i y_i R_{i,j} \\ &\geq \sum_j x_j c^* \qquad (\text{since } x_j \geq 0) \\ &\geq c^* \qquad (\text{since } \sum_j x_j = 1) \end{aligned}$$

- Argue that $\max_{y \in \Delta_m} h(y)$ is same as the optimal value of the below LP.

$$\begin{aligned} &\max z \\ &\text{subject to} \\ &\sum_{i=1}^{m} y_i R_{i,1} \geq z \\ &\sum_{i=1}^{m} y_i R_{i,2} \geq z \\ &\qquad \vdots \\ &\sum_{i=1}^{m} y_i R_{i,n} \geq z \\ &\sum_{i=1}^{m} y_i = 1 \\ &y_i \geq 0 \text{ for each } 1 \leq i \leq m. \end{aligned}$$

  Let the LP optimal value be z^*. For any $y \in \Delta_n$, we know that $h(y) \leq \sum_{i=1}^{m} y_i R_{i,j}$ for any j. Hence, we get that $(y, z = h(y))$ is a feasible solution for the LP for any $y \in \Delta_n$. Thus, $z^* \geq h(y)$ for all $y \in \Delta_n$.

Let (y^*, z^*) be an optimal solution for the LP. Then,

$$h(y^*) = \min_j \{\sum_{i=1}^{m} y_i^* R_{i,j}\} \geq z^*.$$

Thus,

$$\max_{y \in \Delta_m} h(y) = z^*.$$

- Write the dual LP for the above LP. Show that the dual optimal value is same as

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} y^T R x.$$

Writing the LP in one of the standard forms.

$$\begin{aligned} &\max z \\ &\text{subject to} \\ &z - \sum_{i=1}^{m} y_i R_{i,1} \leq 0 \\ &z - \sum_{i=1}^{m} y_i R_{i,2} \leq 0 \\ &\vdots \\ &z - \sum_{i=1}^{m} y_i R_{i,n} \leq 0 \\ &\sum_{i=1}^{m} y_i = 1 \\ &y_i \geq 0 \text{ for each } 1 \leq i \leq m. \end{aligned}$$

The dual LP.

$$\begin{aligned} &\min w \\ &\text{subject to} \\ &w - \sum_{j=1}^{n} x_j R_{1,j} \geq 0 \\ &w - \sum_{j=1}^{n} x_j R_{2,j} \geq 0 \\ &\vdots \\ &w - \sum_{j=1}^{n} x_j R_{m,j} \geq 0 \\ &\sum_{j=1}^{n} x_j = 1 \\ &x_j \geq 0 \text{ for each } 1 \leq j \leq n. \end{aligned}$$

This can be rearranged to get

$$\begin{aligned}
&\min w \\
&\text{subject to} \\
&\sum_{j=1}^{n} x_j R_{1,j} \leq w \\
&\sum_{j=1}^{n} x_j R_{2,j} \leq w \\
&\vdots \\
&\sum_{j=1}^{n} x_j R_{m,j} \leq w \\
&\sum_{j=1}^{n} x_j = 1 \\
&x_j \geq 0 \text{ for each } 1 \leq j \leq n.
\end{aligned}$$

Let us define

$$g(x) := \max_i \{\sum_{j=1}^{n} x_j R_{i,j}\}.$$

As argued in part 1, $g(x)$ can be written as

$$g(x) = \max_{y \in \Delta_m} \{\sum_{i=1}^{m} y_i \sum_{j=1}^{n} x_j R_{i,j}\} = \max_{y \in \Delta_m} y^T R x.$$

Again as argued in part 1, the optimal value for the above program is equal to

$$\min_{x \in \Delta_n} g(x) = \min_{x \in \Delta_n} \max_{y \in \Delta_m} y^T R x.$$

Que 6 [3+3 +5] (Hitting-set). We have n objects, $V = \{v_1, v_2, \ldots, v_n\}$, each with a cost, say $c_1, c_2, \ldots, c_n$. We are given some target subsets $T_1, T_2, \ldots, T_k \subseteq V$ each with 2 or 3 objects.

Example. $V = \{v_1, v_2, v_3, v_4, v_5\}$ with costs $\{20, 15, 10, 10, 20\}$. $T_1 = \{v_1, v_4\}, T_2 = \{v_2, v_4, v_5\}, T_3 = \{v_1, v_2, v_3\}$.

