A Transformational Methodology for Proving Termination of Logic Programs[1]

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▷ A methodology for proving the termination of well-moded logic programs is developed by reducing the termination problem of logic programs to that of term rewriting systems. A transformation procedure is presented to derive a term rewriting system from a given well-moded logic program such that the termination of the derived rewrite system implies the termination of the logic program for all well moded queries under a class of selection rules. This facilitates applicability of a vast source of termination orderings proposed in the literature on term rewriting, for proving termination of logic programs. The termination of various benchmark programs has been established using this approach. Unlike other mechanizable approaches, the proposed approach does not require any preprocessing and works well even in the presence of mutual recursion. The transformation has also been implemented as a front-end to Rewrite Rule Laboratory (RRL) and has been used in establishing termination of nontrivial Prolog programs such as a prototype compiler for **ProCoS**ś, PL_0 language. ◁

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1. Introduction

Termination is an important property of imperative as well as declarative programs and proving termination is one of the main steps in arriving at a sound methodology and for proving the correctness of programs. Recently, termination of logic programs has attracted a lot of attention and many approaches are reported in the literature – see De Schreye and Decorte [14] for a comprehensive survey. In this paper, we present a transformational approach for proving termination of logic programs by reducing the termination problem of logic programs to that of term rewriting systems. The termination problem of term rewriting systems has been well studied and many useful techniques and tools have been developed for proving termination of term rewriting systems. The prime motivation of our approach is to facilitate the use of this vast source of termination techniques and tools in proving termination of logic programs.

Before describing our method, let us discuss the differences between the paradigms of logic programming and term rewriting and see why termination techniques of rewriting cannot be adapted for logic programs in a straightforward way.

1. Unification is the basic step in the computations of logic programs whereas matching plays a similar role in term rewriting. The backward propagation of substitutions due to unification complicates termination analysis of logic programs and the termination techniques of rewriting are not directly applicable to logic programs (e.g., a logic program containing a clause $\mathtt{p(f(X))} \leftarrow \mathtt{p(X)}$ does not terminate for query $\leftarrow \mathtt{p(Y)}$ whereas the corresponding term rewriting system containing rule $\mathtt{p(f(X))} \rightarrow \mathtt{p(X)}$ terminates on all terms).
2. Logic programs have local variables (variables which occur in the body but not in the head of a clause) playing the crucial role of sideways information passing, while in term rewriting literature, it is generally assumed that all the variables in the right hand side term also occur in the left hand side term of the rewrite rules. The reason for avoiding extra variables on the right-hand sides of rewrite rules is that they trivially lead to nontermination. Almost all the termination techniques of rewriting works only under this restriction.
3. In general, logic programs are not directed, in the sense that there is no notion of *input* and *output*. A variable (or an argument position) can be used as either input or output; for example, with the factorial program, it is possible to ask 'what is factorial of 6 ?' ($\leftarrow \mathtt{factorial(6,0)}$) as well as 'what is the value of $\mathtt{I}$ if the factorial of $\mathtt{I}$ is 720 ?' ($\leftarrow \mathtt{factorial(I,720)}$). In fact, this invertibility is often seen as the principal difference between logic and functional programming paradigms. Term rewriting is directional in the sense that left-hand side terms are replaced by the corresponding right-hand side terms.

We present a transformation procedure to derive a term rewriting system from a given logic program such that termination of the derived term rewriting system implies termination of the logic program and thereby reduce the termination problem of logic programs to that of term rewriting systems. To get the directionality, we assume that every predicate has an associated 'mode' specifying which arguments are 'input' and which are 'output' and consider the class of well-moded programs so that input terms of selected atoms are always ground and there is no backward propagation through input positions. The transformation removes the *local variables* present in the logic programs through a kind of Skolemization procedure using mode information while deriving term rewriting systems. The absence of backward propagation through input positions and the local variables in the derived term rewriting systems facilitate the applicability of termination techniques of rewriting in proving termination of logic programs. The transformation derives the term rewriting system in an incremental fashion by transforming each clause in the program into a set of rewrite rules. In section 5, we establish that the given logic program terminates for all well-moded queries under a class of selection rules if the derived term rewriting system is terminating. To summarize, our method consists of two steps: (i) transforming the given well-moded logic program into a term rewriting system and then (ii) proving termination of the resulting rewrite system using various techniques available in the literature on term rewriting systems.

The transformation procedure is purely syntactical and has been implemented as a front-end to RRL – a theorem prover based on rewrite techniques – which supports techniques such as recursive path ordering for proving termination of term rewriting systems in an interactive fashion. The tool developed has been used in establishing termination of a prototype compiler for **ProCoS** PL_0 language. This compiler has been developed using Hoare's refinement algebra approach. Refinement algebra provides elegant proofs for partial correctness (ensuring that the compiler only generates correct code) of compilers developed in this approach. A proof of termination ensures that the complier indeed generates an output (object code). In this respect, our tool plays an important role in the development of provably correct compilers. The fact that termination of this compiler cannot be established by the other mechanizable approaches available in the literature demonstrates the practicality of our approach.

We follow the notations of Lloyd [29] and Apt [2] for logic programming concepts and Dershowitz and Jouannaud [19] for rewriting concepts. The rest of the paper is organized as follows. In section 2, we give definitions of well-modedness and related concepts. In section 3, the transformation of logic programs into term rewriting systems is explained through examples. Section 4 provides a formal description of the transformation procedure. In section 5, we prove that the termination of the derived term rewriting system implies the termination of the logic program for well-moded queries; a brief review of important termination techniques of rewrite systems is also provided. Section 6 briefly discusses the automation of termination proofs of logic programs using our approach. The paper concludes by a comparative evaluation of the methods in section 7.

