CS760 Topics in Computational Complexity **Jul-Nov 2024**

Assignment 1

Total Marks: 100 *Deadline: Sep 6, Friday, 5 pm*

Note: Please write your answers precisely and succinctly. You are not supposed to discuss the problems with anyone else. If you need hints/clarifications, ask on Piazza or in the class.

1. (5 marks) Prove that the following problem is undecidable. Given two C++ programs, whether they have the same input output behavior (that is, on any input, one program halts if and only if the other one halts, and moreover their outputs are same).

2. (5 marks) Consider an infinite Turing Machine (ITM), which same as a TM except that the set of states Q is infinite and the number of accepting states is finite. Argue that ITM is not a reasonable model of computation. One way to argue can be to show an undecidable language that is accepted by ITM.

3. (5 marks) Prove that for a system of linear equation, if there is a solution then there is solution whose size is polynomial in the input size. Hint: formula for matrix inverse and determinant

4. (10 marks) Consider a decision version of the discrete log problem, given numbers a, b, c and prime p, is there a number $1 \leq r \leq c$ such that $a^r \equiv b \pmod{p}$? Prove that this problem is in NP$\cap$coNP. Note that there may be more than one values of r that satisfy the given equation.

5. (10 marks) Let us define the class exponential time as

   $$\text{EXP} = \cup_{c \geq 1} \text{DTIME}(2^{n^c}).$$

   A language L is called EXP-complete if L is in EXP and every language in EXP can be reduced to L in polynomial time. Argue that the following language is EXP-complete.

   Input: a string $x \in \{0,1\}^*$ and an encoding of a Boolean circuit C with $|x|$ input gates.

   The encoding of the Boolean circuit is not explicit. It is represented by two functions. Function $f(i, j)$ says whether ith gate has a wire coming in from jth gate. Function $g(i)$ says whether ith gate is an AND, OR, or NOT gate. The functions f and g are given as two Boolean circuits.

   Output: the output of encoded circuit C on input string x.

6. (7 marks) Reduce problem 1 to problem 2 in polynomial time.

   Problem 1: Given a directed graph, two vertices s, t, and a number k, is there a path from s to t of length at least k?

   Problem 2: Given an udirected graph, two vertices s, t, and a number k, is there a path between s and t of length at least k?

7. (8 marks) We are given a Boolean formula $\psi(x, y)$ in two sets of Boolean variables $x = \{x_1, x_2, \ldots, x_n\}$ and $y = \{y_1, y_2, \ldots, y_m\}$. For a given assignment to the variables, let $\#\text{clause}(\psi(\cdot))$ denote the number of clauses satisfied by the assignment. We want to find

   $$\min_{x \in \{T,F\}^n} \max_{y \in \{T,F\}^m} \#\text{clause}(\psi(x, y)).$$

   Find an appropriate decision problem for this optimization problem. Can you say if the decision problem is in NP, or in coNP? Give an explanation.