Characterizing and Testing Principal Minor Equivalence of Matrices

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Abstract

Two matrices are said to be principal minor equivalent if they have equal corresponding principal minors of all orders. We give a characterization of principal minor equivalence and a deterministic polynomial time algorithm to check if two given matrices are principal minor equivalent. Earlier such results were known for certain special cases like symmetric matrices, skew-symmetric matrices with 0, 1, -1-entries, and matrices with no cuts (i.e., for any non-trivial partition of the indices, the top right block or the bottom left block must have rank more than 1).

As an immediate application, we get an algorithm to check if the determinantal point processes corresponding to two given kernel matrices (not necessarily symmetric) are the same. As another application, we give a deterministic polynomial-time test to check equality of two multivariate polynomials, each computed by a symbolic determinant with a rank 1 constraint on coefficient matrices.

CCS Concepts

• **Theory of computation → Pseudorandomness and derandomization**.

Keywords

Principal Minor Equivalence, Polynomial Identity Testing

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1 Introduction

The determinant of a matrix is a fundamental object of study in mathematics that has found numerous applications throughout computer science, physics, and other fields. A *minor* of a matrix is the determinant of one of its square submatrices and its order is the size of the corresponding submatrix. A *principal minor* of a matrix is a minor obtained by deleting the same set of rows and columns. Principal minors play an important role in a variety of applications, for example, convexity of functions and positive semidefinite matrices [7], the linear complementarity problem and P-matrices [31], counting number of forests via the Laplacian matrix [3], and inverse eigenvalue problems [14].

In this paper, we investigate a basic question about principal minors – what is the relationship between two $n \times n$ matrices which have equal corresponding principal minors of all orders (i.e., two matrices A and B such that for all $S \subseteq \{1, 2, \dots, n\}$, $\det(A[S,S]) = \det(B[S,S])$). We call two such matrices to be *principal minor equivalent (PME)*. Observe that two matrices are PME if and only if all their corresponding principal submatrices have the same set of eigenvalues. We seek answers of the following two questions.

Question 1 (Characterization). Can we identify a property $\mathcal{P}$ such that two matrices are PME if and only if they satisfy $\mathcal{P}$?

Question 2 (Efficient Algorithm). Can we efficiently check whether two matrices are PME or not?

The question of characterizing the relationship between two PME matrices has been extensively studied [1, 4, 5, 13, 22, 27]. One motivation for studying this question comes from the problem of learning determinantal point processes [8, 25, 39] and the closely related principal minor assignment problem [9, 18, 35]. While our original motivation to study this question came from an application to the polynomial identity testing problem (see Section 1).

To move towards characterizing PME matrices, let us first consider some trivial operations which preserve all the principal minors. Two matrices A and B are called *diagonally similar* if there exists an invertible diagonal matrix D such that $A = DBD^{-1}$. We call two matrices A and B *diagonally equivalent* if A is diagonally similar to B or B^T. It is easy to verify that any two diagonally equivalent matrices are PME. Interestingly, Engel and Schneider [13] showed that the converse is also true when one of the matrices is symmetric. That is, principal minor equivalence of a symmetric matrix with another matrix implies their diagonal equivalence (in fact, diagonal similarity). As one can efficiently check whether two matrices are diagonally equivalent or not, it also yields an efficient algorithm to decide principal minor equivalence in this case.

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In general, principal minor equivalence does not imply diagonal equivalence, as demonstrated by the following example. Consider the following block diagonal matrix A and a block upper triangular matrix B:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}. \tag{1}$$

It is easy to see that A and B are principal minor equivalent oblivious to the entries of A_3, but they are not diagonally equivalent. Such matrices that can be written as a block upper triangular matrix by permuting some rows and corresponding columns are called *reducible* matrices (and irreducible otherwise). For any $n \times n$ matrix A, define a graph with the vertex set $[n]$ and allow an edge (i, j) if and only if the (i, j)-th entry of A is nonzero. We can equivalently define reducible matrices as the ones whose graph has more than one *strongly connected components.* One can show that two matrices are PME if and only if they have the same set of irreducible blocks and their corresponding irreducible blocks are PME (see, for example, [1, Section 5]). Hence, we can restrict our focus to irreducible matrices.

In a series of works, Hartfiel and Loewy [22], and Loewy [27] extended the result of Engel and Schneider [13] to general irreducible matrices with no *cuts.* An $n \times n$ matrix A is said to have a cut $X \subsetneq [n]$, if $2 \leq |X| \leq n-2$ and both the submatrices $A[X, \overline{X}]$ and $A[\overline{X}, X]$ have rank one (the submatrices cannot have rank zero if A is irreducible). They showed that for any irreducible matrix A with no cuts and any matrix B, if A and B are PME, then A and B are also diagonally equivalent. So, the case which remained unclear was that of irreducible matrices with cuts. Engel and Schneider [13, Example 3.7] had given an example of two 4×4 matrices which are PME, but not diagonally equivalent. Clearly, both these matrices must have a cut.

The cut-transpose operation. Recently, Ahmadieh [1, Lemma 4.5] gave a general recipe that for any irreducible matrix A with a cut, finds another matrix B that is PME to A, but not necessarily diagonally equivalent to A. For this they define an operation on matrices with a cut, which we refer as cut-transpose. Consider a matrix A and let X be a cut of A. From the definition of a cut, A must be of the following form:

$$A = \begin{pmatrix} M & p \cdot q^T \\ u \cdot v^T & N \end{pmatrix},$$

where the submatrix $A[X, X] = M$ and $A[\overline{X}, \overline{X}] = N$ and p, q, u, v are column vectors of appropriate dimensions. Define a *cut-transpose* operation on A with respect to cut X, which transforms A to a new matrix $\mathrm{ct}(A, X)$ as follows:

$$\mathrm{ct}(A, X) = \begin{pmatrix} M & p \cdot u^T \\ q \cdot v^T & N^T \end{pmatrix}.$$

Ahmadieh [1] showed that cut-transpose is a principal minor preserving operation. A natural conjecture would be that any two irreducible PME matrices are related by a sequence of cut-transpose operations. To elaborate, let us define any two matrices A and B as *cut-transpose equivalent* if there is a sequence $A = A_0, A_1, \ldots, A_k$ of matrices such that for each $0 \leq i \leq k-1$, $A_{i+1} = \mathrm{ct}(A_i, X_i)$ for some cut X_i of A_i, and A_k is diagonally equivalent to B. Can one show that two irreducible matrices are PME if and only if they are cut-transpose equivalent?

Boussaïri and Chergui [5] gave a characterization for principal minor equivalent matrices for a special case, when the two matrices are skew-symmetric with entries from $\{-1, 0, 1\}$ and all their off-diagonal entries in the first row are nonzero. Interestingly, this characterization turns out to be cut-transpose equivalence with a restriction. Moreover, they conjectured that the characterization should be true for arbitrary skew-symmetric matrices. In a follow up work, Boussaïri, Chaïchaâ, Chergui, and Lakhlifi [4] proved a similar result for another special case called generalized tournament matrices (non-negative matrices A with $A + A^T = J_n - I_n$, where J_n is all ones matrix). The two settings use a transformations called HL-clan-reversal and clan-inversion, respectively, which coincide with some restrictions of the cut-transpose operation. Both these work build on a combinatorial result [6] about directed graphs with a similar flavor. The combinatorial result, in turn, is a generalization of Gallai's theorem [15] which states that if two partially ordered sets have the same comparability graph, then they are related by a sequence of orientation reversal operations (see [6, 32]). This orientation reversal on a poset is a special instance of cut-transpose on the corresponding skew-symmetric matrix.

This series of works strengthens the confidence in the conjecture that cut-transpose equivalence should be a characterization of PME for arbitrary irreducible matrices. However, their techniques are graph-theoretic and it is not clear how they can be generalized to arbitrary matrices. We instead employ algebraic techniques and show that conjecture is indeed true, thereby completely resolving Question 1. This extends the results for above mentioned special cases and also proves the conjecture of Boussaïri and Chergui [5] about skew-symmetric matrices. Moreover, we show that for any two $n \times n$ irreducible PME matrices A and B, the cut-transpose sequence contains at most $2n$ matrices.

Theorem 1.1. *Let A and B be two $n \times n$ irreducible matrices over any field. Then, A and B are principal minor equivalent if and only if there exists a sequence of $n \times n$ matrices $(A = A_0, A_1, \ldots, A_k)$ with $k < 2n$ such that*

$$\text{for } 0 \leq i \leq k-1,\ A_{i+1} = \mathrm{ct}(A_i, X_i) \text{ for some cut } X_i \text{ of } A_i \tag{2}$$

and A_k is diagonally equivalent to B.

Now, let us come to the question of an efficient algorithm to check if two given matrices are PME (Question 2). If one is allowed the use of randomness, then there is a simple algorithm for this task via a reduction to polynomial identity testing. Consider a $n \times n$ diagonal matrix Y with variables $y_1, y_2, \ldots, y_n$ in the diagonal. Observe that two $n \times n$ matrices A and B are PME if and only if the following is a polynomial identity (i.e., coefficient-wise equality)

$$\det(A + Y) = \det(B + Y).$$

There is a simple randomized algorithm for polynomial identity testing: just plug-in some random numbers for the variables and then check the equality (see [11, 37, 41]). There is no deterministic polynomial time algorithm known for polynomial identity testing in general, but we can still ask if there is one for this special case. We answer this question positively. Recall the earlier discussion about reducible matrices and note that testing PME for two matrices reduces to the same for their corresponding irreducible blocks.

THEOREM 1.2. *There exists a deterministic polynomial-time algorithm that for any two given $n \times n$ matrices A and B over any field, decides whether the corresponding principal minors of A and B are equal or not. If they are equal, then as a certificate, the algorithm outputs cut-transpose sequences for the corresponding irreducible blocks of the two matrices as guaranteed by Theorem 1.1.*

Now we mention a couple of applications of this result.