2. Preliminaries

2.1. Well-modedness and Related Concepts

In this section, we define the notion of well-moded logic programs (queries) and prove some properties of well-moded programs (queries). The moding information essentially specifies which arguments are input arguments and which are output arguments in a predicate.

Definition 1. A *mode* m of an n-ary predicate p is a function from $\{1, \cdots, n\}$ to the set $\{in,\ out\}$. The set $\{i \mid m(i) = in\}$ is the set of input positions of p and $\{o \mid m(o) = out\}$ is the set of output positions of p.

Notation: The terms $invar(L)$ and $outvar(L)$ denote the sets of variables occurring in the input and output positions of a literal L respectively and $Var(L) = invar(L) \cup outvar(L)$.

Remark 1. It may be noted that some predicates may be used in different modes in a single program. We use different subscripts to a predicate to differentiate between different modings (usages).

In the rest of the paper, we assume that the moding information of all the predicates is available. However, this does not mean that the programmer has to supply this information as there are many techniques available in the literature (e.g., [15]) for deriving moding information from a given logic program.

The notion of well-moded programs has been invented to constrain the "flow of data" and thereby get SLD-derivations with certain desirable properties. One of these desirable properties is the data-drivenness of computations, i.e., input terms of every selected atom are ground. Since groundness of input terms of the selected atom depends on the selection rule employed, two alternatives are possible: (i) to fix a selection rule and consider the class of programs for which the input terms of the selected atom are ground under this selection rule and (ii) to consider the class of programs for which there exists at least one selection rule such that the input terms of the selected atom are ground. Obviously, the class of programs considered in the second alternative is larger than the class considered in the first alternative. Here, we adopt the second alternative and define the notion of well-modedness independent of the selection rule using the concepts of producers and consumers. Then, we give a characterization of the class of selection rules suitable for execution of any given well-moded program (i.e., computations under those selection rules are data-driven). De Schreye and Decorte [14] calls our well-moded programs *well-moded** programs to differentiate between the two notions of well-moded programs.

Definition 2. Let $A \leftarrow B_1, \cdots, B_k$ be a clause and X be a variable occurring in B_i. The atom B_i is a *consumer* of X if $X \in invar(B_i)$, otherwise B_i is a *producer* of X (i.e., if $X \in Var(B_i) - invar(B_i)$). The head A is a *producer*[3] of variable X if $X \in invar(A)$, and A is a *consumer* of X if $X \in Var(A) - invar(A)$.

Remark 2. Note that we say that B_i is a *producer* of X if $X \in Var(B_i) - invar(B_i)$ rather than $X \in outvar(B_i)$ since we have to consider the possibility of a variable, X, occurring in input as well as output positions of an atom, say A. We resolve the conflict in the situation by saying that A is a consumer of X as we want all the input terms to be ground at the time of selection; in other words, X should have another producer which binds X to a ground term before A is selected.

Definition 3. The producer-consumer relation of a clause c: $A \leftarrow B_1, \cdots, B_k$ is defined as $\{ \langle B_i,\ B_j \rangle \mid B_i$ and B_j are producer and consumer of a variable X in c respectively $\}$.

Definition 4. (Well-moded programs and queries)
A clause c is *well-moded* if (a) its producer-consumer relation is *acyclic* and (b) every variable in c has at least one producer. A program P is *well-moded* if every clause in it is well-moded. A *well-moded query* is nothing but a well-moded clause without head.

The following example illustrates these definitions.

Example 1. Consider the following quick-sort program; here, $\leq$ and $<$ are the built-in's with the usual moding information.

```
moding:  q (in, out);  s (in, in, out, out) and a (in, in, out)

1.   q(nil,nil) ←
2.   q(c(H,L),S) ← s(L,H,A,B),q(A,A1),q(B,B1),a(A1,c(H,B1),S)
3.   s(nil,Y,nil,nil) ←
4.   s(c(X,Xs),Y,c(X,Ls),Bs) ← X ≤ Y,s(Xs,Y,Ls,Bs)
5.   s(c(X,Xs),Y,Ls,c(X,Bs)) ← X > Y,s(Xs,Y,Ls,Bs)
6.   a(nil,X,X) ←
7.   a(c(H,X),Y,c(H,Z)) ← a(X,Y,Z)
```

Except for the second clause, the producer-consumer relation of all other clauses is empty. For the second clause, it is $\{ \langle \mathtt{s(L,H,A,B)},\ \mathtt{q(A,A1)} \rangle,\ \langle \mathtt{s(L,H,A,B)},\ \mathtt{q(B,B1)} \rangle,\ \langle \mathtt{q(A,A1)},\ \mathtt{a(A1,c(H,B1),S)} \rangle,\ \langle \mathtt{q(B,B1)},\ \mathtt{a(A1,c(H,B1),S)} \rangle \}$. It is easy to see that for every clause, (i) the producer-consumer relation is acyclic and (ii) all the variables in it have at least one producer. So the program is well-moded. □

[3] Our notion of *producer* is similar to the notion of *generator* used in Conery and Kibler [10] for studying AND/OR parallelism in logic programming.

Some simple properties of well-moded programs

In this subsection, we prove some simple properties of well-moded programs.

Lemma 1. If $H \leftarrow B_1, \cdots, B_n$ is a well-moded clause and X is a variable in $outvar(H)$, then (a) $X \in invar(H)$ or (b) there exist a B_i in the body such that $X \in outvar(B_i)$.

