Parallel Algorithms for Perfect Matching

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Abstract

The perfect matching problem has a randomized NC-algorithm based on the Isolation Lemma of Mulmuley, Vazirani and Vazirani. We give an almost complete derandomization of the Isolation Lemma for perfect matchings in bipartite graphs. This gives us a deterministic quasi-NC-algorithm for the bipartite perfect matching problem.

The outline presented here emphasizes a geometric point of view. We think that this will be useful also for the perfect matching problem in general graphs.

1 Introduction

The perfect matching problem, PM, asks whether a given graph contains a perfect matching. The problem has a polynomial-time algorithm due to Edmonds [Edm65]. However, its parallel complexity is still not completely resolved. The problem can be solved by randomized efficient parallel algorithms due to Lovász [Lov79], i.e., it is in RNC. However, it is not known whether randomness is necessary, i.e., whether it is in NC.

The construction version of the problem, Search-PM, asks to construct a perfect matching in a graph if one exists. This version is also in RNC due to Karp, Upfal, and Wigderson [KUW86] and Mulmuley, Vazirani, and Vazirani [MVV87]. The latter algorithm uses the celebrated Isolation Lemma. It works with a weight assignment on the edges of the graph. A weight assignment is called *isolating* for a graph G if the minimum weight perfect matching in G is unique, if one exists. The Isolation Lemma states that a randomly chosen weight function is isolating with high probability. Given an isolating weight assignment with polynomially bounded integer weights for a graph G, it is easy to construct a perfect matching in G in NC.

Lemma 1.1 (Isolation Lemma [MVV87])**.** *For a graph $G(V,E)$, let $w \in \{1, 2, \ldots, 2|E|\}^E$ be a uniformly random weight assignment on its edges. Then w is isolating with probability $\geq 1/2$.*

Derandomizing this lemma means to construct such a polynomially bounded weight assignment deterministically in NC. This remains a challenging open question. A general version of this lemma, which considers a family of sets and requires a unique minimum weight set, has

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also been studied. The general version is related to the polynomial identity testing problem and circuit lower bounds [AM08].

The Isolation Lemma has been derandomized for some special classes of graphs, e.g., planar bipartite graphs [DKR10, TV12], strongly chordal graphs [DK98], and graphs with a small number of perfect matchings [GK87, AHT07]. Here, we present an almost complete derandomization of the Isolation Lemma for bipartite graphs. In Section 3, we construct an isolating weight assignment for these graphs with quasi-polynomially large weights. As a consequence we get that for bipartite graphs, PM and $\mathsf{Search\text{-}PM}$ are in quasi-NC^2. In particular, they can be solved by uniform Boolean circuits of depth $O(\log^2 n)$ and size $n^{O(\log n)}$ for graphs with n nodes.

Theorem 1.2. *For bipartite graphs,* PM *and* $\mathsf{Search\text{-}PM}$ *are in* quasi-NC^2*.*

There are several ways to prove this result. There is a purely combinatorial proof that uses *Hall's Theorem*, which we discuss in the Appendix. This might be the proof that would be easiest to follow for many readers. However, here, in Section 2 and 3, we present a geometric proof that is based on the *perfect matching polytope* of a graph. We do this for several reasons. First of all, this was the way we got the result. We still think that this is the most intuitive way to understand what is going on. Second, two of the authors [GT16] generalized the result to linear matroid intersection. This generalization is heavily based on the geometric view. Finally, because of the success of the geometric viewpoint in the previous cases, we think that it will also help to solve the ultimate goal: to get the perfect matching problem for general graphs in NC, or quasi-NC. Therefore, we think that it is a good idea to study the perfect matching polytope.

2 Perfect Matching Polytope

We provide definitions and a characterization of the perfect matching polytope in Section 2.1. Then, in Section 2.2, we look at the faces of this polytope and prove Lemma 2.2, the crucial technical result that makes our approach work.