Polynomial Identity Testing. As mentioned earlier our motivation for the principal minor equivalence problem came from the polynomial identity testing (PIT) problem. Given two multivariate polynomials in a succinct representation, the PIT problem asks to decide whether the two polynomials are identical (i.e., all corresponding coefficients are equal). One of the widely studied and useful representation for multivariate polynomials is the *determinantal representation.* We say that a polynomial $f(x_1, \ldots, x_m) \in \mathbb{F}[x_1, \ldots, x_m]$ has a determinantal representation of size n if there exists matrices $A_0, A_1, \ldots, A_m \in \mathbb{F}^{n\times n}$ such that $f = \det(A_0 + \sum A_i x_i)$. The determinantal representation is known to be almost as expressive as algebraic circuits (see [40] for more details). The PIT problem admits a randomized polynomial-time algorithm [11, 37, 41]. Obtaining a deterministic algorithm for PIT remains a challenging open problem that would have interesting implications in proving lower bounds, and many other algorithmic applications (see, for example, [38]). Unable to solve it for the general setting, the problem has been studied for various restricted settings.

One such restricted setting is symbolic determinant under rank one restriction. Here we ask for testing whether $\det(A_0+\sum_{i=1}^{m} A_i x_i) = 0$, for given matrices A_i, where $\mathrm{rank}(A_i) = 1$ for $1 \le i \le m$. There has been a lot of interest in this particular setting because of its connections with some combinatorial optimization problems like bipartite matching and linear matroid intersection (see [12, 28, 33]), and algebraic problems like maximum rank matrix completion (see [17, 23, 30]). The connection with combinatorics also gives a deterministic polynomial time algorithm for identity testing in this setting. In fact, there is also an efficient blackbox PIT (quasi-polynomial time) known for this case [20] (blackbox means that the algorithm cannot see the input, it can only evaluate the given polynomial at any point).

When we have an efficient algorithm to test whether a given polynomial from a class is zero, the next natural question one can ask is to test whether two given polynomials from that class are equal. If the class of polynomials is closed under addition, the equality question easily reduces to testing zeroness of a given polynomial (from the same class). Many well studied classes of polynomials have this property, for example, sparse polynomials, bounded-depth circuits, constant fan-in depth-3 circuits etc. On the other hand, there are classes like ROABPs, which are not closed under addition [24], and for which the equality testing question has been studied independently [19]. Symbolic determinant with rank one restriction is another such class. To the best of our knowledge, the class is not known to be closed under addition. Given that zeroness testing is known for this class, a natural extension would be to ask if two given polynomials from this class are equal. To the best of our knowledge, no non-trivial (deterministic) algorithm was known for testing equality of two polynomials from this class (symbolic determinant with rank one restriction). We show that this problem reduces to testing principal minor equivalence, and hence, has a deterministic polynomial-time algorithm.

THEOREM 1.3. *There exists a deterministic polynomial time algorithm such that given two sequences of $n\times n$ matrices $(A_0, A_1, \ldots, A_m)$ and $(B_0, B_1, \ldots, B_m)$ over any field, with the rank of A_i and B_i being at most 1 for $1 \le i \le n$, it decides whether $\det(A_0 + A_1 y_1 + \ldots + A_m y_m) = \det(B_0 + B_1 y_1 + \ldots + B_m y_m)$.*

Determinantal Point Processes. As mentioned earlier, one motivation to study principal minors come from determinantal point processes (DPP). DPP are a family of probabilistic models which originated in physics [29], and which has subsequently found a wide range of applications in machine learning [25], for example, document summarization, recommender systems, information retrieval etc. (see references given in [16, 39]). Conventionally, a DPP is defined using principal minors of an $n \times n$ symmetric positive semidefinite matrix K, called a kernel, whose eigenvalues are between 0 and 1. The DPP corresponding to kernel matrix K is a probability distribution on subsets Y of $\{1, 2, \ldots, n\}$ such that for any subset $J \subseteq \{1, 2, \ldots, n\}$,

$$\Pr[J \subseteq Y] = \det(K[J]),$$

where $K[J]$ is the principal submatrix of K corresponding to set J (see [26]). DPPs are useful in settings where one needs to generate a diverse set of objects (larger principal minor means the vectors associated with the subset span a larger volume).

Symmetric DPPs (as defined above with a symmetric kernel matrix) have a significant expressive power, however they come with a limitation. Symmetric DPPs can model only repulsive interactions. That is, any pair of items has a negative correlation – selection of one item reduces the chances of selection of another item. To overcome this limitation, nonsymmetric determinantal point process has been proposed, that is, DPP with a nonsymmetric kernel matrix K. A nonsymmetric kernel matrix can model both positive and negative correlations. Lately, there have been a few works on nonsymmetric DPPs [2, 8, 16, 21, 34]. One of the crucial questions in the study of DPPs is to understand how are two kernel matrices related which produce the same DPP, which was explicitly asked in some works on learning DPPs [8, 9]. This is precisely the principal minor equivalence problem. While it was already understood in the case of symmetric DPPs, we answer it for nonsymmetric DPPs in this work. Theoerm 1.1 gives a characterization of the set of matrices K' such that $DPP(K') = DPP(K)$ for a given kernel matrix K (not necessarily symmetric). Theorem 1.2 gives a deterministic polynomial time algorithm to test whether two given kernel matrices will produce the same DPP.

2 Notation and Preliminaries

We use $[n]$ to denote the set of positive integers $\{1, 2, \ldots, n\}$. For any $X \subseteq [n]$, $\overline{X}$ denotes the complement set X. For two sets S and T, $S\Delta T$ denotes the symmetric difference of S and T. For a set X and an element e, we use $X + e$ to denote the set $X \cup \{e\}$ and $X - e$ to denote the set $X \setminus \{e\}$.

Suppose that

$$w_1 = (w_{1,1}, w_{1,2}, \ldots, w_{1,k_1})^T, \ldots, w_\ell = (w_{\ell,1}, w_{\ell,2}, \ldots, w_{\ell,k_\ell})^T$$

are ℓ vectors over a field $\mathbb{F}$. Then, we use $(w_1 \mid \cdots \mid w_\ell)$ to denote the concatenation of the vectors $w_1, \ldots, w_\ell$ as follows

$$(w_1 \mid \cdots \mid w_\ell) = (w_{1,1}, \ldots, w_{1,k_1}, \ldots w_{\ell,1}, \ldots, w_{\ell,k_\ell})^T.$$

For an $n \times n$ matrix A and $S, T \subseteq [n]$, $A[S,T]$ denotes the submatrix of A with rows indexed by elements in S and columns indexed by elements in T. For $S \subseteq [n]$, let $A[S]$ denote the submatrix $A[S,S]$. When $S = \{i\}$, then $A[i,T] = A[S,T]$. We follow a similar notation when T is a singleton. For a square matrix A, by A^{adj}, we denote the adjoint, or adjugate, of A.

2.1 Principal Minor Equivalence

Suppose that A and B are two $n \times n$ matrices over any field. The matrix A is said to be *principal minor equivalent* to B if the corresponding principal minors of A and B are *equal*, i.e. for all $S \subseteq [n]$, $\det(A[S,S]) = \det(B[S,S])$. We use $A \overset{\text{PME}}{=} B$ to denote that A is the principal minor equivalent to B.

The following lemma shows that the principal minor equivalence relation between two matrices remains unchanged under adjoint operation and shift by appropriate diagonal matrices. It is a straightforward consequence of [22, Lemma 4].

Lemma 2.1. *Let A and B be two $n \times n$ matrices over a field $\mathbb{F}$. Let D be an $n \times n$ diagonal matrix over $\mathbb{F}$ such that $A + D$ and $B + D$ are non-singular. Then, $A \overset{\text{PME}}{=} B$ if and only if $(A + D)^{\text{adj}} \overset{\text{PME}}{=} (B + D)^{\text{adj}}$.*

2.2 Reducible and Irreducible Matrix

Definition 2.2 (Reducible and Irreducible matrix)**.** A matrix is called *reducible* if it can be written as a block upper triangular matrix after permuting the rows and the corresponding columns. A matrix that is not reducible is called *irreducible*.

Equivalently, if we replace the nonzero off-diagonal entries with one and the diagonal entries with zero, then a reducible matrix corresponds to the adjacency matrix of a directed graph having more than one strongly connected component.

From the above definition, it is easy to see that any matrix A with all nonzero off-diagonal entries is an irreducible matrix. The above definition directly gives us the following observation.

Observation 2.3. *Let A be an $n \times n$ matrix over a field $\mathbb{F}$ such that the row and columns of A are indexed by $[n]$. Let G_A be a directed graph defined as follows: the vertex set in $[n]$, and a tuple (i, j) is an edge of G_A if and only if $i \neq j$ and $A[i, j] \neq 0$. Let $I_1, I_2, \ldots, I_s$ be the strongly connected components of A. Then, after permuting the rows and the corresponding columns, the matrix A can be made a block upper triangular matrix, and the diagonal blocks $A[I_1], A[I_2], \ldots, A[I_s]$ are irreducible matrices.*

For two reducible matrices A and B, the next lemma helps to reduce the testing of whether $A \overset{\text{PME}}{=} B$ to multiple instances of testing whether two irreducible matrices have the same corresponding principal minors. The following lemma is a direct consequence of [1, Corollary 5.4].

Lemma 2.4. *Let A and B two $n \times n$ matrices over a field $\mathbb{F}$. Suppose that after permuting the rows and the corresponding columns, A can be written as a block upper triangular matrix with s diagonal blocks $A_1, A_2, \ldots, A_s$ where each A_i is irreducible and the rows and columns of A_i are indexed by set $T_i \subseteq [n]$. Then, $A \overset{\text{PME}}{=} B$ if and only if the following holds.*

(1) After permuting some rows and the corresponding columns, B can be written as a block upper triangular matrix with s diagonal blocks $B_1, B_2, \ldots, B_s$ such that each B_i is irreducible and the rows and columns of B_i are indexed by set T_i.
(2) For each $i \in [s]$, $A_i \overset{\text{PME}}{=} B_i$.

2.3 Cut of a Matrix

Definition 2.5 (Cut of a matrix)**.** Let A be an $n \times n$ matrix over a field $\mathbb{F}$ such that $n \geq 4$. A subset $X \subset [n]$ is called a *cut* in A if $2 \leq |X| \leq n - 2$ and the rank of the submatrices $A[X, \overline{X}]$ and $A[\overline{X}, X]$ are at most one.