Proof : By definition 4, every variable in a well-moded clause has at least one producer, i.e., either (a) H is a producer of X or (b) some B_i is a producer of X. By definition 2, H is a producer of X only if $X \in invar(H)$ and B_i is a producer of X only if $X \in (Var(B_i) - invar(B_i)) = (outvar(B_i) - invar(B_i))$. □

Lemma 2. If $H \leftarrow$ is a well-moded unit clause, then $outvar(H) \subseteq invar(H)$.

Proof : Follows from definitions 2, 4, and the above lemma. □

Since the producer-consumer relation of a well-moded clause c: $H \leftarrow B_1, \cdots, B_n$ is *acyclic*, it defines a partial order $<$ on the atoms in the body of c as follows: $B_i < B_j$ if $\langle B_i,\ B_j \rangle$ is in the producer-consumer relation of c.

Definition 5. An element $a \in A$ is *minimal* in the poset $(A, \leq)$ if $\forall b \in A,\ b \not< a$.

Lemma 3. If c : $H \leftarrow B_1, \ldots, B_n$ is a well-moded clause and B_i is a minimal element under the partial order induced by the producer-consumer relation of c, then $invar(B_i) \subseteq invar(H)$.

Proof : Since B_i is a minimal element under the partial order defined by the producer-consumer relation of c, there is no pair $\langle B_j,\ B_i \rangle$ in the producer-consumer relation. Therefore, input variables of B_i (if any) are not produced by other atoms in the body. Hence, they should be produced by H; that is, $invar(B_i) \subseteq invar(H)$. □

Lemma 4. If $\leftarrow B_1, \ldots, B_n$ is a well-moded query and B_i is a minimal element under the partial order defined by the producer-consumer relation, then the input terms of B_i are ground.

Proof : By definition 4, well-moded query is a well-moded clause without head and by the above lemma $invar(B_i) \subseteq invar(head) = \phi$. Therefore the set of input variables of B_i is empty and hence the input terms of B_i are ground. □

Remark 3. The above definition of well-moded programs (queries) is very concise and this notion is quite general compared to the existing notions. The earlier notions are closely linked to the Prolog's selection rule and the producers of a variable should precede the consumers of that variable in the textual order [16]. One of our aims is to define the notion of well-modedness independent of selection rule and we achieve this by placing a condition that the producer-consumer relation is acyclic. This gives quite a general notion. However, it is not a complete generalization of the existing notions which take advantage of the fixed selection rule and can permit certain kinds of cycles in the producer-consumer relation. For instance, the following Prolog clause with modes `p (in, out, out, out); a1 (in, out); a2 (in, out)` and `a3 (in, out, out)` is well-moded w.r.t. the definition given in [16], though its producer-consumer relation has a cycle (`Z` is produced by `a2(Y,Z)` and consumed by `a3(Z,W,Y)` and `Y` is produced by `a3(Z,W,Y)` and consumed by `a2(Y,Z)`).

```
p(X, Y,Z,W) :- a1(X, Y), a2(Y, Z), a3(Z, W,Y)
```

Selection rules implied by moding

In this subsection, we characterize the class of selection rules suitable for execution of well-moded programs. First, we formally define the selection rule.

Definition 6. A *computation rule* (or *selection rule*) [29] is a function from the set of goals to the set of atoms such that the value of the function for a goal is an atom, called *selected* atom, in that goal.

If $G_0, G_1, \cdots, G_n$ is an SLD-derivation such that $G_i = G_j,\ i \neq j$ then the selected atoms of G_i and G_j are the same[4][29]. This notion can be extended to the clauses in the following way: given a clause, the selection rule gives an evaluation order among the atoms in the body of the clause. It can be captured by a partial order (if $l_i < l_j$ in the partial order, it means that l_i should be selected before l_j is selected).

In the following, we only consider selection rules satisfying the following property:

> Consider the following SLD-resolution step. Let $G_i = \leftarrow q_1(\cdots), \ldots, q_m(\cdots)$ be a goal, $q_j(\cdots)$ be the *selected atom*, $A \leftarrow B_1, \cdots, B_l$ be the input clause and σ be the mgu of $q_j(\cdots)$ and A. Then the goal G_{i+1} will be $\leftarrow q_1(\cdots)\sigma, \ldots, q_{j-1}(\cdots)\sigma,\ B_1\sigma, \ldots, B_l\sigma,\ q_{j+1}(\cdots)\sigma, \ldots, q_m(\cdots)\sigma$. We assume that the evaluation order induced by the selection rule satisfies the following: If the evaluation order for goals G_i, G_{i+1} and $\leftarrow B_1, \ldots, B_l$ are $\prec_{G_i}$, $\prec_{G_{i+1}}$ and $\prec_{Body}$ respectively, then
>
> 1. $B_{l_1}\sigma \prec_{G_{i+1}} B_{l_2}\sigma$ *if and only if* $B_{l_1} \prec_{Body} B_{l_2}$.
> 2. $B_{l_3}\sigma \prec_{G_{i+1}} q_{j_2}(\cdots)\sigma$ *if and only if* $1 \leq l_3 \leq l$ *and* $q_j(\cdots) \prec_{G_i} q_{j_2}(\cdots)$

[4] Apt [2] considered a more general notion of selection rule that takes history of the derivation also into account in selecting an atom from the goal. With such a selection rule it is possible to select two different atoms in G_i and G_j even though $G_i = G_j$. A selection rule which selects leftmost and rightmost atoms alternatively is an example.

3. $q_{j_1}(\cdots)\sigma \prec_{G_{i+1}} q_{j_2}(\cdots)\sigma$ *if and only if* $j_1 \neq j \neq j_2$ *and* $q_{j_1}(\cdots) \prec_{G_i} q_{j_2}(\cdots)$

The above conditions essentially say that to obtain the evaluation order of G_{i+1}, extend the evaluation order of G_i with the evaluation order of the input clause and make the order of the new atoms, B's, smaller than those atoms in G_i which were greater than the selected atom. For example, in Prolog's computation strategy, the leftmost atom is selected every time and the body of the input clause is added at the left end of the goal. Thus, the newly added atoms (leftmost) will be selected first.