2.1 Definition of the perfect matching polytope

Perfect matchings have an associated polytope, called the *perfect matching polytope*. The perfect matching polytope $\mathrm{PM}(G)$ of a graph $G(V, E)$ is a polytope in the (real) edge space,[1] i.e., $\mathrm{PM}(G) \subseteq \mathbb{R}^E$. For any perfect matching M of G, consider its incidence vector $x^M = (x^M_e)_{e \in E} \in \mathbb{R}^E$ given by

$$x^M_e = \begin{cases} 1, & \text{if } e \in M, \\ 0, & \text{otherwise.} \end{cases}$$

This vector is referred as a *perfect matching point* for any perfect matching M. The *perfect matching polytope* of a graph G is defined to be the convex hull of all its perfect matching points,

$$\mathrm{PM}(G) = \mathrm{conv}\{\, x^M \mid M \text{ is a perfect matching in } G \,\}.$$

The corners of $\mathrm{PM}(G)$ are exactly the perfect matching points of G.

[1] Here we depart from the usual definition of edge space as a vector space over $\mathbb{Z}/2\mathbb{Z}$.

Any weight function $w\colon E \to \mathbb{R}$ on the edges of a graph G can be naturally extended to $\mathbb{R}^E$ as a linear function: for any $x = (x_e)_{e\in E} \in \mathbb{R}^E$, define

$$w(x) = w \cdot x = \sum_{e\in E} w(e)\, x_e.$$

Clearly, for any perfect matching M, we have $w(M) = w(x^M)$. In particular, let M^* be a perfect matching in G of minimum weight. Then

$$w(M^*) = \min\{\, w(x) \mid x \in \mathrm{PM}(G)\,\}.$$

The task to isolate a perfect matching can now be rephrased as: *construct a weight function w such that $w(x)$ has a unique minimum point in the perfect matching polytope* $\mathrm{PM}(G)$.

Cycles play an important role in the context of perfect matchings, and also in our arguments. One reason is that the symmetric difference of two perfect matchings is a union of disjoint cycles. Each of these cycles has edges alternating from the two perfect matchings. For an even cycle $C = (e_1, e_2, \ldots, e_p)$, we define its *circulation vector* $\chi^C = (\chi^C_e)_{e\in E}$ by

$$\chi^C_e = \begin{cases} (-1)^j & \text{if } e = e_j, \text{ for some } 1 \le j \le p, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the definition actually depends on the starting edge e_1. For our purpose, it does not matter which edge of a cycle is chosen as e_1. Let $\delta(v)$ denote the set of edges incident on vertex v. Observe that χ^C satisfies

$$\sum_{e\in\delta(v)} \chi^C_e = 0 \qquad v \in V. \tag{1}$$

Now we come back to the perfect matching polytope. It is well known that for bipartite graphs, it has a simple characterization in terms of linear inequalities.

Lemma 2.1 (See [LP86]). *Let $G(V,E)$ be a bipartite graph and $x \in \mathbb{R}^m$. Then $x \in \mathrm{PM}(G)$ if and only if*

$$\sum_{e\in\delta(v)} x_e \;=\; 1 \qquad v \in V, \tag{2}$$

$$x_e \;\ge\; 0 \qquad e \in E. \tag{3}$$

Proof. Let Q be the polytope described by (2) and (3). Clearly, any perfect matching point is in Q. Thus, $\mathrm{PM}(G) \subseteq Q$. Note also that any integral point $x \in Q$ is a perfect matching point. The non-trivial part of the lemma is to show that all corners of the polytope Q are integral. We argue that any non-integral point in Q is not a corner.

Let $x \in Q$ be a non-integral point. It follows from (2) and (3) that for any $x \in Q$, we have $x_e \le 1$, for each $e \in E$. Hence, there exists an edge $e_1 \in E$ such that $0 < x_{e_1} < 1$. Let e_1 be incident on a vertex v_1. As x satisfies (2), there must be another edge e_2 incident on v_1 such that $0 < x_{e_2} < 1$. Let v_2 be the other end point of e_2. Then there must be another edge e_3 incident on v_2 such that $0 < x_{e_3} < 1$. We can keep finding such neighboring edges until we get an edge we have already seen. This will give us a cycle C such that $0 < x_e < 1$, for each $e \in C$.

Now, choose an $\varepsilon > 0$ such that $\varepsilon \leq x_e$, for each $e \in C$. Let χ^C be the circulation vector of C. We define two new points $y = x + \varepsilon\chi^C$ and $z = x - \varepsilon\chi^C$. By the choice of ε, we have $y_e, z_e \geq 0$, for all $e \in E$. As χ^C satisfies (1), y and z also satisfy (2). Thus, $y, z \in Q$. As x is the mid-point of the two points $y, z \in Q$, it cannot be a corner of Q. $\square$

Note that for an even cycle C, by equation (1) its circulation vector χ^C lies parallel to the hyperplane defined by (2).