In particular, if A is an irreducible matrix and X is cut in A, then $\text{rank}(A[X, \overline{X}]) = \text{rank}(A[\overline{X}, X]) = 1$.

By definition, if X is a cut of a matrix A then so is $\overline{X}$. For a matrix A, a cut X in A is called a *minimal cut* if there exists no other cut X' in A such that $X' \subseteq X$. Note that any cut of size two is always a minimal cut.

Next, we show that the set of cuts of a matrix remains the same if we take its adjugate after adding an appropriate diagonal matrix.

Lemma 2.6. *Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Let D be an $n \times n$ diagonal matrix over $\mathbb{F}$ such that $A + D$ is non-singular. Then, A and $(A + D)^{\text{adj}}$ have the same set of cuts.*

For proof, see full version [10, Appendix].

2.4 Diagonal Similarity

Suppose that A and B are two $n \times n$ matrices over a field $\mathbb{F}$. We say that A is *diagonally similar* to B, denoted by $A \overset{\text{DS}}{=} B$, if there exists an $n \times n$ invertible diagonal matrix D over $\mathbb{F}$ such that $B = DAD^{-1}$. We say that A and B are *diagonally equivalent*, denoted by $A \overset{\text{DE}}{=} B$, if $A \overset{\text{DS}}{=} B$ or $A \overset{\text{DS}}{=} B^T$.

In the following claim, we describe how to efficiently check whether two matrices are diagonally similar or not.

Claim 2.7. *Given two $n \times n$ matrices A and B over $\mathbb{F}$, in polynomial time, we can decide whether $A \overset{\text{DS}}{=} B$.*

For proof, see full version [10, Claim 2.7].

One can observe that if $A \overset{\text{DE}}{=} B$, then $A \overset{\text{PME}}{=} B$. Next, we consider the converse direction. Hartfiel and Loewy [22, Theorem 3] showed that when $n = 2$ or 3, and A is an irreducible matrix, $A \overset{\text{PME}}{=} B$ implies that $A \overset{\text{DE}}{=} B$. Later, Lowey [27, Theorem 1] showed that if A is an irreducible matrix and has no cut, then $A \overset{\text{PME}}{=} B$ implies $A \overset{\text{DE}}{=} B$. Therefore, by combining them, we have the following lemma:

Lemma 2.8. *Let A and B two $n \times n$ matrices over a field $\mathbb{F}$ such that A is irreducible and $A \overset{\text{PME}}{=} B$. Then, the following holds:*

(1) if $n = 2$ or 3, then $A \overset{\text{DE}}{=} B$.
(2) if $n \geq 4$ and A has no cut, then $A \overset{\text{DE}}{=} B$.

The next lemma shows that the diagonal similarity relation also holds for the adjugate of the matrices obtained by adding an appropriate diagonal matrix. It directly follows from [22, Lemma 4].

Lemma 2.9. *Let A and B be two $n \times n$ matrices over $\mathbb{F}$. Let D be an $n \times n$ diagonal matrix such that both $A + D$ and $B + D$ are invertible. Then, $A \overset{DS}{=} B$ if and only if $(A + D)^{\text{adj}} \overset{DS}{=} (B + D)^{\text{adj}}$.*

2.5 Cut-Transpose Operation

In the previous section, we have seen that under diagonal similarity, the values of the principal minors of a matrix remain unchanged. Now, we describe another operation under which also the values of the principal minors remain the same. This operation was defined by Ahmadieh [1, Lemma 4.5], and we refer to it as cut-transpose.

Definition 2.10 (Cut-transpose operation)**.** Let A be an $n \times n$ irreducible matrix represented as follows, and $X \subseteq [n]$ be a cut of A. Let $q, u \in \mathbb{F}^{|\overline{X}|}$ such that q is the first non-zero row of $A[X, \overline{X}]$, u is the first non-zero column of $A[\overline{X}, X]$. Let $p, v \in \mathbb{F}^{|X|}$ such that $A[X, \overline{X}] = p \cdot q^T$ and $A[\overline{X}, X] = u \cdot v^T$.

$$A = \begin{pmatrix} A[X] & p \cdot q^T \\ u \cdot v^T & A[\overline{X}] \end{pmatrix}.$$

Then, the *cut-transpose operation on A with respect to X*, denoted by $\text{ct}(A, X)$, transforms A to the following matrix:

$$\text{ct}(A, X) = \begin{pmatrix} A[X] & p \cdot u^T \\ q \cdot v^T & A[\overline{X}]^T \end{pmatrix}.$$

Remark 2.11. Note that in the above definition, we take one particular rank-one decomposition for submatrices $A[X, \overline{X}]$ and $A[\overline{X}, X]$. For every nonzero $\alpha, \beta \in \mathbb{F}$, the rank-one submatrices $A[X, \overline{X}]$ and $A[\overline{X}, X]$ are equal to $(\alpha p) \cdot (q/\alpha)^T$ and $(\beta u) \cdot (v/\beta)^T$, respectively. Depending on what rank one decomposition we choose, we can get a different matrix after applying this operation. However, all these obtained matrices are *diagonally similar* to each other. Also, one advantage of choosing the above decomposition is that $\text{ct}(\text{ct}(A, X), X) = A$.

For a $k \times \ell$ matrix M with $\text{rank}(M) \leq 1$, in polynomial time, we can find $p \in \mathbb{F}^k$ and $q \in \mathbb{F}^\ell$ such $M = p \cdot q^T$. Hence, we can find the cut-transpose of a matrix with respect to a given cut in polynomial time.

Now, we mention some properties of the cut-transpose operation. First, we show that under cut-transpose operation, the values of the principal minors of a matrix remain the same.

Lemma 2.12. *Let A be an $n \times n$ irreducible matrix over a field $\mathbb{F}$ with a cut $X \subset [n]$. Then, $A \overset{PME}{=} \text{ct}(A, X)$.*

For proof, see full version [10, Appendix]. Next, we show that the cut-transpose operation and the adjoint operation commute with each other up to diagonal similarity.

Lemma 2.13. *Let A be an $n \times n$ irreducible matrix over a field $\mathbb{F}$. Then, a cut $X \subseteq [n]$ of A is also a cut of A^{adj} and*

$$\text{ct}(A, X)^{\text{adj}} \overset{DS}{=} \text{ct}(A^{\text{adj}}, X).$$

For proof, see full version [10, Appendix]. Next, we define the *cut-transpose equivalence* relation. Theorem 1.1 says that it characterizes principal minor equivalence for irreducible matrices.

Definition 2.14. Let A and B be two irreducible matrices over a field $\mathbb{F}$ with their rows and columns indexed by I. Let $\mathcal{X} = (X_1, X_2, \ldots, X_k)$ be a sequence of subsets of I. We say that A and B are *cut-transpose equivalent* with respect to *cut sequence $\mathcal{X}$* if it produces a sequence of matrices $(A_0 = A, A_1, A_2, \ldots, A_k)$ with the following property:

$$\forall i \in [k],\ A_i = \text{ct}(A_{i-1}, X_i) \text{ where } X_i \text{ is a cut in } A_{i-1}, \text{ and } A_k \overset{DE}{=} B.$$

The following lemma demonstrates how the cut-transpose relation extends to the adjoint whose proof can found in the full version [10, Lemma 2.15].

Lemma 2.15. *Let A, B be two $n \times n$ irreducible matrices over a field $\mathbb{F}$ and $\mathcal{X}$ be sequence of subsets of $[n]$. Let D be a diagonal matrix such that $A + D$ and $B + D$ are invertible. Then, A and B are cut-transpose equivalent with respect to $\mathcal{X}$ if and only if $(A + D)^{\text{adj}}$ and $(B + D)^{\text{adj}}$ are cut-transpose equivalent with respect to $\mathcal{X}$.*

3 Characterization of Prinicipal Minor Equivalence for Irreducible Matrices

In this section, we show that two irreducible matrices are principal minor equivalent if and only if they are cut-transpose equivalent. Formally, we show Theorem 1.1. The forward direction directly follows from Lemma 2.12 and the fact that diagonally equivalent matrices are principal minor equivalent. For the other direction, first, we argue that we only need to show the theorem for matrices whose all off-diagonal entries are non-zero.

Suppose there exists a diagonal matrix D such that $A+D$ and $B+D$ are invertible and off-diagonal entries of $(A + D)^{\text{adj}}$ and $(B + D)^{\text{adj}}$ are non-zero. Since $A \overset{PME}{=} B$, from Lemma 2.1, $(A+D)^{\text{adj}} \overset{PME}{=} (B+D)^{\text{adj}}$. From Lemma 2.15, if $(A + D)^{\text{adj}}$ and $(B + D)^{\text{adj}}$ are cut-transpose equivalent with respect to a cut sequence $\mathcal{X}$ of size at most $2n$ then so are A and B. This implies that if the Theorem 1.1 holds for matrices with non-zero off-diagonal entries, then it also holds for general irreducible matrices. We show the existence of such D in Claim 4.1. Hence, in the rest of this section, we assume that A and B have non-zero off-diagonal entries, without loss of generality.

If $n \leq 3$ or A does not have any cut, then Theorem 1.1 directly follows from Lemma 2.8. So, we assume that $n \geq 4$ and A has a cut. We prove the theorem using induction on n. The base case of $n = 4$ directly follows from the following lemma. See full version [10, Appendix] for the proof.

Lemma 3.1. *Let A be a 4×4 matrix over $\mathbb{F}$ with all off-diagonal entries are nonzero. Let B be another 4×4 matrix over $\mathbb{F}$ such that $A \overset{PME}{=} B$. Then, one of the following two holds:*

(1) $A \overset{DE}{=} B$.
(2) There exists a common cut in A and B. Furthermore, for any common cut X of A and B, $\text{ct}(A, X) \overset{DE}{=} B$.

Now, we have two $n \times n$ matrices A and B with non-zero off-diagonal entries and at least one cut such that $B \overset{PME}{=} A$. First, we show some relation between minimal cuts of A and B that enables us to apply induction. Precisely, we show that if A and B have the same principal minors, and A has a cut, then a minimal cut of A is also a cut of B if its size is greater than two. Otherwise, if the size of a minimal cut S of A is two, then either it is also a cut of B, or there exists a cut X in B such that the cut-transpose of B with respect to X has cut S. To show this relationship between cuts, we first show the following three results.