The following definition characterizes the class of selection rules suitable for a given well-moded program.

Definition 7. A selection rule is *implied* by the moding information of a well-moded program if and only if the partial order induced by the selection rule for each clause in the program is an *extension* of the producer-consumer relation of that clause.

When P is a well-moded program and Q is a well-moded query, we say that a selection rule S is implied by $P \cup \{Q\}$ if S is implied by the moding information of P and the evaluation order given by S for Q is an extension of the producer-consumer relation of Q.

One can easily check whether a given selection rule is implied by the moding information of a given well-moded program, by checking whether the evaluation order given for each clause by the selection rule is an extension of the producer-consumer relation of that clause.

Theorem 1. It is decidable whether a given selection rule is implied by the moding information of a given well-moded program.

Proof : Since every logic program has finitely many clauses, it is always possible to check whether the evaluation order given for each clause by the selection rule is an extension of the producer-consumer relation of that clause. In fact it is easy to visualize an algorithm with time complexity $O(n)$, where n is the number of pairs in the relation for verifying the property. □

Example 2. It is very easy to see that the textual order of the atoms in the body of each clause in the quick-sort program (given in Example 1) is a linear extension of the producer-consumer relation. Therefore, Prolog's left-to-right selection rule is *implied* by the moding information. □

Example 3. The following permutation program is not well-moded according to earlier definitions (e.g. the definition given in [16, 32] – consumer `ap`$_1$`(X1s,c(X,X2s),Xs)` of variables X1s and X2s precedes their producer $\texttt{ap}_2$(X1s, X2s, Zs)) but is well-moded according to our definition and the right-to-left selection rule is implied by the moding information.

moding: perm (out, in); $\texttt{ap}_1$ (in, in, out) and $\texttt{ap}_2$ (out, out, in)

```
1.  ap₁(nil,X,X) ←
2.  ap₁(c(H,X),Y,c(H,Z)) ← ap₁(X,Y,Z)
3.  ap₂(nil,X,X) ←
4.  ap₂(c(H,X),Y,c(H,Z)) ← ap₂(X,Y,Z)
5.  perm(nil,nil) ←
6.  perm(Xs,c(X,Ys)) ← ap₁(X1s,c(X,X2s),Xs), ap₂(X1s,X2s,Zs), perm(Zs,Ys)
```

The clauses defining the predicates $\texttt{ap}_1$ and $\texttt{ap}_2$ are essentially the clauses in the standard append program. □

In the following, we establish that the computations of well-moded programs under the implied selection rules are data-driven.

Definition 8. Let P be a well-moded program and Q be a well-moded query. An evaluation (SLD-derivation) of $P \cup \{Q\}$ is said to be *data driven* if at every resolution step, the selected atom is ground on all its input positions.

Theorem 2. If P is a well-moded program, Q is a well-moded query and S is a selection rule implied by $P \cup \{Q\}$, then *every SLD-derivation of* $P \cup \{Q\}$ *is a data-driven evaluation.*

Proof : See Appendix. □

2.2. Term rewriting systems

In this subsection, we briefly explain the basic concepts of term rewriting systems.

Definition 9. A *term rewriting system* (TRS, for short) $\mathcal{R}$ is a pair $(\mathcal{F}, R)$ consisting of a set $\mathcal{F}$ of function symbols and a set R of rewrite rules of the form $l \to r$ satisfying:

1. $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, the set of terms built from functions in $\mathcal{F}$ and variables in $\mathcal{X}$,
2. left-hand-side l is not a variable and
3. $Var(r) \subseteq Var(l)$.

A rule $l \to r$ applies to term t in $\mathcal{T}(\mathcal{F}, \mathcal{X})$ if a subterm s of t matches with l through some substitution σ, i.e., $s \equiv l\sigma$ and the rule is applied by replacing the subterm s in t by $r\sigma$ resulting in a new term u. This is formalized in the following definitions.

Definition 10. A *context* $C[,\ldots,]$ is a term in $\mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$. If $C[,\ldots,]$ is a context containing n occurrences of $\Box$ and $t_1,\ldots,t_n$ are terms then $C[t_1,\ldots,t_n]$ is the result of replacing the occurrences of $\Box$ from left to right by $t_1,\ldots,t_n$. A context containing precisely 1 occurrence of $\Box$ is denoted $C[\,]$.

Definition 11. The *rewrite relation* $\Rightarrow_{\mathcal{R}}$ induced by a TRS $\mathcal{R}$ is defined as follows: $s \Rightarrow_{\mathcal{R}} t$ if there is a rewrite rule $l \rightarrow r$ in $\mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s \equiv C[l\sigma]$ and $t \equiv C[r\sigma]$.
We say that s *reduces to* t in *one rewrite* (or reduction) *step* if $s \Rightarrow_{\mathcal{R}} t$ and say s *reduces to* t if $s \Rightarrow^*_{\mathcal{R}} t$, where $\Rightarrow^*_{\mathcal{R}}$ is the transitive-reflexive closure of $\Rightarrow_{\mathcal{R}}$).

Definition 12. A term rewriting systems $\mathcal{R}$ is *terminating* if there is no infinite rewriting derivation $t_1 \Rightarrow_{\mathcal{R}} t_2 \Rightarrow_{\mathcal{R}} t_3 \Rightarrow_{\mathcal{R}} \cdots$.

