A Statistical Model for Smooth Shapes in Kendall Shape Space

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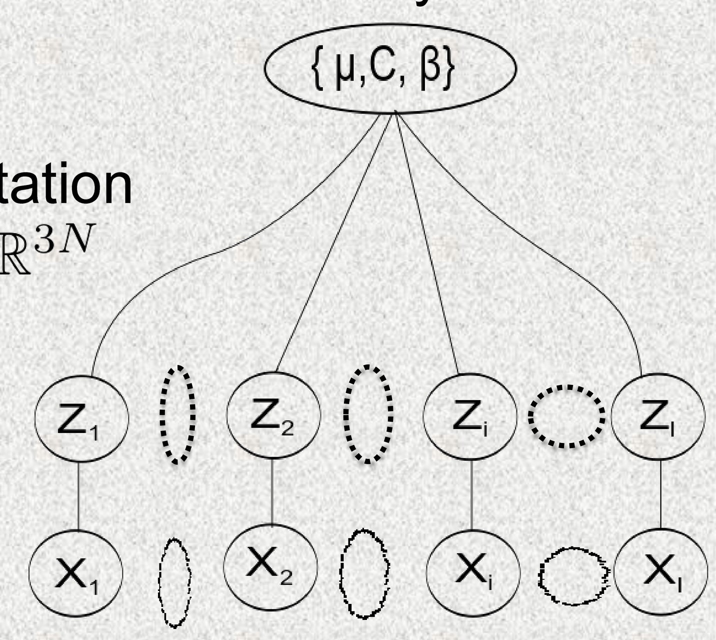
Motivation

- We want to learn the statistical variability of shapes of objects within a population [*Cootes 1995 CVIU, Goodall 1991*] automatically from given (imperfect) object segmentations
  - Such models have many uses, e.g., as priors in object segmentation, for hypothesis testing, for classification, etc.
- Object *shape* is an equivalence class of object boundaries, where the equivalence relation is given by similarity transformations
- Represent shape by a set of points placed near object boundary. Pointsets, representing objects shapes, have equal cardinality. This calls for analysis in *Kendall shape space* [*Kendall 1989*]
- We extend the state of the art on pointset-based statistical shape modeling by:
  - Using a Riemannian statistical model, in Kendall shape space, capturing population shape mean and covariance
  - Using a model that enforces spatial smoothness on shapes
- We use a computationally-efficient shape dissimilarity between object shape and object segmentation / boundary
- We fit the model by Monte Carlo expectation maximization (EM), relying on sampling (smooth) shapes in Kendall shape space

Proposed Statistical Modeling Scheme

- Shape in 3D represented using a pointset of cardinality *N*
- Data pointset for individual i is $\mathbf{X}_i$
  = a set of points on object boundary obtained from a given object segmentation
- Shape pointset for individual i is $\mathbf{Z}_i \in \mathbb{R}^{3N}$
  This is unknown.
- Group mean is $\mu \in \mathbb{R}^{3N}$
- Group covariance is $C \in \mathbb{R}^{3N \times 3N}$
- Smoothness prior on individual shape controlled by parameter β



- Shapes constrained to have centroid at origin and unit norm
- Let $\bar{\mathbf{X}} := \{\mathbf{X}_i\}_{i=1}^{I}$ and $\bar{\mathbf{Z}} := \{\mathbf{Z}_i\}_{i=1}^{I}$
- Probability density function of $\bar{\mathbf{X}}$ contains two parts:
  - Likelihood term: conditional probability of $\mathbf{X}_i$ given $\mathbf{Z}_i$
  - Prior term: probability distribution of $\mathbf{Z}_i$ given μ, C, β

$$P(\bar{\mathbf{X}}|\mu, C, \beta) = \prod_{i=1}^{I} P(\mathbf{X}_i|\mu, C, \beta) = \prod_{i=1}^{I} \int P(\mathbf{X}_i, \mathbf{Z}_i|\mu, C, \beta) d\mathbf{Z}_i$$