The following lemma establishes the relation between cuts of a matrix and its cut-transpose whose proof can be found in the full version [10, Lemma 3.2].

Lemma 3.2. *Let A be an $n \times n$ matrix over $\mathbb{F}$ with nonzero off-diagonal entries. Let $S \subseteq [n]$ be a cut in A. Then, for any $T \subseteq [n]$ the following holds.*

(1) If T or $\overline{T}$ is a subset of S or $\overline{S}$, then T is a cut in A if and only if T is a cut in $\mathrm{ct}(A, S)$.
(2) Otherwise, T is a cut in A if and only if $T \Delta S$ is a cut in $\mathrm{ct}(A, S)$.

Proof. We start with the proof of the first part of the lemma.

Proof of the first part. Assume that $T \subseteq S$ and T is a cut in A. Then, the matrix A has the following structure:

$$A = \begin{array}{c} \\ T \\ S\setminus T \\ \overline{S} \end{array} \begin{array}{c} \begin{array}{ccc} T & S\setminus T & \overline{S} \end{array} \\ \begin{pmatrix} * & u_1 \cdot v_1^T & u_1 \cdot v_2^T \\ p_1 \cdot q_1^T & * & u_2 \cdot v_2^T \\ p_2 \cdot q_1^T & p_2 \cdot q_2^T & * \end{pmatrix} \end{array}$$

such that

$$u_1, q_1 \in \mathbb{F}^{|T|}, \quad v_1, u_2, p_1, q_2 \in \mathbb{F}^{|S|-|T|}, \quad \text{and} \quad v_2, p_2 \in \mathbb{F}^{|\overline{S}|},$$

and '$*$' marked submatrices can be arbitrary. After applying the cut-transpose operation on A with respect to the cut S, using Remark 2.11,

$$\mathrm{ct}(A,S)[T,\overline{T}] \overset{\text{DS}}{=} u_1 \cdot (v_1 \mid p_2)^T \text{ and } \mathrm{ct}(A,S)[\overline{T},T] \overset{\text{DS}}{=} (p_1 \mid v_2) \cdot q_1^T.$$

Therefore, T is also a cut in $\mathrm{ct}(A, S)$. The converse follows because $\mathrm{ct}(\mathrm{ct}(A, S), S) = A$.

Now we assume that $T \subseteq \overline{S}$. Note that the set of cuts in A is the same as the set of cuts in A^T. Since $T \subseteq \overline{S}$, from the above discussion, T is a cut in A^T if and only if T is a cut in $\mathrm{ct}(A^T, \overline{S})$. Observe that $\mathrm{ct}(A^T, \overline{S}) \overset{\text{DS}}{=} \mathrm{ct}(A, S)$. Thus, when $T \subseteq \overline{S}$, the set T is a cut in A if and only if T is a cut in $\mathrm{ct}(A, S)$. The proof for the remaining cases directly follows from these.

Proof of the second part. Assume that neither T nor $\overline{T}$ is a subset of S or $\overline{S}$, and T is a cut in A. This implies that $S \setminus T$, $S \cap T$ and $T \setminus S$ are nonempty. Since S is a cut in A, the matrix A has the following structure.

$$A = \begin{array}{c} \\ S\setminus T \\ S\cap T \\ T\setminus S \\ \overline{S\cup T} \end{array} \begin{array}{c} \begin{array}{cccc} S\setminus T & S\cap T & T\setminus S & \overline{S\cup T} \end{array} \\ \begin{pmatrix} * & * & u_1 \cdot v_1^T & u_1 \cdot v_2^T \\ * & * & u_2 \cdot v_1^T & u_2 \cdot v_2^T \\ p_1 \cdot q_1^T & p_1 \cdot q_2^T & * & * \\ p_2 \cdot q_1^T & p_2 \cdot q_2^T & * & * \end{pmatrix} \end{array} \tag{3}$$

such that

$$u_1, q_1 \in \mathbb{F}^{|S\setminus T|}, v_1, p_1 \in \mathbb{F}^{|T\setminus S|}, v_2, p_2 \in \mathbb{F}^{|\overline{S\cup T}|}, \text{ and } u_2, q_2 \in \mathbb{F}^{|S\cap T|}.$$

Since T is also a cut, the columns and rows of $A[S \setminus T, S \cap T]$ are multiples of u_1 and q_2^T respectively. Hence, $A[S \setminus T, S \cap T] = (\alpha u_1) \cdot q_2^T$ for some $\alpha \neq 0$. Since $\mathrm{rank}(A[\overline{T}, T]) = 1$, $A[\overline{S \cup T}, T \setminus S] = p_2 \cdot (v_1^T/\alpha)$. Similarly, for some non-zero β,

$$A[S \cap T, S \setminus T] = (\beta u_2) \cdot q_1^T \text{ and } A[T \setminus S, \overline{S \cup T}] = p_1 \cdot (v_2^T/\beta).$$

Thus, the matrix A has the following form.

$$A = \begin{array}{c} \\ S\setminus T \\ S\cap T \\ T\setminus S \\ \overline{S\cup T} \end{array} \begin{array}{c} \begin{array}{cccc} S\setminus T & S\cap T & T\setminus S & \overline{S\cup T} \end{array} \\ \begin{pmatrix} * & (\alpha u_1) \cdot q_2^T & u_1 \cdot v_1^T & u_1 \cdot v_2^T \\ (\beta u_2) \cdot q_1^T & * & u_2 \cdot v_1^T & u_2 \cdot v_2^T \\ p_1 \cdot q_1^T & p_1 \cdot q_2^T & * & p_1 \cdot (v_2^T/\beta) \\ p_2 \cdot q_1^T & p_2 \cdot q_2^T & p_2 \cdot (v_1^T/\alpha) & * \end{pmatrix} \end{array} \tag{4}$$

From Eq. (4), applying cut-transpose operation on A with respect to the cut S, we get that

$$\begin{aligned} \mathrm{ct}(A,S)[T\Delta S, \overline{T\Delta S}] &\overset{\text{DS}}{=} (\alpha u_1 \mid v_1) \cdot (q_2 \mid \alpha^{-1} p_2)^T, \text{ and} \\ \mathrm{ct}(A,S)[\overline{T\Delta S}, T\Delta S] &\overset{\text{DS}}{=} (\beta u_2 \mid v_2) \cdot (q_1 \mid \beta^{-1} p_1)^T. \end{aligned}$$

Therefore, $S \Delta T$ is a cut in $\mathrm{ct}(A, S)$.

Converse follows, because $\mathrm{ct}(\mathrm{ct}(A, S), S) = A$ and $(T\Delta S)\Delta S = T$. □

In the following lemma, we state a property about a minimal cut of size greater than two.

Lemma 3.3. *Let A be an $n \times n$ matrix over $\mathbb{F}$ such that the off-diagonal entries of A are nonzero. Let S be a minimal cut in A of size greater than two. Let T be a nonempty subset of $\overline{S}$, $X \subsetneq S \cup T$ and $\widetilde{X} = (S \cup T) \setminus X$. Then, if X is a cut in $A[S \cup T]$, then either $S \subseteq X$ or $S \subseteq \widetilde{X}$.*

In particular, if $T = \{t\}$ for some $t \in \overline{S}$, then the matrix $A[S + t]$ have no cut.

Proof. For the sake of contradiction, assume that $X \subsetneq S \cup T$ is a cut of $A[S \cup T]$ such that neither $S \subseteq X$ nor $S \subseteq \widetilde{X}$. Since $|S| \geq 3$, either $|S \cap X| \geq 2$ or $|S \cap \widetilde{X}| \geq 2$. This implies that a cut exists in $A[S \cup T]$, which contains at least two elements of S. Hence, without loss of generality, we can assume that $|S \cap X| \geq 2$. If $T \setminus X$ is empty, then $\widetilde{X} \subseteq S$ and hence $|\widetilde{X} \cap S| = |\widetilde{X}| \geq 2$. Also, $T \setminus \widetilde{X} = T$ is non-empty. This implies that $A[S \cup T]$ has a cut such that it has at least two elements from S, and T has at least one element that is not present in it. Without loss of generality, we assume that $|X \cap S| \geq 2$ and $T \setminus X \neq \emptyset$. Since S is not a subset of X, $S \setminus X$ is non-empty.

Let $t \in T \setminus X = T \cap \widetilde{X}$. Let

$$A[S \cap X, t] = u_1, A[S \setminus X, t] = u_2, A[t, S \cap X] = q_1^T \text{ and } A[t, S \setminus X] = q_2^T.$$

Since X is a cut of $A[S+T]$ and $t \in \widetilde{X}$, the columns of $A[S \cap X, S \setminus X]$ are multiples of u_1. Similarly, the rows of $A[S \setminus X, S \cap X]$ are multiples of q_1^T. Since S is a cut of A, A has the following structure.

$$A = \begin{array}{c} \\ S\cap X \\ S\setminus X \\ t \\ \overline{S+t} \end{array} \begin{array}{c} \begin{array}{cccc} S\cap X & S\setminus X & t & \overline{S+t} \end{array} \\ \begin{pmatrix} * & u_1\cdot v_1^T & u_1 & u_1\cdot v_2^T \\ p_1\cdot q_1^T & * & u_2 & u_2\cdot v_2^T \\ q_1^T & q_2^T & * & * \\ p_2\cdot q_1^T & p_2\cdot q_2^T & * & * \end{pmatrix} \end{array} \tag{5}$$

where $v_1, p_1 \in \mathbb{F}^{|S\setminus X|}$ and $v_2, p_2 \in \mathbb{F}^{|\overline{S}|+1}$. From Eq. (5), $A[S\cap X, \overline{S\cap X}] = u_1\cdot(v_1 \mid 1 \mid v_2)^T$ and $A[\overline{S\cap X}, S\cap X] = (p_1 \mid 1 \mid p_2)\cdot q_1^T$. This implies $S\cap X \subset S$ is a cut in A which contradicts the minimality of S.