The goal is to select a *hitting-set* $H \subseteq V$ with minimum total cost, such that H contains at least one object from every target subset T_i. In the above example, $H = \{v_1, v_2\}$ is a hitting set with cost 35 and $H = \{v_3, v_4\}$ is a hitting-set with cost 20.

- Write the natural LP for the minimum cost hitting-set, with primal variables $x_1, x_2, \ldots, x_n$ and k constraints.

$$\begin{aligned}
\min \sum_{i=1}^{n} c_i x_i \text{ subject to} \\
x_i \geq 0 \text{ for } 1 \leq i \leq n \\
\sum_{i \in T_j} x_i \geq 1 \text{ for } 1 \leq j \leq k
\end{aligned}$$

- Write the dual LP with variables $y_1, y_2, \ldots, y_k$.

$$\max \sum_{i=1}^{k} y_j \text{ subject to}$$
$$y_j \geq 0 \text{ for } 1 \leq j \leq k$$
$$\sum_{j:i \in T_j} y_j \leq c_i \text{ for } 1 \leq i \leq n$$

Consider the following primal-dual algorithm.

1. Initialize H as empty set.
2. Consider any target subset T_j, from which we have no object in H.
3. Increase the dual variable y_j till the point that the dual constraint for one of the objects in T_j becomes tight.
4. Whichever object becomes tight, put it in H (if multiple objects become tight, put all of them).
5. If there is any target subset T_j for which we have no object in H, go to 2. Otherwise return H.

- Prove that the hitting-set output by the algorithm has cost at most 3 times the optimal cost.

The cost of the hitting-set H output by the algorithm is $c(H) = \sum_{i \in H} c_i$. Note that an object i is taken into H only when the ith dual constraint becomes tight. Hence, for every $i \in H$, we have

$$c_i = \sum_{j:i \in T_j} y_j.$$

The cost of H will be

$$c(H) = \sum_{i \in H} \sum_{j:i \in T_j} y_j = \sum_{j=1}^{k} \sum_{\substack{i \in H, \\ i \in T_j}} y_j.$$

Now, recall that there are at most three objects in any T_j, and thus, $\sum_{\substack{i \in H, \\ i \in T_j}} y_j \leq 3 y_j$. We get,

$$c(H) \leq 3 \sum_{j=1}^{k} y_j$$

But, we know that the objective value for any dual solution is at most the primal LP optimal value, which in turn is at most the cost of the minimum hitting-set. Thus, we have $c(H) \leq 3\times$ cost of optimal hitting-set.

Que 7 [5+3+5] (Disjoint paths). Given a directed graph $G(V, E)$, with a source vertex s and a destination vertex t. We want to find out the maximum number of edge-disjoint paths from s to t. We write the following linear program.

$$\begin{aligned}
&\max p && \text{subject to} \\
&0 \leq x_e \leq 1 && \text{for each } e \in E \\
&-\sum_{e \in \text{in}(v)} x_e + \sum_{e \in \text{out}(v)} x_e = 0 && \text{for each } v \neq s, t. \\
&p - \sum_{e \in \text{in}(t)} x_e = 0 && \\
&\sum_{e \in \text{out}(s)} x_e - p = 0 &&
\end{aligned}$$

- Write the dual LP.

$$\begin{array}{ll} \min \sum_{e\in E} z_e & \text{subject to} \\ z_e \geq 0 & \text{for each } e \in E \\ z_e + y_a - y_b \geq 0 & \text{for each } e = (a,b) \in E \\ y_t - y_s = 1 & \end{array}$$

- Show that every s-t cut corresponds to a feasible solution of the dual. An s-t cut is a subset of vertices S, which contains s and does not contain t.

  For any s-t cut S containing s and not t, we construct the following dual solution.

  $$\begin{aligned} y_v &= 0 \text{ for } v \in S, \\ y_v &= 1 \text{ for } v \notin S, \\ z_e &= 1 \text{ for } e = (a,b) \in E \text{ such that } a \in S, b \notin S \text{ (cut edge)}. \end{aligned}$$

  It is straightforward to verify feasibility.

- The size of an s-t cut is the number of edges going from S to outside S. Argue that the dual optimal value is at most the minimum size of an s-t cut.

  For any s-t cut, consider the above dual feasible solution. The objective value for the above solution will be precisely the number of edges going from S to outside S. This is true, in particular, for the minimum size s-t cut. Hence, dual LP optimal value can be at most minimum size of an s-t cut.