CS760 Topics in Computational Complexity **2024-25 Sem I**

Lecture 11: 06-09-2024

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Complexity Zoo - Wiki with almost all (over 500) complexity classes listed.
HW: Prove 2-SAT is in NL

Different problems in Polynomial hierarchy

Exact Independent set Problem: Given a graph, and a number k, is the max independent set size $= k$?

Min DNF problem: Given a DNF (OR of ANDs) ψ and k, is there another DNF ϕ equivalent to ψ and size(ϕ) $\leq k$?

Succinct Tournament Reachability Problem:
Tournament: directed graph - $\forall\ i, j$, exactly one of (i, j) and (j, i) is an edge.
This problem can be represented in this way:
Given a Boolean formula ψ on $2n$ variables,
$\psi(i, j) = 1$ if $i \to j$
$\psi(i, j) = 0$ if $j \to i$
Given s, t, is t reachable from s?

Polynomial Hierarchy

First let us define Σ_2^p.
A language L is in Σ_2^p if there is poly-time TM and a polynomial q s.t.:
$x \in L$ if and only if $\exists u_i \in \{0,1\}^{q(|x|)}$, $\forall v_i \in \{0,1\}^{q(|x|)}$, $M(x, u, v) = 1$

Talking about Exact Independent set Problem:
M takes two subsets u, v
- check whether $|u| = k$, and u is independent
- and if $|v| > k$ and should not be independent.
If both of the above are satisfied then M outputs 1.

The **Exact Independent set Problem** is in Σ_2^p.
The **Min DNF problem** is also in Σ_2^p.

Next, let us define:

$$\Pi_2^p = \{\{0,1\}^* \setminus L : L \in \Sigma_2^p\}$$

The **Succinct Tournament Reachability Problem** is in Π_2^p (not easy to see).

More classes in polynomial hierarchy:
We say that a language L is in Σ_i^p if there exists a polynomial time TM and a polynomial q such that
$x \in$ L if and only if $\exists u_1 \in \{0,1\}^{q(|x|)}\ \forall u_2 \in \{0,1\}^{q(|x|)}\ \exists u_3\ \forall u_4 \cdots Q_i u_i\ M(x, u_1, u2, \ldots, u_i) = 1$.
Here Q_i is either $\exists$ or $\forall$ depending on parity of i.
Observe that
- $\Sigma_1^p =$ NP
- $\Pi_1^p =$ coNP
- $\Sigma_i^p \subseteq \Sigma_{i+1}^p$

Polynomial Hierarchy (PH) Definition.:

$$PH = \bigcup_{i \geq 1} \Sigma_i^p \subseteq PSPACE$$

Note that a similar union of classes Π_i^p will also give us PH, because $\Sigma_i^p \subseteq \Pi_{i+1}^p \subseteq \Sigma_{i+2}^p$.

Collapse of Polynomial Hierarchy (PH)

We say that PH collapses to i^{th} level if for some i,:

$$PH = \Sigma_i^p = \Pi_i^p$$

Theorem: If $\Sigma_{i+1}^p = \Sigma_i^p$ then PH = Σ_i^p

Theorem: $\Sigma_i^p = \Pi_i^p$ iff PH = $\Sigma_i^p = \Pi_i^p$

Theorem: P = NP $\implies$ PH = P

Q) Given a graph G, is it true that for any given list of k colors $L(v)$ for each vertex v, it is possible to list-color s.t. every maximal clique is not monochromatic?

This can be written as: ∀ Lists ∃ coloring ∀ cliques.
This is in Π_3^p-complete (Karp reduction, many-one).

Σ_2^pcomplete example:
$\psi_{x,y}$ is a 3-CNF
$\exists x \in \{0,1\}^n \forall y \in \{0,1\}^n \ \psi_{x,y}$

Theorem: P = NP $\implies$ PH = P
Proof: Assume P = NP.

We will show that Σ_i^p = P for every $i \geq 1$, using induction on i.
Base case: $i = 1$. Then Σ_i^p = NP = P.
Induction hypothesis: assume Σ_{i-1}^p = P.
Induction step: We will show Σ_i^p = P. Take $L \in \Sigma_i^p$. Then there is a poly-time TM M such that

$$x \in L \text{ if and only if } \exists u_1 \forall u_2 \exists u_3 \cdots Q_i u_i \ M(u_1, u_2, \dot{;} u_i) = 1.$$

Let us define another language L' as follows.

$$L' = \{(x, u_1) : \forall u_2 \exists u_3 \cdots Q_i u_i M(x, u_1, u_2, u_3, \ldots, u_i) = 1\}.$$

Observe that $L' \in \Pi_{i-1}^p = \Sigma_{i-1}^p$ = P. Thus there must be a poly-time TM M' such that

$$(x, u_1) \in L \text{ if and only if } M'(x, u_1) = 1.$$

Hence, we can write $L = \{x : \exists u_1 M'(x, u_1) = 1\}$. This means that $L \in$ NP = P.

Oracle Definition

NP $\subseteq$ NP$^{\text{SAT}}$
~~$\Pi_2^p \subseteq$ NP$^{\text{SAT}}$?~~ **Wrong**

$\Sigma_2^p \subseteq \mathrm{NP}^{\mathrm{SAT}}$

- $\exists u_1 \forall u_2 \; M(x, u_1, u_2) = 1$.

- We can non-deterministically guess u_1 and we can use the SAT oracle to test whether $\forall u_2 \; M(x, u_1, u_2) = 1$.

  - $\Sigma_2^p \subseteq \mathrm{NP}^{\mathrm{SAT}} \subseteq \Sigma_2^p$ (homework).

- $\Sigma_i^p = \mathrm{NP}^{\Sigma_{i-1}^p} = \mathrm{NP}^{\Pi_{i-1}^p}$.

- $\Pi_i^p = \mathrm{coNP}^{\Sigma_{i-1}^p} = \mathrm{coNP}^{\Pi_{i-1}^p}$.

We know that SAT can be solved in $O(2^n)$ time. Can we prove that SAT cannot be solved in linear time? Can we prove SAT cannot be solved in logspace? Both these questions are open. But using hierarchy theorems, we can show that

Theorem: SAT cannot be solved by a TM that takes time $O(n^{1.1})$ and space $O(n^{0.1})$.