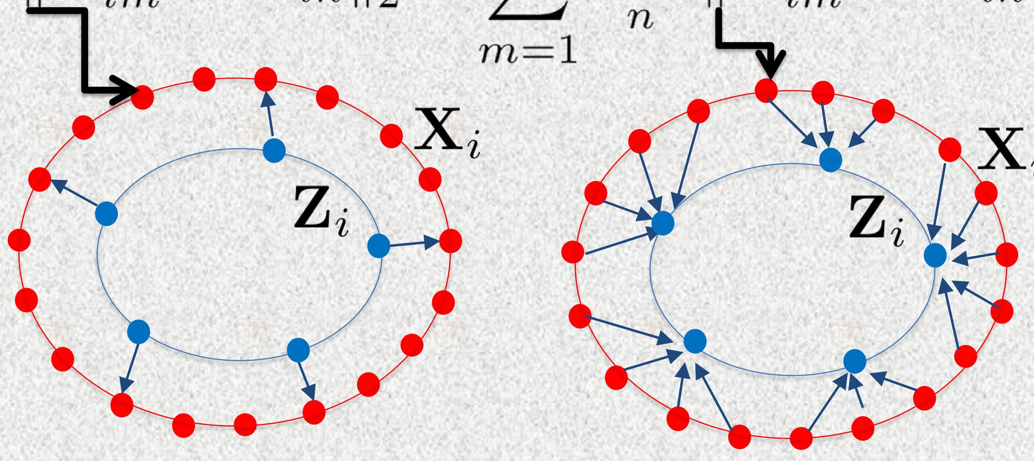
$$= \prod_{i=1}^{I} \int \underbrace{P(\mathbf{X}_i|\mathbf{Z}_i)}_{\text{Likelihood term}} \underbrace{P(\mathbf{Z}_i|\mu, C, \beta)}_{\text{Prior term}} d\mathbf{Z}_i$$

Data Likelihood and Alignment

- Proposed model uses a new shape-dissimilarity measure $\delta(\mathbf{X}_i, \mathbf{Z}_i)$ between data / boundary pointset $\mathbf{X}_i$ and shape pointset $\mathbf{Z}_i$ which are of different cardinalities
- This measure handles disoriented data, by aligning the data pointset to the shape pointset, with respect to translation *t*, rotation *R*, and scale *s*

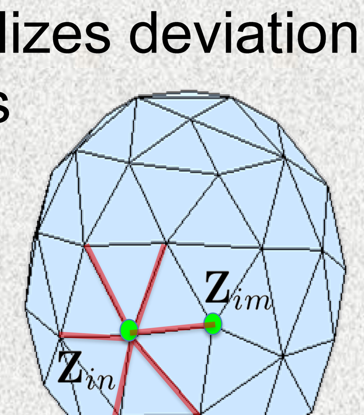
$$\delta(\mathbf{X}_i, \mathbf{Z}_i) := \min_{s,R,t} \left(\sum_{n=1}^{N} \min_m \|\mathbf{X}_{im}^{sRt} - \mathbf{Z}_{in}\|_2^2 + \sum_{m=1}^{M_i} \min_n \|\mathbf{X}_{im}^{sRt} - \mathbf{Z}_{in}\|_2^2 \right)$$

- We model
  $P(\mathbf{X}_i|\mathbf{Z}_i)$
  $:= \exp(-\delta(\mathbf{X}_i, \mathbf{Z}_i))/\gamma$



Spatial Smoothness on Shapes

- Proposed Markov prior for shape smoothness penalizes deviation of each point's location from it's neighbors' locations
- Let $\mathcal{N}_n$ = Markov neighborhood system
- Gibbs energy $\sum_n \sum_{m \in \mathcal{N}_n} 0.5\beta \parallel \mathbf{Z}_{in} - \mathbf{Z}_{im} \parallel_2^2$
- Gibbs energy can be written as the quadratic form $0.5\mathbf{Z}_i^T \Omega \mathbf{Z}_i$ where Ω is a sparse matrix



Statistical Model on Smooth Shapes

- Proposed model captures the mean and covariance structure of a population of smooth shapes
- It relies on an approximate Normal law [*Pennec 2006 JMIV*] in the tangent space of Kendall shape space

$$P(\mathbf{Z}_i|\mu, C, \beta) := \frac{1}{\eta(\mu, C, \beta)} \exp\left(-\frac{1}{2}(\mathrm{Log}_\mu(\mathbf{Z}_i))^T C^{-1} \mathrm{Log}_\mu(\mathbf{Z}_i) - \frac{1}{2}\mathbf{Z}_i^T \Omega \mathbf{Z}_i\right)$$