For general graphs, the polytope described by (2) and (3) can have vertices which are not perfect matchings. Thus, the description does not capture the perfect matching polytope for general graphs.

2.2 Faces of the Perfect Matching Polytope

Since $w(x)$ is a linear function, the points in $\mathrm{PM}(G)$ that minimize $w(x)$ will form a face of $\mathrm{PM}(G)$. Note that a corner is also a face. The corners of this minimizing face will all correspond to minimum weight perfect matchings. Any face of a polytope can be obtained by replacing some of the inequalities in its description by equalities. In the case of the perfect matching polytope, these inequalities are just the non-negativity constraints (3). Thus, by Lemma 2.1, for any face F of $\mathrm{PM}(G)$, there exists a set $S \subseteq E$ of edges such that F is described by (2) and (3), and $x_e = 0$ for $e \in S$.

Now, for any weight function w, let F_w be the face of $\mathrm{PM}(G)$ minimizing $w(x)$. Let

$$S_w = \{\, e \in E \mid F_w \text{ satisfies } x_e = 0 \,\}.$$

Intuitively, the edges in S_w do not participate in any minimum weight perfect matching with respect to w. Define $E_w = E - S_w$ and $G_w = (V, E_w)$. Hence, G_w is the subgraph of G that contains only those edges that participate in some minimum weight perfect matching in G.

The following lemma is crucial for our weight construction. It shows that for any cycle C in G_w, its circulation vector χ^C lies parallel to face F_w, which implies $w(\chi^C) = 0$.

Lemma 2.2. *Let w be a weight function on the edges of a graph G. Let C be a cycle in the subgraph G_w. Then $w(\chi^C) = 0$.*

Proof. Recall that F_w is described by (2) and (3), and $x_e = 0$, for $e \in S_w$. Observe that χ^C also satisfies $x_e = 0$ for all $e \in S_w$, and furthermore, $\sum_{e \in \delta(v)} \chi^C_e = 0$ for all $v \in V$ by equation (1). Therefore χ^C lies parallel to F_w.

By definition, all points $x \in F_w$ have the same weight, i.e., $w(x) = c_0$, for some constant c_0. Hence, vector w is orthogonal to the hyperplane F_w. We conclude that w is also orthogonal to χ^C, and therefore $w(\chi^C) = 0$. $\square$

3 Constructing an Isolating Weight Assignment

We will construct the weight function in rounds. In every round, we slightly modify the current weight function to get a smaller minimizing face. In more detail, if w_i is the weight function in the i-th round, then in the next round, we will consider the weight function $w_{i+1} = Nw_i + w'$, for some weight function w' and a number N which is larger than the weights in w'. Note that Nw_i gets precedence over w' and thus, the face $F_{w_{i+1}}$ is contained in F_{w_i}. We stop when the minimizing face is just a single point. Then the weight function isolates a unique minimum weight perfect matching.

To argue that the number of rounds is small, we use the cycle circulation vectors. Suppose C is a cycle in graph G_{w_i}. From Lemma 2.2, we have $w_i(\chi^C) = 0$. Now, we update the weight function such that $w_{i+1}(\chi^C) \neq 0$. Then again from Lemma 2.2, we get $C \notin G_{w_{i+1}}$, i.e., at least one edge of C is missing from $E_{w_{i+1}}$. This gives us a way to destroy cycles. Note that if $F_{w_{i+1}} \subseteq F_{w_i}$, then $E_{w_{i+1}} \subseteq E_{w_i}$. The strategy is to keep eliminating cycles until we obtain a w such that G_w has no cycles. We claim that F_w will be a point then. Because if not, then F_w has at least two perfect matching points and union of two perfect matchings gives a set of cycles. Clearly, these cycles belong to G_w.

It is not clear whether one can construct a weight function w with small weights which ensures $w(\chi^C) \neq 0$ for all cycles C. However, there are standard ways to construct such a weight function for any small set of cycles, see for example [FKS84].