Now we prove the other part of the lemma. Suppose this T is a singleton set, i.e. $T = \{t\}$ for some $t \in \overline{S}$. For the sake of contradiction, assume that there exists a cut X in $A[S+t]$. Then, from the first part of the lemma, either $S \subseteq X$ or $S \subseteq \widetilde{X}$ where $\widetilde{X} = (S+t)\setminus X$. Without loss of generality, assume $S \subseteq X$. Then $|\widetilde{X}| \leq 1$. This is a contradiction since X is a cut in $A[S+t]$. Therefore, $A[S\cup T]$ has no cut when T is a singleton set. □

Lemma 3.4. *Let A be an $n\times n$ matrix over $\mathbb{F}$ with nonzero off-diagonal entries. Let $S \subseteq [n]$ be a cut in the matrix A and $t \in S$, and suppose $X \subseteq \overline{S}$ is a cut in $A[\overline{S}+t]$. Then, X is also a cut in the matrix A.*

Proof. The off-diagonal entries of A are nonzero. The sets X and S are cuts in $A[S+t]$ and A, respectively. This implies that the matrix A can be written as follows.

$$A = \begin{array}{c} \\ X \\ \overline{S}\setminus X \\ t \\ S-t \end{array} \begin{array}{c} \begin{array}{cccc} X & \overline{S}\setminus X & t & S-t \end{array} \\ \begin{pmatrix} * & u_1\cdot v_1^T & u_1 & u_1\cdot v_2^T \\ p_1\cdot q_1^T & * & u_2 & u_2\cdot v_2^T \\ q_1^T & q_2^T & * & * \\ p_2\cdot q_1^T & p_2\cdot q_2^T & * & * \end{pmatrix} \end{array}$$

where

$$u_1, q_1 \in \mathbb{F}^{|X|}, \quad v_1, u_2, p_1, q_2 \in \mathbb{F}^{|\overline{S}\setminus X|}, \quad v_2, p_2 \in \mathbb{F}^{|S|-1},$$

and '$*$'marked submatrices can be arbitrary. From the above structure of A, observe that

$$A[X,\overline{X}] = u_1\cdot(v_1 \mid 1 \mid v_2)^T \text{ and } A[\overline{X},X] = (p_1 \mid 1 \mid p_2)\cdot q_1^T.$$

Therefore, X is a cut in A. □

Now, we get back to show the relationship between the minimal cuts of two PME matrices. First, we handle the case when the size of the minimal cut of A is two.

Lemma 3.5. *Let A and B be two $n\times n$ matrices over field $\mathbb{F}$ with nonzero off-diagonal entries such that $A \overset{\text{PME}}{=} B$ and $S = \{s_1, s_2\}$ is a cut in A of size 2. Then either S is a cut in B or for each $i \in \{1,2\}$, B has a cut X_i, defined as*

$$X_i = \{t \in \overline{S} \mid A[S+t] \overset{\text{DS}}{=} B[S+t]\} \cup \{s_i\},$$

and S is a cut in $\mathrm{ct}(B, X_i)$.

Proof. Without loss of generality, let $S = \{1,2\}$. Let $3 \leq t \leq n$. Since $A \overset{\text{PME}}{=} B$, it follows that $A[\{1,2,t\}] \overset{\text{PME}}{=} B[\{1,2,t\}]$. From Lemma 2.8, we have $A[\{1,2,t\}] \overset{\text{DE}}{=} B[\{1,2,t\}]$. Hence, there exists a diagonal matrix D_t with $D_t[1,1] = 1$ such that

$$D_tA[\{1,2,t\}]D_t^{-1} = B[\{1,2,t\}] \text{ or}$$
$$D_tA[\{1,2,t\}]D_t^{-1} = B[\{1,2,t\}]^T.$$

For any t for which the former condition holds, we will have

$$\frac{B[1,t]}{B[2,t]} = \frac{A[1,t]A[2,1]}{A[2,t]B[2,1]} = \frac{A[1,3]A[2,1]}{A[2,3]B[2,1]} \text{ and} \tag{6}$$
$$\frac{B[t,1]}{B[t,2]} = \frac{A[t,1]A[1,2]}{A[t,2]B[1,2]} = \frac{A[3,1]A[1,2]}{A[3,2]B[1,2]}. \tag{7}$$

The last two equalities hold because $\{1,2\}$ is a cut in A. For any t for which the later condition holds, we will have

$$\frac{B[1,t]}{B[2,t]} = \frac{A[t,1]A[1,2]}{A[t,2]B[2,1]} = \frac{A[3,1]A[1,2]}{A[3,2]B[2,1]} \text{ and} \tag{8}$$
$$\frac{B[t,1]}{B[t,2]} = \frac{A[1,t]A[2,1]}{A[2,t]B[1,2]} = \frac{A[1,3]A[2,1]}{A[2,3]B[1,2]}. \tag{9}$$

If equations (6) and (7) hold for every $3 \leq t \leq n$, or if equations (8) and (9) hold for every $3 \leq t \leq n$, then $\{1,2\}$ will be a cut of B.

Suppose that is not true. It follows that

$$\frac{A[1,3]A[2,1]}{A[2,3]} \neq \frac{A[3,1]A[1,2]}{A[3,2]}.$$

Let $P \subseteq \{3,4,\ldots,n\}$ be the set of indices for which equations (6), (7) hold and let $Q := \{3,4,\ldots,n\}\setminus P$ be the set of indices for which equations (8), (9) hold.

We will show that $P\cup\{1\}$ is a cut in B. Consider two indices $s \in P$ and $t \in Q$. Consider the set $T = \{1,2,s,t\}$. Since equations (6) and (7) hold for s and do not hold for t, we have that

$$\frac{B[1,t]}{B[2,t]} \neq \frac{B[1,s]}{B[2,s]} \text{ or } \frac{B[t,1]}{B[t,2]} \neq \frac{B[s,1]}{B[s,2]}.$$

Hence, $\{1,2\}$ is not a cut in $B[T]$ and $B[T] \overset{\text{DS}}{\neq} A[T]$. But, we have that $A[T] \overset{\text{PME}}{=} B[T]$. Hence, there must be a cut in $B[T]$ (Lemma 3.1) In fact, $B[T]$ will have more than one cut. Because if $B[T]$ has a unique cut, say $\{1,t\}$, then that will also be a unique cut of $A[T]$ (Lemma 3.1). But, $A[T]$ has a cut $\{1,2\}$.

So, we conclude that $B[T]$ has cuts $\{1,s\}$ and $\{1,t\}$. Hence, we can write

$$B[s,t]/B[1,t] = B[s,2]/B[1,2] \text{ and}$$
$$B[t,s]/B[2,s] = B[t,1]/B[2,1]$$

Using these equations for every $s \in P$ and every $t \in Q$, we get that $X = P\cup\{1\}$ is a cut in B. Similarly, we can show that $X' = P\cup\{2\}$ is a cut in B. From Lemma 3.2, $X\Delta X' = \{1,2\}$ is a cut of $\mathrm{ct}(B,X)$ and $\mathrm{ct}(B,X')$. □

In the following lemma, we show that a minimal cut of A of size greater than two is also a cut of B.

Lemma 3.6. *Let A and B be two $n\times n$ matrices over $\mathbb{F}$ with nonzero off-diagonal entries. Let $A \overset{\text{PME}}{=} B$, and $S \subseteq n$ be a minimal cut in A of size greater than two. Then, S is also a cut in B.*

Proof. Let $s \in \overline{S}$. We show that for all $t \in \overline{S+s}$, the set $T_t := \{s,t\}$ is a cut in $B[S+T_t]$. This will imply that

$$B[S,t] = \alpha \cdot B[S,s] \text{ and } B[t,S] = \beta \cdot B[s,S]$$

for some non-zero $\alpha, \beta \in \mathbb{F}$. Hence, S is a cut in B.

Since S is a minimal cut in A of size greater than two, from Lemma 3.4, there are no cuts in both the matrices $A[S+s]$ and $A[S+t]$. We have that $A \overset{\text{PME}}{=} B$. Therefore, applying Lemma 2.8, $A[S+s] \overset{\text{DE}}{=} B[S+s]$ and $A[S+t] \overset{\text{DE}}{=} B[S+t]$. This implies that both $B[S+s]$ and $B[S+t]$ have no cuts.

For the sake of contradiction, assume that T_t is not a cut in $B[S+T_t]$. Note that T_t is a cut in $A[S+T_t]$ of size two. Then, from Lemma 3.5, there exists a cut $X \subseteq S+T_t$ in the matrix $B[S+T_t]$ such that $s \in X$ but $t \notin X$. Since $|S+T_t| \geq 5$, either $|X| > 2$ or the size of $\widetilde{X} := (S+T_t) \setminus X$ is greater than 2. If $|X| > 2$, then $X - s$ is a cut in $B[S+t]$ as X is a cut of $B[S+T_t]$. Otherwise, $\widetilde{X} - t$ is a cut in $B[S+s]$. In both the cases, we have contradictions. Thus, T_t is a cut in $B[S+T_t]$ for all $t \in \overline{S+s}$. This completes our proof. □

Let S be a minimal cut of A. If S is of size greater than two, then from Lemma 3.6, S is also a cut of B. Otherwise, if $|S| = 2$ and S is not a cut of B, then from Lemma 3.5, there exists a cut X in B such that S is a cut of $\mathrm{ct}(B,X)$. Hence, from now on, we can assume that A has a minimal cut S, which is also a cut of B.

Now, we go to the inductive step. Since $A \overset{\text{PME}}{=} B$, any principal submatrix of A and the corresponding principal submatrix of B are also principal minor equivalent. We fix one principal submatrix corresponding to set $\overline{S}+t$ for some s in S and try to get a cut sequence for A and B using the cut sequence for $A[\overline{S}+s]$ and $B[\overline{S}+s]$, which we get from induction hypothesis. For this, we show Claim 3.8. Before that, we state the following observation.