- We can approximate this by a Gaussian with mean μ, covariance $C_{\text{reg}} = (C^{-1} + \Omega)^{-1}$

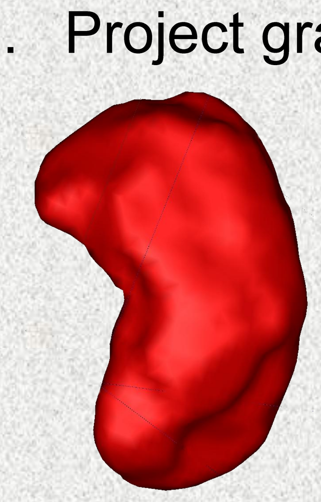
Model Fitting using Monte Carlo EM

- Optimize parameters to maximize $P(\bar{\mathbf{X}}|\{\mu, C\})$
- E step at iteration *t*
  $Q(\{\mu, C\}; \{\mu^t, C^t\}) := E_{P(\bar{\mathbf{Z}}|\bar{\mathbf{X}}, \{\mu^t, C^t\})}[-\log P(\bar{\mathbf{X}}, \bar{\mathbf{Z}}|\{\mu, C\})]$
- M Step at iteration *t*
  $\{\mu^{t+1}, C^{t+1}\} := \arg\min_{\mu, C} Q(\{\mu, C\}; \{\mu^t, C^t\})$
- In E step, approximate expectation by simulating shape pointsets $\bar{\mathbf{Z}}^s \sim P(\bar{\mathbf{Z}}|\bar{\mathbf{X}}, \{\mu^t, C^t\})$ to get
  $Q(\{\mu, C\}; \{\mu^t, C^t\}) \approx \frac{1}{S} \sum_{s=1}^{S} -\log P(\bar{\mathbf{X}}, \bar{\mathbf{Z}}^s|\{\mu, C\})$

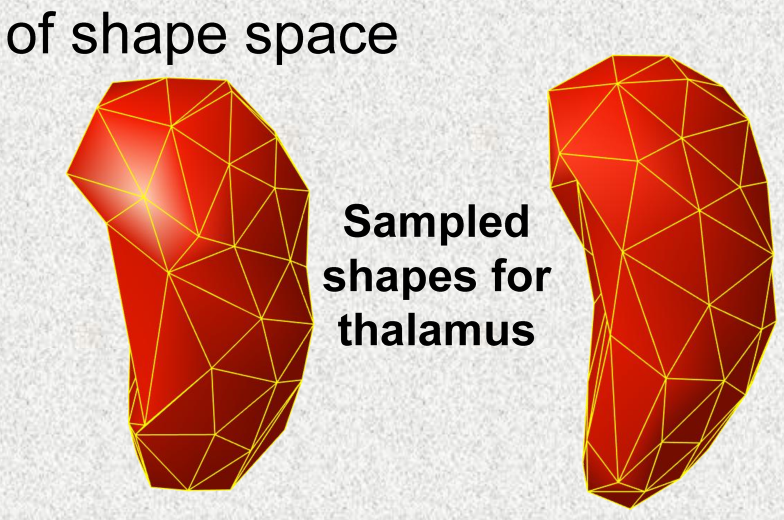
Sampling Shapes in Kendall Shape Space

- Sample shape pointsets $\{\mathbf{Z}_i^s\}_{s=1}^{S}$ from posterior $P(\mathbf{Z}_i|\mathbf{X}_i, \{\mu^t, C^t\})$ by using Hamiltonian Monte Carlo (HMC) in Kendall shape space
- Sample shapes by first sampling in tangent space of shape space at μ, and then taking the exponential map
- HMC needs gradient of log posterior with respect to shape $\mathbf{Z}_i$