3. Transforming a Logic Program into a Rewrite System

Our main objective is to reduce the termination problem of logic programs to the termination problem of term rewriting systems so that we can use many techniques available in the rewriting literature. It may be noted that term rewriting systems do not have local variables and all the termination results of term rewriting systems crucially depend on this property. With this as the motivation, we eliminate local variables by introducing Skolem functions.

For each n-ary predicate p having a moding with k output positions, we introduce k new function symbols $p^1,\ldots,p^k$ of arity $n-k$. These k-function symbols correspond to the k output positions of the predicate p. (If $k=0$, we introduce a n-ary function symbol p^0.) Then, we construct a set of rewrite rules to compute these new functions. For example, we associate two binary function symbols $\mathtt{r}^1$ and $\mathtt{r}^2$ with the predicate `r` with moding `(in, in, out, out)` in Example 8 in the sequel. These functions take first two arguments of `r` as inputs and give one output each corresponding to the third and fourth arguments of `r`. Rewrite rules for these functions are constructed in Example 8.

In the following, we illustrate the transformational approach through a series of examples.

Example 4. Consider the following multiplication program.

```
moding: add (in, in, out)  and  mult (in, in, out)
```

```
add(0, Y, Y) ←
add(s(X), Y, s(Z)) ← add(X, Y, Z)
mult(0, Y, 0) ←
mult(s(X), Y, Z) ← mult(X, Y, Z1), add(Z1, Y, Z)
```

From these clauses and the moding information, we obtain the following rewriting rules:

1. Since the output of predicate add for inputs 0 and Y is Y, we get $\mathtt{add}^1(\mathtt{0}, \mathtt{Y}) \to \mathtt{Y}$.
2. Since the output of mult for inputs 0 and Y is 0, we get $\mathtt{mult}^1(\mathtt{0}, \mathtt{Y}) \to \mathtt{0}$.
3. In the second clause, the output of add for inputs s(X) and Y is s(Z), where Z is the output of add for the inputs X and Y. We get $\mathtt{add}^1(\mathtt{s}(\mathtt{X}), \mathtt{Y}) \to \mathtt{s}(\mathtt{add}^1(\mathtt{X}, \mathtt{Y}))$.
4. In the last clause, the output of mult for inputs s(X) and Y is Z, where Z is the output of add for the inputs Z1 and Y, where Z1 is the output of mult for inputs X and Y. So we get $\mathtt{mult}^1(\mathtt{s}(\mathtt{X}), \mathtt{Y}) \to \mathtt{add}^1(\mathtt{Z1}, \mathtt{Y})$, where $\mathtt{Z1} = \mathtt{mult}^1(\mathtt{X}, \mathtt{Y})$.
   The resulting rule is $\mathtt{mult}^1(\mathtt{s}(\mathtt{X}), \mathtt{Y}) \to \mathtt{add}^1(\mathtt{mult}^1(\mathtt{X}, \mathtt{Y}), \mathtt{Y})$. □

Example 5. Consider the permutation program given in Example 3. It was shown that the right-to-left selection rule is implied by the moding. First, we get the following rules from clauses 1 to 5.

$$\mathtt{ap}_1^1(\mathtt{nil}, \mathtt{X}) \to \mathtt{X}$$
$$\mathtt{ap}_1^1(\mathtt{c}(\mathtt{H}, \mathtt{X}), \mathtt{Y}) \to \mathtt{c}(\mathtt{H}, \mathtt{ap}_1^1(\mathtt{X}, \mathtt{Y}))$$
$$\mathtt{ap}_2^1(\mathtt{X}) \to \mathtt{nil}$$
$$\mathtt{ap}_2^1(\mathtt{c}(\mathtt{H}, \mathtt{Z})) \to \mathtt{c}(\mathtt{H}, \mathtt{ap}_2^1(\mathtt{Z}))$$
$$\mathtt{ap}_2^2(\mathtt{X}) \to \mathtt{X}$$
$$\mathtt{ap}_2^2(\mathtt{c}(\mathtt{H}, \mathtt{Z})) \to \mathtt{ap}_2^2(\mathtt{Z})$$
$$\mathtt{perm}^1(\mathtt{nil}) \to \mathtt{nil}$$

Let us now consider clause 6. The output of predicate perm for input $\mathtt{c}(\mathtt{X}, \mathtt{Ys})$ is Xs, where Xs is the output of predicate $\mathtt{ap}_1$ for inputs $\mathtt{X1s}, \mathtt{c}(\mathtt{X}, \mathtt{X2s})$, i.e., Xs is $\mathtt{ap}_1^1(\mathtt{X1s}, \mathtt{c}(\mathtt{X}, \mathtt{X2s}))$. Here, X1s and X2s are outputs of predicate $\mathtt{ap}_2$ for input Zs and Zs is the output of perm for input Ys. Therefore, we get the rewrite rule $\mathtt{perm}^1(\mathtt{c}(\mathtt{X}, \mathtt{Ys})) \to \mathtt{ap}_1^1(\mathtt{ap}_2^1(\mathtt{perm}^1(\mathtt{Ys})), \mathtt{c}(\mathtt{X}, \mathtt{ap}_2^2(\mathtt{perm}^1(\mathtt{Ys}))))$.
□

When a nonvariable term appears in an output position of a body atom, we need to introduce *inverse functions* as illustrated in the following example.