Lemma 3.1. *For any set $\mathcal{C}$ of s cycles in graph G, one can find a weight function w with weights bounded by $n^2 s$, such that $w(\chi^C) \neq 0$, for any $C \in \mathcal{C}$.*

Proof Idea. Let $e_1, e_2, \ldots, e_m$ be the edges of G. Define a weight function w by $w(e_i) = 2^{i-1}$, for $i = 1, 2, \ldots, m$. One can show that one of the functions $\{\, w \bmod j \mid 2 \leq j \leq n^2 s \,\}$ has the desired property. $\square$

We will start with small length cycles, whose number is small. In each round, we will double the cycle lengths we handle. We will show that we work with at most n^4 cycles in one round. Thus, the weights we get from Lemma 3.1 will be bounded by n^6. Consider the following weighting scheme. Let $N > n^6$ and $k = \log n - 2$. Let w_0 be a weight function such that $w_0(\chi^C) \neq 0$ for all cycles C of length 4 in G. For $i = 1, 2, \ldots, k-1$, define

w_i': a weight function such that $w_i'(\chi^C) \neq 0$ for all cycles C in $G_{w_{i-1}}$ of length $\leq 2^{i+2}$.

w_i: $N w_{i-1} + w_i'$.

From Lemma 2.2, there are no cycles of length $\leq 2^{i+2}$ in G_{w_i}. Thus, G_{w_k} has no cycles and F_{w_k} is a point.

Lemma 3.2. *Weight function w_k is isolating.*

By the construction, the weights in w_k are bounded by $O(n^{6 \log n})$. The following lemma bounds the number of cycles in each round.

Lemma 3.3. *Let H be a graph with n nodes that has no cycles of length $\leq r$, for some even $r \geq 4$. Then H has $\leq n^4$ cycles of length $\leq 2r$.*

Proof. Let C be a cycle of length $\leq 2r$ in G. We choose 4 vertices u_0, u_1, u_2, u_3 on C which divide it into 4 almost equal parts. We *associate* the tuple (u_0, u_1, u_2, u_3) with C. We claim that C is the only cycle associated with (u_0, u_1, u_2, u_3). For the sake of contradiction, let there be another such cycle C'. Let $p \neq p'$ be paths of C and C', respectively, that connect the same u-nodes. As the four segments of C are of equal length, we have $|p| \leq r/2$ and similarly $|p'| \leq r/2$. Thus p and p' create a cycle of length $\leq r$, which is a contradiction. Hence, a tuple is associated with only one cycle. The number of tuples of four nodes is bounded by n^4, and so is number of cycles of length $\leq 2r$. $\square$

4 Extending the Technique

We discuss some settings where a similar approach works, respectively might work.

4.1 b-factors

A b-factor is a generalization of perfect matching. For a graph $G(V, E)$ and a vector $b \in \mathbb{N}^V$, a b-*factor* is a set of edges such that vertex v has exactly b_v edges incident to it. Note that a 1-factor is a perfect matching.

One can generalize our approach to isolate a b-factor in bipartite graphs. There is a simple reduction from b-factors to perfect matching which works in NC (see, for example, [LP86, Section 10.1]). However, by directly constructing an isolating weight assignment for b-factors, we solve the problem in a *black-box* way, i.e., without looking at the given graph.

For bipartite graphs, the b-factor polytope is similar to the perfect matching polytope and is given by

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &= b_v \qquad v \in V, \\ x_e &\geq 0 \qquad e \in E, \\ x_e &\leq 1 \qquad e \in E. \end{aligned}$$

Note that we have a new inequality here, namely $x_e \leq 1$. Thus, a face is defined by equalities of the kind $x_e = 0$ or $x_e = 1$. Now, let us follow an approach similar to the perfect matching case. For a weight function w, define S_w to be the set of edges for which F_w satisfies $x_e = 0$ or $x_e = 1$. Define G_w again as $(V, E - S_w)$. We take the same definition of the circulation vector of a cycle. The rest of the arguments work essentially in the same way. In particular, we get an analog of Lemma 2.2. Note also that just like perfect matchings, the union of two b-factors gives a set of cycles.