1. Compute gradient with respect to pointset $\mathbf{Z}_i$
2. Project gradient on tangent space of shape space



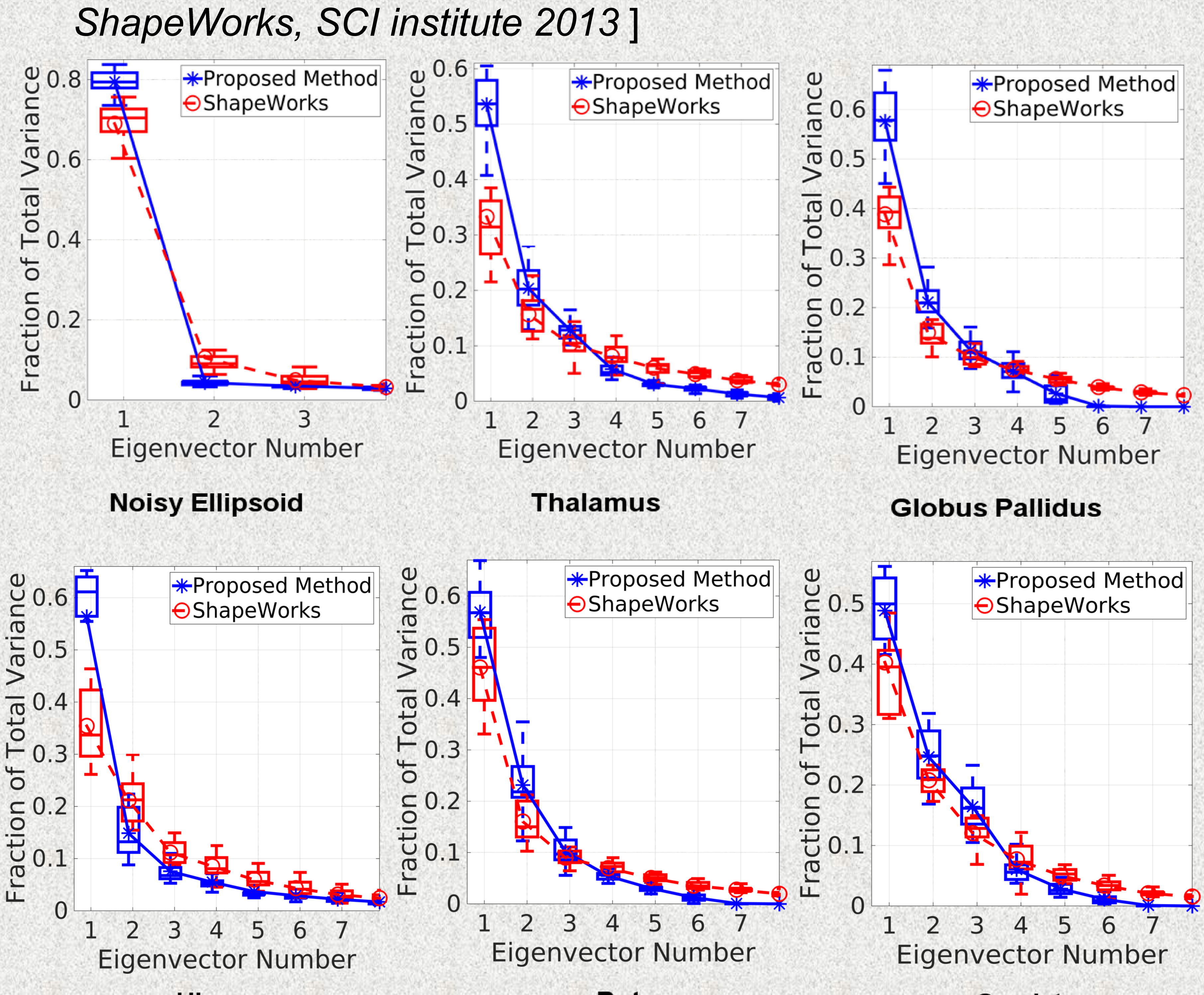
Thalamus segmentation from MR image



Sampled shapes for thalamus

Results

- Simulated data: Imperfect segmentations of 40 ellipsoids
- Clinical data: Subcortical structure segmentations in brain MRI
- Results show that the proposed framework learns more compact statistical models as compared to the state of the art [*ShapeWorks, SCI institute 2013*]



Noisy Ellipsoid **Thalamus** **Globus Pallidus** **Hippocampus** **Putamen** **Caudate**

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