Observation 3.7. *Let A be an $n \times n$ matrix with non-zero off-diagonal entries. Let $S \subset [n]$ and $X \subset S$ such that X is a cut of $A[S]$. Then,*

(1) If X is a cut of A, then $\mathrm{ct}(A[S],X) = \mathrm{ct}(A,X)[S]$.
(2) If $\overline{S}+X$ is a cut of A, then $\mathrm{ct}(A[S],X) = \mathrm{ct}(A,\overline{S}+X)[S]$

Claim 3.8. *Let A and B be two $n \times n$ matrices with non-zero off-diagonal entries and a common cut $S \subset [n]$ such that $A \overset{\text{PME}}{=} B$. Let $s \in S$ and $A[\overline{S}+s]$ and $B[\overline{S}+s]$ be cut-transpose equivalent with respect to cut sequence $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_k)$. Let $A_0 = A$ and $X_i = \begin{cases} \tilde{X}_i \cup S & \text{if } s \in \tilde{X}_i \\ \tilde{X}_i & \text{otherwise} \end{cases}$. Then,*

(1) For each $i \in [k]$, X_i and S are cuts of A_{i-1} where $A_i = \mathrm{ct}(A_{i-1},X_i)$ and $A_k[\overline{S}+s] \overset{\text{DS}}{=} B[\overline{S}+s]$.
(2) If S is a minimal cut of A, then S is also a minimal cut of A_i for each $i \in [k]$.

Proof. Given that there exists a sequence of matrices $(A[\overline{S}+s] = \tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_k)$ such that $\tilde{A}_i = \mathrm{ct}(\tilde{A}_{i-1}, \tilde{X}_i)$ where $\tilde{X}_i$ is a cut of $\tilde{A}_{i-1}$ and $\tilde{A}_k \overset{\text{DS}}{=} B[\overline{S}+s]$ or $B[\overline{S}+s]^T$. If $s \in \tilde{X}_1$, then $X_1 = \tilde{X}_1 \cup S$. Since S is a cut of $A = A_0$ and $(\overline{S}+s) \setminus \tilde{X}_1$ is a cut of $A[\overline{S}+s]$, from Lemma 3.4, $(\overline{S}+s) \setminus \tilde{X}_1$ is a cut of A_0 which in turn implies its complement, that is, X_1 is a cut of A_0. Since $X_1 = \overline{\overline{S}+s+\tilde{X}_1}$, from observation 3.7, $\mathrm{ct}(A_0,X_1)[\overline{S}+s] = \mathrm{ct}(\tilde{A}_0,\tilde{X}_1)$. Note that in this case, $S \subseteq X_1$.

If $s \notin \tilde{X}_1$, then $X_1 = \tilde{X}_1$. Note that from Lemma 3.4, X_1 is also a cut of A_0 and $X_1 \subset \overline{S}$. Hence from observation 3.7, $\mathrm{ct}(A_0,X_1)[\overline{S}+s] = \mathrm{ct}(\tilde{A}_0,\tilde{X}_1)$. Given that $A_1 = \mathrm{ct}(A_0,X_1)$ and $\tilde{A}_1 = \mathrm{ct}(\tilde{A}_0,\tilde{X}_1)$. Hence, $A_1[\overline{S}+s] = \tilde{A}_1$. Since, either $X_1 \subset \overline{S}$ or $S \subset X_1$, from Lemma 3.2, A_1 also has cut S. The minimality of cut S also follows from Lemma 3.2 when S is a minimal cut of A. Iteratively, in a similar way, we can show that for each $i \in \{2,3,\ldots,k\}$, X_i and S are cuts of A_{i-1} and $A_i[\overline{S}+s] = \tilde{A}_i$. Also, we can show that if S is a minimal cut of A, then it is also a minimal cut of A_i for each $i \in \{2,3,\ldots,k\}$. □

Since $A[\overline{S}+s] \overset{\text{PME}}{=} B[\overline{S}+s]$, from induction hypothesis, we get that they are cut-transpose equivalent with respect to a cut sequence of length, say $k \leq 2(|\overline{S}|+1) \leq 2n-2$. Using Claim 3.8, we can get another matrix A' from A through a sequence of cut-transpose operations such that $A'[\overline{S} \cup \{s\}] \overset{\text{DE}}{=} B[\overline{S} \cup \{s\}]$ and S is a minimal cut of A'. We can go even further and assume that $A'[\overline{S} \cup \{s\}] \overset{\text{DS}}{=} B[\overline{S} \cup \{s\}]$. This is because when $A'[\overline{S} \cup \{s\}] \overset{\text{DS}}{=} B[\overline{S} \cup \{s\}]^T$, then we can work with B^T instead of B. The following lemma shows that we can get B from A' by using at most one cut-transpose operation.

The number of cut transpose operations from A to A' is $k \leq 2n-2$. If S has size 2 and it is not a cut in B, then we need one cut-transpose operation from Lemma 3.5. From Lemma 3.9, we might need one more cut-transpose operation from A' to B. This completes the proof of Theorem 1.1 by giving a cut-sequence of size at most $2n$ from A to B.

Lemma 3.9. *Let A and B be two $n \times n$ matrices over $\mathbb{F}$ with nonzero off-diagonal entries and $A \overset{\text{PME}}{=} B$. Let $S \subseteq [n]$ be a minimal cut in A and also a cut in B. Let $s \in S$ such that $A[\overline{S}+s] \overset{\text{DS}}{=} B[\overline{S}+s]$. Then, either $A \overset{\text{DS}}{=} B$ or $\mathrm{ct}(A,\overline{S}) \overset{\text{DS}}{=} B$.*

Proof. Without loss of generality, assume that $S = [i]$ and $s = i$. Then, from the hypothesis, $B[\overline{S}+i] \overset{\text{PME}}{=} A[\overline{S}+i]$. Since S is a minimal cut in A, using Lemma 3.3 and Lemma 2.8, there exists an $(i+1)\times(i+1)$ invertible diagonal matrix D_1 such that $D_1[i+1,i+1] = 1$ and

$$D_1 \cdot A[[i+1]] \cdot D_1^{-1} = B[[i+1]] \text{ or } B[[i+1]]^T.$$

From the hypothesis, there exists another $(n-i+1)\times(n-i+1)$ invertible diagonal matrix D_2 such that $D_2[i,i] = 1$ and

$$B[\overline{S}+i] = D_2 \cdot A[\overline{S}+i] \cdot D_2^{-1}. \tag{10}$$

We assume that the rows and columns of D_2 are indexed by $\overline{S}+i$. Next, we divide our proof into the following two cases.

Case I:. In this case, we assume that

$$D_1 \cdot A[[i+1]] \cdot D_1^{-1} = B[[i+1]], \tag{11}$$

and show $A \overset{\text{DS}}{=} B$. Let D be an $n \times n$ invertible matrix defined as follows: For all $k \in [n]$,

$$D[k,k] = \begin{cases} D_1[k,k] & \text{if } k \in [i] \\ \frac{D_2[k,k]}{D_2[i+1,i+1]} & \text{otherwise}. \end{cases}$$

We will show that B is equal to DAD^{-1}. Since S is a common cut in both the matrices A and B, the rank-one submatrices $A[S,\overline{S}]$ and

$B[S,\overline{S}]$ can be written as follows.

$$A[S,\overline{S}] = A[S,i+1]\cdot\frac{A[i,\overline{S}]}{A[i,i+1]} \text{ and } A[\overline{S},S] = A[\overline{S},i]\cdot\frac{A[i+1,S]}{A[i+1,i]} \quad (12)$$

$$B[S,\overline{S}] = B[S,i+1]\cdot\frac{B[i,\overline{S}]}{B[i,i+1]} \text{ and } B[\overline{S},S] = B[\overline{S},i]\cdot\frac{B[i+1,S]}{B[i+1,i]} \quad (13)$$

From Eq. (10) and Eq. (11),

$$\begin{aligned} B[i,i+1] &= A[i,i+1]\cdot D_2^{-1}[i+1,i+1] \\ B[i,\overline{S}] &= A[i,\overline{S}]\cdot D_2^{-1}[\overline{S}], \text{ and} \\ B[S,i+1] &= D_1[S]\cdot A[S,i+1] \end{aligned}$$

Therefore, using the above equation and Eq. (13),

$$\begin{aligned} B[S,\overline{S}] &= D_1[S]\cdot A[S,i+1]\cdot\frac{D_2[i+1,i+1]\cdot A[i,\overline{S}]\cdot D_2^{-1}[\overline{S}]}{A[i,i+1]} \\ &= D[S]\cdot A[S,\overline{S}]\cdot D^{-1}[\overline{S}] \end{aligned}$$

Similarly, we can show that

$$B[\overline{S},S] = D[\overline{S}]\cdot A[\overline{S},S]\cdot D^{-1}[S].$$

Applying Eq. (11) and Eq. (10), we get that

$$\begin{aligned} B[S] &= D[S]\cdot A[S]\cdot D^{-1}[S] \text{ and} \\ B[\overline{S}] &= D[\overline{S}]\cdot A[\overline{S}]\cdot D^{-1}[\overline{S}]. \end{aligned}$$

Thus, $B = DAD^{-1}$.

