1 Extending Augmenting Path Algorithm to General Graphs

Alternating path algorithms don’t work for general graph as the transitive path property does not hold. On the other hand, finding an alternating walk is easy.

**Definition 6.1 (M-Flower).** For a Matching $M$ in a Graph $G$ an M-Flower is sequence of nodes $(v_0, v_1, ... v_t)$ where $v_0$ is unmatched and $v_0...v_t$ are distinct, $v_t = v_i$ for some even $i$, odd $t$ and $v_0...v_t$ is an $M$-alternating walk $W$. Additionally the sequence of nodes $(v_i...v_t)$ is said to be an M-Blossom.
**Definition 6.2 (Shrinking Blossom).** Let $B$ be an $M$-Blossom $(v_i...v_t)$ in $G/B$ a graph with vertex set $V \setminus \{v_{i+1}...v_t\}$. Any edge $(v_k, u), \forall i + 1 \leq k \leq t - 1$ is replaced by $(v_i, u)$.

**Theorem 6.3.** $M$ is a maximum matching in $G \iff M/B$ is a maximum matching in $G/B$.

*Proof.* $\Rightarrow$ Suppose $M$ is a maximum matching in $G$ but $M/B$ is not a maximum matching in $G/B$.

$\exists$ an $M/B$-Augmenting Path in $G/B$ for $M/B$. From this Augmenting Path we can build an $M$-Augmenting Path in $G$ as follows.

In $G/B$, $P = (v_0, u_1...u_j)$. If $P \cap B = \phi$ $\Rightarrow$ $P$ is an $M$-Augmenting Path of $G$.

If $P \cap B \neq \phi$ and $P$ uses an unmatched edge $(B, u)$, consider its image $(v_k, u)$ in $G, \forall i \leq k \leq t - 1$. Combine the $M$-Alternating Path $(v_i, v_k)$ with $P$, giving a new $M$-Augmenting Path, going around the Blossom clockwise for even $k$ and counter-clockwise for odd $k$. 
Suppose $M$ is a not maximum matching in $G$ but $M/B$ is a maximum matching in $G/B$.

∃ an $M$-Augmenting Path in $G$ for $M$. With no loss of generality assume $v_0 = v_i$ for $B$.

If the $M$-Augmenting Path in $G$ does not use nodes of $B$ then it is also an $M/B$-Augmenting Path in $G/B$.

Otherwise, we can get a matching of same size by switching the matching edges on path $(v_0...v_i)$.

Consider $M$-Augmenting Path $(u_0,u_1...u_k)$ in $G$. Both $u_0$ and $u_k$ are unmatched. Only one node can be unmatched in $B$, so only one of them $\in B$. Assume $u_0 \notin B$ with no loss of generality.

Let $j > 0, u_j \in B$ be the smallest such $j$. Then $(u_0,u_1...u_j,B)$ is an $M/B$-Augmenting Path in $G/B$, by the definition of $G/B$. This contradicts the fact that $M/B$ is a maximum matching in $G/B$.

**Definition 6.4** (Auxiliary Graph). For a graph $G \equiv (V,E)$ and a matching $M(\subseteq E)$ in $G$, the Auxiliary Graph $G' \equiv (V,E')$ is a directed graph where $E' \equiv \{(u,w) | \exists v \in V, (u,v) \notin M \land (v,w) \in M\}$

Let $U \equiv \{\text{the set of vertices of } G \text{ unmatched in } M\}$.

Let $N(U)$ be the neighbourhood of $U$ in $G$.

Let $P'$ be the shortest path in $G'$ from $U$ to $N(U)$. $P'$ corresponds to a $M$-alternating walk $P$ in $G$.

If $P$ is not a simple path, let $P$ be $(v_0...v_t)$. Let $j$ be the minimum such that $v_j = v_i, i < j$.

**Claim 6.5.** $(v_i...v_j)$ is a Blossom

**Claim 6.6.** $j - i$ is odd

**Proof.** If $j - i$ is even, then we get a node that is matched twice if $j$ is even and contradicts the minimality of $P$ if $j$ is odd.

\[ \square \]
Claim 6.7. \((v_i, v_{i+1})\) is unmatched

Proof. If not then \(v_{j-1} = v_{i+1}\) which contradicts the minimality of \(j\).

Algorithm 1:

Input: a matching \(M\) in graph \(G\)
Output: \(M\)-augmenting path if \(M\) is not a maximum matching

1. Construct auxiliary directed graph \(G'\) of \(G\) and \(M\)
2. Find shortest path \(P'\) in \(G'\) from \(U\) to \(N(U)\).
3. if \(P\) is a simple path then
   4. Return \(P\) (\(P\) is an \(M\)-Augmenting Path in \(G\))
5. else The corresponding walk \(P\) in \(G\) contains a Blossom \(B\)
6. Recurse on \(G/B\).

1.1 Running Time

Number of Augmentations \(\leq n/2\)

Number of Shrinkings \(\leq n/2 - 1\)

Constructing \(G'\) and finding shortest path = \(O(m)\)

Total Running Time = \(O(n^2m)\)

State of the Art best is \(O(m\sqrt{n})\)
Red blue alternating path - Let $G$ be an undirected graph with red/blue edges. Given 2 nodes $s$ and $t$, find a path from $s$ to $t$ with a red-blue alternating path.

→ easy for bipartite graphs → no odd cycles → reduces to perfect matching

→ directed graphs → NP-Hard

HW: Show that for undirected general graphs, the Red blue alternating path problem reduces to the Perfect matching problem.