Example 6. Consider the permutation program given in Example 3 with moding perm (in, out); $\mathtt{ap}_1$ (out, out, in) and $\mathtt{ap}_2$ (in, in, out). It is easy to see that we get the following rules from the clauses 1 to 5.

$$\mathtt{ap}_1^1(\mathtt{X}) \to \mathtt{nil}$$
$$\mathtt{ap}_1^1(\mathtt{c}(\mathtt{H}, \mathtt{Z})) \to \mathtt{c}(\mathtt{H}, \mathtt{ap}_1^1(\mathtt{Z}))$$
$$\mathtt{ap}_1^2(\mathtt{X}) \to \mathtt{X}$$
$$\mathtt{ap}_1^2(\mathtt{c}(\mathtt{H}, \mathtt{Z})) \to \mathtt{ap}_1^2(\mathtt{Z})$$
$$\mathtt{ap}_2^1(\mathtt{nil}, \mathtt{X}) \to \mathtt{X}$$
$$\mathtt{ap}_2^1(\mathtt{c}(\mathtt{H}, \mathtt{X}), \mathtt{Y}) \to \mathtt{c}(\mathtt{H}, \mathtt{ap}_2^1(\mathtt{X}, \mathtt{Y}))$$

$\mathtt{perm}^1(\mathtt{nil}) \rightarrow \mathtt{nil}$

Let us now consider clause 6. The output of predicate perm for input Xs is c(X,Ys), where Ys is the output of predicate perm for input Zs and X is a part of the second output of predicate $\mathtt{ap}_1$ for input Xs. How do we extract X and X2s from c(X,X2s) ? They can be extracted through inverse functions of c. The operators car and cdr can be used as inverse functions of c and we get a rewrite rule $\mathtt{perm}^1(\mathtt{Xs}) \rightarrow \mathtt{c}(\mathtt{car}(\mathtt{ap}_1^2(\mathtt{Xs})), \mathtt{perm}^1(\mathtt{ap}_2^1(\mathtt{ap}_1^1(\mathtt{Xs}), \mathtt{cdr}(\mathtt{ap}_1^2(\mathtt{Xs})))))$. To evaluate the functions car and cdr, we add the following rules: $\mathtt{car}(\mathtt{c}(\mathtt{H},\mathtt{T})) \rightarrow \mathtt{H}$ and $\mathtt{cdr}(\mathtt{c}(\mathtt{H},\mathtt{T})) \rightarrow \mathtt{T}$. Thus, for clause (6), we get the following rules.

$$\begin{array}{l} \mathtt{perm}^1(\mathtt{Xs}) \rightarrow \mathtt{c}(\mathtt{car}(\mathtt{ap}_1^2(\mathtt{Xs})), \mathtt{perm}^1(\mathtt{ap}_2^1(\mathtt{ap}_1^1(\mathtt{Xs}), \mathtt{cdr}(\mathtt{ap}_1^2(\mathtt{Xs}))))) \\ \mathtt{car}(\mathtt{c}(\mathtt{H},\mathtt{T})) \rightarrow \mathtt{H} \\ \mathtt{cdr}(\mathtt{c}(\mathtt{H},\mathtt{T})) \rightarrow \mathtt{T}. \end{array}$$

□

As shown in the above example, inverse functions are needed when a non-variable term occurs in an output position of an atom in the body. Appropriate inverse functions are generated as follows:

1. Build a representation of the nonvariable output term.
2. Identify for each variable in this term (and consumed by other atoms in the clause), a path from root to an occurrence of that variable in the above tree. If there are multiple occurrences of a variable in the term, one can arbitrarily choose one of these occurrences and take the path from root to that particular occurrence.
3. Traverse the path upwards (from leaf to root), collecting suitable inverse function symbols, which will be used in constructing the right-hand-sides of the rewrite rules as illustrated in the following example.

Example 7. Consider the following clause:

moding: a (in, out); b (in, in, out); and c (in, out)

$\mathtt{a}(\mathtt{X},\mathtt{Y}) \leftarrow \mathtt{b}(\mathtt{X},0,\mathtt{f}(\mathtt{X},\mathtt{g}(\mathtt{h}(0,\mathtt{Z}),\mathtt{X}),\mathtt{X},1)), \mathtt{c}(\mathtt{Z},\mathtt{Y})$

A nonvariable term f(X, g(h(0, Z), X), X, 1) is occurring in the output position of b and the local variable Z occurring in this term is consumed by atom c(Z, Y). Function symbols f, g and h occur in the path from the root to variable Z and the appropriate inverse functions are collected as follows:

Since Z is the second argument of h, the inverse function $\mathtt{h2}^{-1}$ is collected (and a rewrite rule $\mathtt{h2}^{-1}(\mathtt{h}(\mathtt{X1},\ \mathtt{X2})) \rightarrow \mathtt{X2}$ is added to the rewrite system). Since the subterm h(0, Z) is the first argument of g, the inverse function symbol $\mathtt{g1}^{-1}$ is collected (and a rewrite rule $\mathtt{g1}^{-1}(\mathtt{g}(\mathtt{X1},\ \mathtt{X2})) \rightarrow \mathtt{X1}$ is added). Since g(...) is the second argument of f, the inverse function symbol $\mathtt{f2}^{-1}$ is collected (and a rewrite rule $\mathtt{f2}^{-1}(\mathtt{f}(\mathtt{X1},\ \mathtt{X2},\ \mathtt{X3},\ \mathtt{X4})) \rightarrow \mathtt{X2}$ is added). The transformation procedure derives the following rewrite rules for the above clause:

$\texttt{a}^1(\texttt{X}) \rightarrow \texttt{c}^1(\texttt{h2}^{-1}(\texttt{g1}^{-1}(\texttt{f2}^{-1}(\texttt{b}^1(\texttt{X},\texttt{0})))))$
$\texttt{h2}^{-1}(\texttt{h}(\texttt{X1},\ \texttt{X2})) \rightarrow \texttt{X2}$
$\texttt{g1}^{-1}(\texttt{g}(\texttt{X1},\ \texttt{X2})) \rightarrow \texttt{X1}$
$\texttt{f2}^{-1}(\texttt{f}(\texttt{X1},\ \texttt{X2},\ \texttt{X3},\ \texttt{X4})) \rightarrow \texttt{X2}$ □

In the above examples, all the variables occurring in the output positions of an atom in the body are also occurring either in output positions of the head or in input positions of some other atom in the body. And the rewrite systems derived in both the examples capture the termination of the corresponding logic programs correctly. The above transformation is basically capturing the data flow in the program execution. When there are some variables occurring only in output positions of atoms in the body, we need to *add additional rewrite rules* as illustrated below.