Case II:. In this case, we assume that

$$D_1\cdot A[[i+1]]\cdot D_1^{-1} = B[[i+1]]^T, \quad (14)$$

and show $B \overset{\text{DS}}{=} \mathrm{ct}(A,\overline{S})$. Let D be an $n\times n$ invertible diagonal matrix defined as follows: For all $k\in[n]$,

$$D[k,k] = \begin{cases} D_1^{-1}[k,k] & \text{if } k\in[i] \\ \frac{D_2[k,k]}{D_2[i+1,i+1]} & \text{otherwise}. \end{cases}$$

We will prove that B is equal to $D\cdot\mathrm{ct}(A,\overline{S})\cdot D^{-1}$. Since S is a cut, the matrix A has the following structure.

$$A = \begin{array}{c c} & \begin{array}{cc} S & \overline{S} \end{array} \\ \begin{array}{c} S \\ \overline{S} \end{array} & \begin{pmatrix} A[S] & A[S,i+1]\cdot\frac{A[i,\overline{S}]}{A[i,i+1]} \\ \frac{A[\overline{S},i]}{A[i+1,i]}\cdot A[i+1,S] & A[\overline{S}] \end{pmatrix} \end{array}.$$

Thus, $\mathrm{ct}(A,\overline{S})$ can be written as follows.

$$\mathrm{ct}(A,\overline{S}) = \begin{array}{c c} & \begin{array}{cc} S & \overline{S} \end{array} \\ \begin{array}{c} S \\ \overline{S} \end{array} & \begin{pmatrix} A[S]^T & A[i+1,S]^T\cdot\frac{A[i,\overline{S}]}{A[i,i+1]} \\ \frac{A[\overline{S},i]}{A[i+1,i]}\cdot A[S,i+1]^T & A[\overline{S}] \end{pmatrix} \end{array}.$$

From Eq. (10) and Eq. (14), we have that

$$\begin{aligned} B[i,i+1] &= A[i,i+1]\cdot D_2^{-1}[i+1,i+1] \\ B[S,i+1] &= D_1^{-1}[S]\cdot A[i+1,S]^T \\ B[i,\overline{S}] &= A[i,\overline{S}]\cdot D_2^{-1}[\overline{S}]. \end{aligned}$$

Using the above equation and Eq. (13),

$$\begin{aligned} B[S,\overline{S}] &= D_1^{-1}[S]\cdot A[i+1,S]^T\cdot\frac{D_2[i+1,i+1]\cdot A[i,\overline{S}]\cdot D_2^{-1}[\overline{S}]}{A[i,i+1]} \\ &= D[S]\cdot\mathrm{ct}(A,\overline{S})[S,\overline{S}]\cdot D^{-1}[\overline{S}] \end{aligned}$$

Similarly, we can show that

$$B[\overline{S},S] = D[\overline{S}]\cdot\mathrm{ct}(A,\overline{S})[\overline{S},S]\cdot D^{-1}[S].$$

Applying Eq. (14) and Eq. (10), we get that

$$\begin{aligned} B[S] &= D[S]\cdot\mathrm{ct}(A,\overline{S})[S]\cdot D^{-1}[S], \text{ and} \\ B[\overline{S}] &= D[\overline{S}]\cdot\mathrm{ct}(A,\overline{S})[\overline{S}]\cdot D^{-1}[\overline{S}]. \end{aligned}$$

Thus, $B = D\cdot\mathrm{ct}(A,\overline{S})\cdot D^{-1}$. □

4 Algorithm for Principal Minor Equivalence Testing

In this section, we give a proof of Theorem 1.2 by giving polynomial time algorithm for testing whether two matrices are principal minor equivalent. For reducible matrices, the problem reduces to smaller instances of principal minor equivalence testing for irreducible matrices from Lemma 2.4. Using observation 2.3, we can find these instances in polynomial time. Hence, it is sufficient to give a polynomial time algorithm for irreducible matrices.

In Algorithm 1, given two irreducible matrices A and B as input, we output a cut sequence with respect to which A and B are cut-transpose equivalent if $A \overset{\text{PME}}{=} B$ otherwise, we output "No". The algorithm is directly based on the proof of characterization result. As mentioned earlier, we first reduce to an instance where all the off-diagonal entries are non-zero. The following claim describes how to get such an instance whose proof can be found in the full version [10, Claim 4.1].

Claim 4.1. *Let $\mathbb{F}$ be a field of size greater than $10n^5$. Let A and B be two $n\times n$ irreducible matrices over $\mathbb{F}$. Then, in $\mathrm{poly}(n)$ time, we can find a diagonal matrix $D\in\mathbb{F}^{n\times n}$ such that $A+D$ and $B+D$ are nonsingular and all entries of $(A+D)^{\mathrm{adj}}$ and $(B+D)^{\mathrm{adj}}$ are nonzero.*

Remark 4.2. When the size of the underlying field $\mathbb{F}$ is not greater than $10n^5$, we can construct an extension $\mathbb{K}$ of $\mathbb{F}$ such that $|\mathbb{K}| > 10n^5$ and work with the larger field $\mathbb{K}$. We can also construct such an extension $\mathbb{K}$ in time $\mathrm{poly}(n)$.

Now, we show how to find a minimal cut in an irreducible matrix efficiently.

Lemma 4.3. *Let A be an $n\times n$ irreducible matrix over a field $\mathbb{F}$. Then, we can test whether A has a cut in $\mathrm{poly}(n)$ time. Moreover, if there exists a cut in A, then a minimal cut of A can be computed using $\mathrm{poly}(n)$ time.*

Proof. Let $2^{[n]}$ denote the set of all subsets of $[n]$. We first show that the functions $g_1, g_2 : 2^{[n]}\to\mathbb{Z}$, defined as

$$\forall X\in 2^{[n]},\ g_1(X) := \mathrm{rank}(A[X,\overline{X}]) \text{ and } g_2(X) := \mathrm{rank}(A[\overline{X},X]),$$

are submodular functions. For each $i\in[n]$, let V_i be the subspace of $\mathbb{F}^n$ spanned by the ith row vector of A and the characteristic vector χ_i for the set $\{i\}$. Let $f : 2^{[n]} \to \mathbb{Z}$ be the function defined as

$$\forall X \in 2^{[n]},\ f(X) = \dim\left(\sum_{e \in X} V_e\right).$$

It is not hard to verify that the function f is a submodular function. Observe that a subset of row vectors of $A[X, \overline{X}]$ indexed by $T \subseteq X$ are linearly independent if and only if the set $\{\chi_e \mid e \in X\} \sqcup \{A[e', [n]] \mid e' \in T\}$ are linearly independent. Therefore, for all $X \in 2^{[n]}$,

$$f(X) = g_1(X) + |X|.$$

Since f is a submodular function, g_1 is a submodular function. Similarly, we can show that g_2 is also a submodular function.

Since g_1 and g_2 are submodular functions, their sum $g = g_1 + g_2$ is also a submodular function. For any set $T = \{t_1, t_2\} \sqcup \{t_3, t_4\}$ with four distinct elements from $[n]$, let g_T be a function defined on subsets of $\overline{T}$ such that

$$\forall X \subseteq \overline{T},\ g_T(X) = g(X \cup \{t_1, t_2\}).$$

For any $X \subseteq \overline{T}$ and $a, b \in \overline{T}$,

$$\begin{aligned} g_T(X \cup \{a\}) + g_T(X \cup \{b\}) &= g(X \cup \{a, t_1, t_2\}) + g(X \cup \{b, t_1, t_2\}) \\ &\geq g(X \cup \{t_1, t_2\}) + g(X \cup \{a, b, t_1, t_2\}) \\ &\quad (\text{ submodularity of } g) \\ &= g_T(X) + g_T(X \cup \{a, b\}). \end{aligned}$$

From the above, g_T is a submodular function. Note that there exists a cut S in A with $\{t_1, t_2\} \subseteq S$ and $\{t_3, t_4\} \subseteq \overline{S}$ if and only if the minimum value of function g_T is at most 2. One can also observe that for any subset $X \subseteq \overline{T}$, $g_T(X)$ can be computed in poly(n) time. Thus, using the submodular minimization algorithm in [36, Chapter 45], we can compute the minimum the value of g_T for any set $T = \{t_1, t_2\} \sqcup \{t_3, t_4\}$ of four distinct elements from $[n]$ in poly(n) time. There are at most n^4 such subsets T, and we can test whether A has a cut by computing the minimum value of g_T for all such possible subsets T. Thus, we can test whether A has a cut in poly(n) time.

Now, we discuss how to find a minimal cut. For a subset $T = \{t_1, t_2\} \sqcup \{t_3, t_4\}$ with four distinct elements from $[n]$, let g'_T be the function on subsets of $\overline{T}$ such that

$$\forall X \subseteq \overline{T},\ g'_T(X) = (n+1)g_T(X) + |X|.$$

Since both g_T and the cardinality function are submodular, g'_T is also a submodular function. Next observe that for $X \subseteq \overline{T}$, the set X minimizes g'_T if and only if for any $S \subseteq [n]$ with $t_1, t_2 \in S$ but $t_3, t_4 \notin S$ the following holds:

(1) $g(X \cup \{t_1, t_2\}) \leq g(S)$.
(2) if $g(X \cup \{t_1, t_2\}) = g(S)$, then $|X \cup \{t_1, t_2\}| \leq |S|$.

Therefore, a minimizing set of g'_T gives a minimal cut that contains both t_1 and t_2 but not t_3 and t_4, if such a cut exists. Now, using [36, Theorem 45.1], we can compute minimizing sets for the submodular functions g'_T for all possible subsets T, and thus, we get a minimal cut in poly(n) time if A has a cut. □

For details regarding proof of correctness and time complexity of Algorithm 1, see full version [10, Subsections 4.1-4.2].

Algorithm 1 Algorithm to test equal corresponding principal minors of two irreducible matrices

Input: Two $n \times n$ irreducible matrices A and B over $\mathbb{F}$
Output: If $A \overset{\text{PME}}{=} B$, then returns a cut sequence $\mathcal{X}$ of subsets of $[n]$ such that A, B are cut-transpose equivalent with respect to $\mathcal{X}$. Otherwise, returns "No".

Using Claim 4.1, get D and $A' \leftarrow (A+D)^{\text{adj}}$ and $B' \leftarrow (B+D)^{\text{adj}}$.
Finding-Cut-Sequence(A', B', $[n]$)

function Finding-Cut-Sequence(A, B, I)
if $|I| \leq 3$, *or*, A has *no* cut **then**
if A is *not* diagonally equivalent to B **then**
return "No".
else
return empty sequence.
else
$\widetilde{B} \leftarrow B$
Using Lemma 4.3, find a minimal cut $S \subseteq I$ in A.
if $|S| \geq 3$, *and*, S is *not* a cut of B **then**
return "No".
else if $|S| = 2$, *and*, S is *not* a cut of B **then**
$X \leftarrow$ Min-Cut-size-Two(A, B, S, I)
if X = "No" **then**
return "No".
$\widetilde{B} \leftarrow \text{ct}(B, X)$
Let $s \in S$.
$\mathcal{X}' \leftarrow$ Finding-Cut-Sequence($A(\overline{S}+s)$, $\widetilde{B}(\overline{S}+s)$, $\overline{S}+s$).
if $\mathcal{X}'$ = "No" **then**
return "No".
Let $\mathcal{X}' = (X'_1, X'_2, \ldots, X'_k)$.
$A_0 \leftarrow A$.
for $i = 1$ to k **do**
if $s \in X'_i$ **then**
$X_i \leftarrow X'_i \cup S$
else
$X_i \leftarrow X'_i$
$A_i \leftarrow \text{ct}(A_{i-1}, X_i)$.
if $A_k \overset{\text{DE}}{=} \widetilde{B}$ **then**
$\mathcal{X} \leftarrow (X_1, X_2, \ldots, X_k)$
else if $\text{ct}(A_k, \overline{S}) \overset{\text{DE}}{=} \widetilde{B}$ **then**
$\mathcal{X} \leftarrow (X_1, X_2, \ldots, X_k, \overline{S})$
else
return "No".
if $|S| = 2$, *and*, S is *not* a cut of B **then**
$\mathcal{X} \leftarrow (\mathcal{X}, X)$.
return $\mathcal{X}$.