4.2 Matroid intersection

Another generalization of bipartite matching is matroid intersection. Here, we are given two matroids on the same ground set and we are interested in common base sets. As mentioned in the Introduction, two of the authors [GT16] generalized the current approach to isolate a common base set of the two given matroids. The description of the common base polytope requires exponentially many constraints. Still, one can find 'nice' descriptions for its faces and that is the non-trivial part. The next step is to find an appropriate definition for the circulation vectors. Basically, the circulation vectors should be defined in a way so that they lie parallel to a given face. After this, following a similar line of argument gives an isolating weight function.

4.3 Perfect matching in general graphs

The case of perfect matching in general graphs is intriguing. The perfect matching polytope requires exponentially many constraints to characterize it. Together with (2) and (3), we have a constraint for each odd cut,

$$\sum_{e \in E(T, \overline{T})} x_e \geq 1 \qquad T \subseteq V \text{ is an odd subset of size} \geq 3.$$

To describe a face here, we have equations of the kind $x_e = 0$ and $\sum_{e \in E(T, \overline{T})} x_e = 1$.

The crucial point is how to define a circulation vector. These must be some chosen vectors which lie parallel to the current minimizing face. Their number needs to be small enough so that we can find a weight function assigning nonzero values to them. On the other hand, there number should be large enough so that we get a significant reduction in the dimension of the face. It is not clear if one can define such circulation vectors.

It will be interesting to know what are the most general polytopes for which our isolation technique will work.

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A An Alternative Proof of Lemma 2.2

Here we discuss an alternate, combinatorial proof of our key lemma, Lemma 2.2. This proof, using Hall's theorem, was found by Rao, Shpilka, and Wigderson and was first reported in [GG15]. Afterwards, we compare their proof with ours.

Lemma A.1. *Let $G(V, E)$ be a d-regular bipartite multigraph. Then the edges of G can be partitioned into d many perfect matchings.*

Proof. By induction on d. For $d = 1$, the edges E form precisely one perfect matching. Now let $d > 1$. Then by the pigeonhole principle, G satisfies Hall's criterion; that is, the neighborhood of any set of k vertices on one side of the bipartition has size $\geq k$. Thus G has a perfect matching M by Hall's theorem. Removing M from G results in a $(d-1)$-regular bipartite multigraph. □

Alternate proof of Lemma 2.2. Let q be the minimum weight of a perfect matching in G, and suppose that G has $d > 0$ many minimum weight perfect matchings. Let G'_w be the bipartite multigraph obtained by taking the *disjoint* union of these perfect matchings. Then G'_w is a d-regular bipartite multigraph with the same edges as G_w except for multiplicity. The total weight of G'_w is dq.

Suppose G'_w has a cycle C with $w(\chi^C) \neq 0$. Coloring the edges along C alternately red and blue, the two colors of edges have different total weight. Suppose that the red edges outweigh the blue edges. Let G''_w be the multigraph obtained from G'_w by removing all the red edges and adding a single duplicate of each blue edge. Then G''_w is also a d-regular bipartite multigraph but with total weight $< dq$. By Lemma A.1, graph G''_w is the disjoint union of d many perfect matchings, but now at least one of these matchings must have weight $< q$, and this matching is also a perfect matching of G. Hence, we have a contradiction. □

The two proofs of Lemma 2.2 relate closely to each other. Suppose $x_1, \ldots, x_d$ are the perfect matching points in $\mathbb{R}^E$ corresponding to the minimum-weight perfect matchings of G. Let $s = \sum_{i=1}^{d} x_i$ be the sum of these points. Note that s is the characteristic vector of the multigraph G'_w in the proof above. Let $c = s/d$ be the centroid of these points. The point c clearly lies in the face F_w.

Suppose we orient C so that $\chi^C_e = -1$ for red edges e and $\chi^C_e = +1$ for blue edges e. This would make $w(\chi^C) < 0$ by assumption. Then swapping red edges for blue ones along C amounts to displacing s by χ^C. That is, G''_w has characteristic vector $s + \chi^C$. Dividing this vector by d, we get $c' = c + \frac{1}{d}\chi^C$. All constraints of F are satisfied by c', so $c' \in F$. On the other hand, $w(c') = w(c) - w(\chi^C) < w(c)$, so c' lies outside F. This contradiction essentially comes from the fact that χ^C was assumed not to be parallel to F_w.