Example 8. Consider the following (nonterminating) logic program:

`moding: a (in, out); b (in, out); c (in, out)` and `r (in, in, out, out)`.

$\texttt{a}(\texttt{X},\texttt{f}(\texttt{X})) \leftarrow$
$\texttt{b}(\texttt{X},\texttt{X}) \leftarrow$
$\texttt{c}(\texttt{X},\texttt{Y}) \leftarrow \texttt{a}(\texttt{X},\texttt{Z}),\texttt{r}(\texttt{X},\texttt{Z},\texttt{Y},\texttt{Z1})$
$\texttt{r}(\texttt{X},\texttt{Y},\texttt{O1},\texttt{O2}) \leftarrow \texttt{b}(\texttt{X},\texttt{O1}),\texttt{c}(\texttt{Y},\texttt{O2})$

We get the following rewrite system according to the above transformation:

$\texttt{a}^1(\texttt{X}) \rightarrow \texttt{f}(\texttt{X})$
$\texttt{b}^1(\texttt{X}) \rightarrow \texttt{X}$
$\texttt{c}^1(\texttt{X}) \rightarrow \texttt{r}^1(\texttt{X},\texttt{a}^1(\texttt{X}))$
$\texttt{r}^1(\texttt{X},\texttt{Y}) \rightarrow \texttt{b}^1(\texttt{X})$
$\texttt{r}^2(\texttt{X},\texttt{Y}) \rightarrow \texttt{c}^1(\texttt{Y})$

It is easy to see that this rewrite system captures the data-flow in the program execution correctly. This rewrite system is terminating (it can be proved using recursive path ordering with precedence $\texttt{r}^2 > \texttt{c}^1 > \texttt{a}^1 > \texttt{f}, \quad \texttt{c}^1 > \texttt{r}^1 > \texttt{b}^1$) while the above program is a nonterminating one.

For the query $\leftarrow \texttt{c}(\texttt{t},\texttt{Y})$, where $\texttt{t}$ is a ground term, the program has an infinite SLD-derivation

$\leftarrow\ \texttt{c}(\texttt{t},\texttt{Y})$
$\leftarrow\ \texttt{a}(\texttt{t},\texttt{Z}),\ \texttt{r}(\texttt{t},\texttt{Z},\texttt{Y},\texttt{Z1})$
$\leftarrow\ \texttt{r}(\texttt{t},\texttt{f}(\texttt{t}),\texttt{Y},\texttt{Z1})$
$\leftarrow\ \texttt{b}(\texttt{t},\texttt{Y}),\ \texttt{c}(\texttt{f}(\texttt{t}),\texttt{Z1})$
$\leftarrow\ \texttt{c}(\texttt{f}(\texttt{t}),\texttt{Z1})$
$\vdots$
$\leftarrow\ \texttt{c}(\texttt{f}(\texttt{f}(\texttt{t})),\texttt{Z}')$
$\vdots$

In the rewrite system, we compute $\mathtt{c}^1(\mathtt{t})$ for some ground term $\mathtt{t}$ through the rewrite derivation $\mathtt{c}^1(\mathtt{t}) \Rightarrow \mathtt{r}^1(\mathtt{t}, \mathtt{a}^1(\mathtt{t})) \Rightarrow \mathtt{r}^1(\mathtt{t}, \mathtt{f}(\mathtt{t})) \Rightarrow \mathtt{b}^1(\mathtt{t}) \Rightarrow \mathtt{t}$, whereas for the execution of $\mathtt{c}(\mathtt{t},\mathtt{Y})$ in the logic programming paradigm, we need to execute $\mathtt{r}(\mathtt{t},\mathtt{f}(\mathtt{t}),\mathtt{Y},\mathtt{Z1})$, which in turn needs the execution of $\mathtt{b}(\mathtt{t},\mathtt{Y})$ and $\mathtt{c}(\mathtt{f}(\mathtt{t}),\mathtt{Z1})$ – thus, leading to an infinite SLD-derivation. Computing $\mathtt{c}^1$ in rewriting involves only a partial computation of $\mathtt{r}$ (i.e., computation of $\mathtt{r}^1$ only) while this kind of partial computation is not possible in logic programming and execution of $\mathtt{r}$ does not stop after computing $\mathtt{r}^1$ but continues to evaluate $\mathtt{r}^2$ (here, it needs to execute $\mathtt{c}$ which leads to the above loop). □

In the above example, the computation of the second output of $\mathtt{r}(\cdots)$ causes the looping of the logic program and the computation does not contribute anything useful in the evaluation of the initial query. The corresponding rewrite system does not enter any loop as it does not involve the computation of the second output of $\mathtt{r}(\cdots)$. To cover all the computation paths in the SLD-trees, we include additional rewrite rules reflecting the (possible) nontermination due to the computation of unnecessary values (the computation of these values does not provide any information directly or indirectly to the head). The derivation of these rewrite rules will be clear in the next section.

