

## Lecture 9: 05-09-2022

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**Definition 9.1.** A graph  $G$  is said to be an  $(n, d, \lambda)$ -expander if

- $|V(G)| = n$ ,
- $G$  is  $d$ -regular, and
- $\lambda$  is the second largest eigenvalue  $\lambda(G)$  of  $G$  in absolute value.

**Definition 9.2** (Spectral expanders). A sequence  $\{G_n\}_{n \geq 0}$  of  $d$ -regular graphs is said to be a spectral expander family if for some  $\lambda < 1$ ,  $\lambda(G_n) \leq \lambda$  for all  $n$ .

We saw in last lecture that random walks on expander graphs converge to the uniform distribution in  $O(\log n)$  steps.

This means that there are only  $d^{O(\log n)} = \text{poly}(n)$  paths to explore. Therefore, there is a *deterministic* polynomial time algorithm to determine connectivity of expander graphs.

**Definition 9.3** (Edge expansion). Given a  $d$ -regular graph  $G$ , define the normalized edge expansion

$$h(G) = \min_{\substack{S \subseteq V \\ |S| \leq (n/2)}} \frac{|E(S, \bar{S})|}{d|S|}.$$

It is evident that  $h(G) \leq 1$ . If skip the  $d$  in the denominator, that is just known as the edge expansion.

It is natural to see that  $h(G)$  measures (in some sense) how well-connected a graph is. If it is low, there is some “bottleneck” in the graph where the random walk can get stuck – a set of high measure with very few outgoing edges.

**Definition 9.4** (Combinatorial expanders). A sequence  $\{G_n\}_{n \geq 0}$  of  $d$ -regular graphs is said to be a combinatorial expander family if for some  $h > 0$ ,  $h(G_n) \geq h$  for all  $n$ .

**Theorem 9.5** (Cheeger’s Inequality). For any graph  $G$  with second eigenvalue  $\lambda_2$  and sparsity  $h$ ,

$$\frac{1 - \lambda_2}{2} \leq h \leq \sqrt{2(1 - \lambda_2)}.$$

In particular, spectral expanders are combinatorial expanders and vice-versa.

Markov chain Monte Carlo methods find many uses in problems such as sampling random spanning trees, random independent sets etc. The idea in these is that we start with an arbitrary spanning tree (say), and then randomly move to a “neighbouring” spanning tree – add a random edge not in the spanning tree and remove a random edge from the cycle thus formed. After sufficiently many steps, we are at a(n almost) uniformly random spanning tree. This massive graph composed of spanning trees as vertices ends up being an expander! Because the graph of spanning trees has only exponentially many vertices, we get a polynomial time algorithm to randomly sample spanning trees.

Cheeger’s inequality is also seen to be asymptotically tight by the following examples:

- The  $n$ -cycle  $C_n$ . It has edge expansion  $h(C_n) = 2/n$ , and  $\lambda(C_n) = \cos(2\pi/n) \approx 1 - (2\pi/n)^2$ .
- The hypercube graph  $H_n$ , with vertex set  $\{0, 1\}^n$  and edge between two vertices iff they differ at precisely one coordinate. It has sparsity  $h(H_n) = 1/n$ , and  $\lambda(H_n) = 1 - \frac{1}{n}$ .

What guarantee do we even have that expanders exist? It turns out that a random  $d$ -regular graph is a (combinatorial) expander with high probability! The probabilistic argument is similar to what we had seen for showing existence of magical graphs.

However, how do we construct expander graphs? Our goal is to use expander graphs to reduce randomness in algorithms, so it does not make sense to construct them using the above random argument. In some applications, we also want the algorithm itself to run in polylog time – this requirement makes sense in light of our remarks towards the end of Lecture 6. An strongly explicit construction of expander graphs would be one, where given a vertex and an index  $i$ , we should be able to find the  $i$ th neighbor of the vertex in time  $\text{poly}(\log n)$ .

**Example.** Let  $p$  be a prime and consider the 3-regular graph over  $\mathbb{F}_p$ , where each  $x$  is adjacent to  $x + 1, x - 1, x^{-1}$  (define  $0^{-1}$  as 0). This graph is known to be an expander; the proof goes via some results in number theory.

**Theorem 9.6** (Expander Mixing Lemma). *Let  $G$  be a  $(n, d, \lambda)$ -expander. Then, for any  $S, T \subseteq V$ ,*

$$\left| E(S, T) - \frac{d}{n} |S| |T| \right| \leq d \lambda \sqrt{|S| |T|}.$$

If  $G$  were a random graph, then the expected number of edges between  $S, T$  is precisely  $(d/n)|S||T|$  – of the  $d|S|$  edges out of  $S$ , we expect a  $|T|/n$  fraction to be incident on  $T$ .