5 PIT for Sum of Two DET1

In this section, we show Theorem 1.3. Given two sequences of $n \times n$ matrices $(A_0, A_1, \ldots, A_m)$ and $(B_0, B_1, \ldots, B_m)$ over a field $\mathbb{F}$ such that the rank of A_i and B_i is at most 1 for $1 \leq i \leq n$, the goal is to decide whether two polynomials $P_1 = \det(A_0 + A_1 y_1 + \ldots + A_m y_m)$ and $P_2 = \det(B_0 + B_1 y_1 + \ldots + B_m y_m)$ are the same in

Algorithm 2 Function for handling $|S| = 2$ case in function Cut-transpose of Algorithm 1

function Min-Cut-size-Two(A, B, I, S)
  $P \leftarrow \emptyset$, and $Q \leftarrow \emptyset$
  Let $s \in S$.
  for $t \in I \setminus S$ **do**
    if $A(S+t) \overset{\text{DS}}{=} B(S+t)$ **then**
      $P \leftarrow P \cup \{t\}$.
    else if $A(S+t) \overset{\text{DS}}{=} B(S+t)^T$ **then**
      $Q \leftarrow Q \cup \{t\}$.
    else
      return "No".
  $X \leftarrow P \cup \{s\}$.
  if X is *not* a cut of B **then**
    return "No".
  else
    return X.

poly(m, n) time. First, we consider the case when A_0 and B_0 are the zero matrix. Then, we reduce the general case where there are no constraints on A_0 and B_0 to this case. Then, we give a polynomial time reduction from this problem to the problem of equivalence testing of principal minors of two $m \times m$ matrices. For integers p and q, let 0_p and $0_{p,q}$ denote the $p \times p$ and $p \times q$ matrix, respectively, with all zeros.

5.1 $A_0 = B_0 = 0_n$.

Let $A_j = u_{1,j} \cdot v_{1,j}^T$ and $B_j = u_{2,j} \cdot v_{2,j}^T$ for each $j \in [m]$ where $u_{1,j}, v_{1,j}, u_{2,j}, v_{2,j} \in \mathbb{F}^n$. Let U_i, V_i be $n \times m$ matrices such that their jth column are $u_{i,j}$ and $v_{i,j}$, respectively, for $i \in \{1, 2\}$ and $j \in [m]$. Let Y be an $m \times m$ diagonal matrix with indeterminate y_i as the ith diagonal entry. Then,

$$A_1 y_1 + \ldots + A_m y_m = U_1 Y V_1^T \text{ and } B_1 y_1 + \ldots + B_m y_m = U_2 Y V_2^T. \quad (15)$$

For a subset T of $[m]$, let $y_T = \prod_{e \in T} y_e$, $U_{i,T} = U_i[[n], T]$ and $V_{i,T} = V_i[[n], T]$ for $i \in \{1, 2\}$. Using the Cauchy-Binet formula for multiplying two rectangular matrices,

$$\det(U_i Y V_i^T) = \sum_{T \subseteq [m], |T| = n} \left(\det(U_{i,T}) \det(V_{i,T}) y_T\right) \quad \text{for } i \in \{1, 2\}.$$

Hence, by comparing coefficients of monomials of P_1 and P_2, we get

$$P_1 = P_2 \iff \det(U_{1,T}) \det(V_{1,T}) = \det(U_{2,T}) \det(V_{2,T}) \quad \forall T \subseteq [m] \text{ with } |T| = n. \quad (16)$$

Now, we discuss how to test the latter part mentioned above. First, we find a set T of size $[n]$ such that $\det(U_{1,T}) \det(V_{1,T})$ is non-zero using a matroid intersection algorithm for matroids represented by U_1 and V_1 in poly(m, n) time. If such T doesn't exist, then $P_1 = 0$. Similarly, we can check whether P_2 is zero and decide whether $P_1 = P_2$. Suppose such a set T exists and without loss of generality, let $T = [n]$. If $\det(U_{1,[n]}) \det(V_{1,[n]}) \neq \det(U_{2,[n]}) \det(V_{2,[n]})$, then $P_1 \neq P_2$ from Eq. (16).

Suppose $\det(U_{1,[n]}) \det(V_{1,[n]}) = \det(U_{2,[n]}) \det(V_{2,[n]})$. Now, we have to check this for other sets T of size n. Let $U_i' = U_{i,[n]}^{-1} \cdot U_i$ and $V_i' = V_{i,[n]}^{-1} \cdot V_i$ for $i = 1, 2$. Since $U_i = U_{i,[n]} \cdot U_i'$, $V_i = V_{i,[n]} \cdot V_i'$ for $i = 1, 2$ and $\det(U_{1,[n]}) \det(V_{1,[n]}) = \det(U_{2,[n]}) \det(V_{2,[n]})$, for any set T of size n,

$$\det(U_{1,T}) \det(V_{1,T}) = \det(U_{2,T}) \det(V_{2,T}) \iff \det(U_{1,T}') \det(V_{1,T}') = \det(U_{2,T}') \det(V_{2,T}') \quad (17)$$

Note that $U_{i,[n]}' = V_{i,[n]}' = I_n$. For $i = 1, 2$, let $\widehat{U}_i$ and $\widehat{V}_i$ be the $n \times (m-n)$ matrices defined as $U_i'[[n], [m] \setminus [n]]$ and $V_i'[[n], [m] \setminus [n]]$, respectively. For $i \in \{1, 2\}$ and a set $T = T_1' \sqcup T_2'$ of size n with $T_1' \subseteq [n], T_2' \subseteq [m] - [n]$ such that $T_2' = \{n + e \mid e \in T_2\}$ where $T_2 \subseteq [m-n]$,

$$\det(U_{i,T}') = \sigma(T) \det(U_i'[[n] \setminus T_1', T_2']) \text{ and } \det(V_{i,T}') = \sigma(T) \det(V_i'[[n] \setminus T_1', T_2']) \quad (18)$$

where $\sigma : \binom{[m]}{n} \to \{1, -1\}$ is some sign function on n sized subsets of $[m]$. Since $U_i'[[n] \setminus T_1', T_2'] = \widehat{U}_i[T_1, T_2]$ and $V_i'[[n] \setminus T_1', T_2'] = \widehat{V}_i[T_1, T_2]$ where $T_1 = [n] \setminus T_1'$, using Eqs. (16) to (18) we get

$$P_1 = P_2 \iff \det(\widehat{U}_1[T_1, T_2]) \det(\widehat{V}_1[T_1, T_2]) = \det(\widehat{U}_2[T_1, T_2]) \det(\widehat{V}_2[T_1, T_2]) \quad (19)$$
$$\text{for each } T_1 \subseteq [n], T_2 \subseteq [m-n] \text{ with } |T_1| = |T_2|.$$

Let A and B be the $m \times m$ matrices defined as follows:

$$A = \left[\begin{array}{c|c} 0_{m-n} & \widehat{V}_1^T \\ \hline -\widehat{U}_1 & 0_n \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{c|c} 0_{m-n} & \widehat{V}_2^T \\ \hline -\widehat{U}_2 & 0_n \end{array}\right].$$

Let us consider the principal minors of A and B. If a set T is a subset of $[m-n]$ or $[m] - [m-n]$, then the corresponding principal minors of both A and B are zero. Consider a set $T = T_1' \sqcup T_2$ such that $T_2 \subseteq [m-n]$ and $T_1' \subseteq [m] - [m-n]$ such that $T_1' = \{m - n + e \mid e \in T_1\}$ where $T_1 \subseteq [n]$. Then,

$$A[T] = \left[\begin{array}{c|c} 0_{|T_2|} & \widehat{V}_1[T_1, T_2]^T \\ \hline -\widehat{U}_1[T_1, T_2] & 0_{|T_1|} \end{array}\right] \text{ and } B[T] = \left[\begin{array}{c|c} 0_{|T_2|} & \widehat{V}_2[T_1, T_2]^T \\ \hline -\widehat{U}_2[T_1, T_2] & 0_{|T_1|} \end{array}\right].$$

Note that if $|T_1| \neq |T_2|$, then both $\det(A[T])$ and $\det(B[T])$ are zero. If $|T_1| = |T_2|$, then

$$\det(A[T]) = \det(\widehat{U}_1[T_1, T_2]) \det(\widehat{V}_1[T_1, T_2]);$$
$$\det(B[T]) = \det(\widehat{U}_2[T_1, T_2]) \det(\widehat{V}_2[T_1, T_2]). \quad (20)$$

From above discussion and Eq. (20),

$$A \overset{\text{PME}}{=} B \iff \det(\widehat{U}_1[T_1, T_2]) \det(\widehat{V}_1[T_1, T_2]) = \det(\widehat{U}_2[T_1, T_2]) \det(\widehat{V}_2[T_1, T_2]) \quad (21)$$
$$\forall T_1 \subseteq [n], T_2 \subseteq [m-n] \text{ with } |T_1| = |T_2|.$$

From Eq. 19 and Eq. 21, $P_1 = P_2 \iff A \overset{\text{PME}}{=} B$. Note that A and B can be computed in poly(m, n) time. From Theorem 1.1, we can check whether $A \stackrel{\text{PME}}{=} B$ in poly(m) time. This completes the proof of Theorem 1.3 when A_0 and B_0 are the zero matrix.

For the case when there are no constraints on A_0 and B_0, the proof of Theorem 1.3 follows a similar idea with some additional tools. For details, see full version [10, Subsection 5.2]

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