*Proof.* Let  $M$  be the transition matrix of the random walk on  $G$ ; it is equal to  $(1/d)$  times the adjacency matrix of  $G$ . For any set  $X$ , let  $\mathbf{1}_X$  be the indicator vector of  $X$ , with 1s at vertices in  $X$  and 0 elsewhere. Observe that

$$\frac{1}{d} E(S, T) = \mathbf{1}_S^\top M \mathbf{1}_T.$$

Now, we have  $M = \sum_i \lambda_i u_i u_i^\top$  using the spectral theorem, where  $(u_i)_{i=1}^n$  are orthonormal eigenvectors of  $M$  with corresponding real eigenvalues  $(\lambda_i)_{i=1}^n$ . Note in particular that  $\lambda_1 = 1$  and  $u_1$  is the vector with all coordinates having value  $1/\sqrt{n}$ .

Let  $\mathbf{1}_S = \sum_i \alpha_i u_i$  and  $\mathbf{1}_T = \sum_i \beta_i u_i$ . It is seen that  $\alpha_1 = \langle \mathbf{1}_S, u_1 \rangle = |S|/\sqrt{n}$  and  $\beta_1 = |T|/\sqrt{n}$ . Using orthonormality,

$$\begin{aligned} \frac{1}{d} E(S, T) &= \left( \sum_i \alpha_i u_i^\top \right) \left( \sum_i \lambda_i u_i u_i^\top \right) \left( \sum_i \beta_i u_i \right) \\ &= \left( \sum_i \alpha_i u_i^\top \right) \left( \sum_i \beta_i \lambda_i u_i \right) \\ &= \sum_i \alpha_i \beta_i \lambda_i \\ &= \alpha_1 \beta_1 + \sum_{i=2}^n \alpha_i \beta_i \lambda_i \\ &= \frac{|S| |T|}{n} + \sum_{i=2}^n \alpha_i \beta_i \lambda_i. \end{aligned}$$

Therefore,

$$\begin{aligned}
\left| E(S, T) - \frac{d}{n} |S| |T| \right| &= d \left| \sum_{i=2}^n \alpha_i \beta_i \lambda_i \right| \\
&\leq d \lambda \sum_{i=2}^n |\alpha_i \beta_i| \\
&\leq d \lambda \sqrt{\left( \sum_{i=2}^n \alpha_i^2 \right) \left( \sum_{i=2}^n \beta_i^2 \right)} \\
&\leq d \lambda \sqrt{\|\mathbf{1}_S\| \|\mathbf{1}_T\|} = d \lambda \sqrt{|S| |T|}. \quad \square
\end{aligned}$$

We shall now see how to save randomness using expanders.

Let  $\mathcal{A}$  be an algorithm that uses  $k$  independent random bits. Let  $G$  be a  $(2^k, d, \lambda)$ -expander. Starting at an arbitrary vertex  $v_1$ , perform a random walk for  $\ell$  steps through vertices  $v_1, v_2, \dots, v_\ell$ . Run the algorithm on each of these inputs  $v_1, \dots, v_\ell$  (interpreting the  $2^k$  elements of  $V(G)$  as length  $k$  bit strings).

Recall that if running  $\mathcal{A}$  once (using  $k$  bits) has error probability  $\beta$ , running the algorithm  $\ell$  times (using  $k\ell$  bits) reduces this to an error probability of  $\beta^\ell$ . It turns out that running the algorithm  $\ell$  times in the manner described above (using  $O(k + \ell \log d)$  bits) reduces the error probability to  $(\beta + \lambda)^\ell$ !

In purely graph theoretic terms, this says the following.

**Theorem 9.7.** *Let  $G$  be a  $(n, d, \lambda)$ -expander, and let  $B \subseteq V$  be of size  $\beta n$ . Starting at a uniformly random vertex  $v_1$ , consider  $\ell$  steps of the random walk going through vertices  $v_1, v_2, \dots, v_\ell$ . Then,*

$$\Pr[\text{all } v_i \text{ are in } B] \leq (\beta + \lambda)^\ell.$$

*Proof sketch.* Consider the diagonal matrix  $D$  with 1s at vertices in  $B$  and 0 elsewhere. Let  $p^{(0)}$  be the initial (uniform) distribution of  $v_0$ . Verify that the sum of components of  $(DM)^{\ell-1} D p^{(0)}$  is precisely the probability that each  $v_i$  is in  $B$  for  $i = 1, 2, \dots, \ell$ . For any probability vector  $u$  try to upper bound the  $\ell_2$  norm of  $DMDu$  as follows: split  $u$  into its components along and orthogonal to the uniform distribution vector. Verify that after applying  $DMD$ , the norm of the along component reduces by at least  $\beta$  and the orthogonal component by  $\lambda$ . Using this  $\ell$  times gives the bound.  $\square$

**Theorem 9.8** (Alon-Boppana bound). *For any  $(n, d, \lambda)$ -expander,*

$$\lambda \geq \frac{2\sqrt{d-1}}{d} (1 - o(1)).$$

**Definition 9.9** (Ramanujan Graph). *A  $(n, d, \lambda)$ -expander is said to be a Ramanujan graph if*

$$\lambda \leq \frac{2\sqrt{d-1}}{d}.$$

Ramanujan graphs are “ideal” expanders in some sense.

It was proved in 2014 by Adam Marcus, Daniel Spielman, and Nikhil Srivastava that there exist infinite families of bipartite Ramanujan graphs of every degree greater than 2.