Example 8 continued: We include the following rewrite rule to capture the unnecessary computations:

$$\mathtt{c}^1(\mathtt{X}) \to \#(\mathtt{r}^2(\mathtt{X}, \mathtt{a}^1(\mathtt{X})))$$

It is easy to see that the resulting rewrite system is a nonterminating one. □

4. Formal Description of the Transformation Procedure

Though input and output positions of a predicate can mix together in all possible ways, for notational convenience, we write all the input positions first, followed by all the output positions. We write $p(t_{i_1}, \ldots, t_{i_j},\ t_{o_1}, \ldots, t_{o_k})$ to denote an atom $p(\cdots)$ containing the terms $t_{i_1}, \ldots, t_{i_j}$ in input positions and $t_{o_1}, \ldots, t_{o_k}$ in output positions.

The main step in our transformation is the elimination of local variables. Basically, the right-hand sides of rewrite rules are derived from an output term of the head by repeatedly replacing a local variable (to be precise, variables in $Var(c) - invar(head)$) by a term corresponding to one of its producers. When a variable has more than one producer, one has to consider all the possible choices. The following function ELIMINATE-LOC-VARS has been designed to perform the elimination of local variables repeatedly until there is no local variable to replace. This function needs the computation of the set of producers for each local variable.

It is easy to see that computation of an output term in the body does not provide any information directly or indirectly to the head or any other atom if and only if none of the variables in that output term have any consumer. Therefore, the unnecessary computations can be captured by computing the set of variables without consumers. To derive the rewrite rules corresponding to unnecessary computations, we compute the following sets $Consvar$ and $Unsry$ for each clause in the program. In all, the transformation procedure computes the following sets for each clause $c : head \leftarrow body$ in the program.

1. $Prod(X) = \{ \langle p^l(t_{i_1}, \ldots, t_{i_j}),\ t_{o_l} \rangle \mid p(t_{i_1}, \ldots, t_{i_j},\ t_{o_1}, \ldots, t_{o_k})$ is an atom in the body of the clause c, $X \in Var(t_{o_l})$ and $X \notin Var(\{t_{i_1}, \ldots, t_{i_j}\})\}$, the set of producers, for each variable X not occurring in input positions of the head.

2. $Consvar = \{X \in Var(c) - invar(head) \mid X \in outvar(head)$ or X occurs in an input position of an atom in the body $\}$, the set of variables in the clause *consumed at least once.*

3. $Unsry = \{ p^l(t_{i_1}, \ldots, t_{i_j}) \mid Var(t_{o_l}) \cap Consvar = \phi \} \cup \{ q^0(s_{i_1}, \ldots, s_{i_k}) \mid$ predicate q does not have output positions $\}$, where $p(\cdots)$ and $q(\cdots)$ are atoms in the body.
   This set corresponds to the set of computations (we call them *unnecessary computations*) which do not contribute to the outputs of head directly or indirectly.

The algorithm TRANSFORM and the function ELIMINATE-LOC-VARS are formally described below.

algorithm TRANSFORM (P : in; R_P : out);
begin
$R_P := \phi$; {* *R_P contains rewrite rules* *}
for each clause c: $a(t_{i_1}, \cdots, t_{i_k}, t_{o_1}, \cdots, t_{o_{k'}}) \leftarrow B_1, \ldots B_n \in P$ **do**
begin $INhead := Var(\{t_{i_1}, \cdots, t_{i_k}\})$
Compute $Consvar$ and $Unsry$;
Compute $Prod(X)$ for every variable in $Var(c) - INhead$;
for $j := 1$ **to** k' **do**
begin
$S :=$ ELIMINATE-LOC-VARS($\{t_{o_j}\}$); {* *S contains the right-hand sides of the rules in R_P* *}
$R_P := R_P \cup \{ a^j(t_{i_1}, \cdots, t_{i_k}) \rightarrow t \mid t \in S \}$
end;
{* *Following code derives rewrite rules corresponding to* unnecessary *computations.* *}
$S :=$ ELIMINATE-LOC-VARS($Unsry$);
$R_P := R_P \cup \{ a^{k'}(t_{i_1}, \cdots, t_{i_k}) \rightarrow \#(t) \mid t \in S \}$
end
end TRANSFORM.

function ELIMINATE-LOC-VARS(T)

{* *This function goes on replacing the* local variables *in the set of terms T by the* terms corresponding to their producers *as long as there are local variables. Since producer-consumer relation of every well-moded clause is* acyclic, *this function is guaranteed to terminate. Here, $INhead$ is a global variable.* *}

begin $V := Var(T) - INhead$;
while $V \neq \phi$ **do**
begin
for each $X \in V$ **do**
begin $T' := \phi$;
for each $\langle p^l(\cdots), t\rangle \in Prod(X)$ **do**
if $t = X$ **then** $T' := T' \cup T.\{X/p^l(\cdots)\}$ {* *Replace local var X by its producer-term.* *}
else if $t = f(X)$ **then**
begin
$T' := T' \cup T.\{ X/f^{-1}(p^l(\cdots)) \}$; {* *Introduce inverse functions* *}
$R_P := R_P \cup \{ f^{-1}(f(X)) \rightarrow X \}$
end;
$T := T'$
end;
$V := Var(T) - INhead$
end;
Return(T)
end ELIMINATE-LOC-VARS;

The following series of examples illustrate the transformation procedure.

Example 9. For the quick-sort program given in Example 1, the algorithm **TRANSFORM** derives the following term rewriting system:

1.	$\mathtt{q}^1(\mathtt{nil}) \rightarrow \mathtt{nil}$
2.	$\mathtt{q}^1(\mathtt{c(H,L)}) \rightarrow \mathtt{a}^1(\mathtt{q}^1(\mathtt{s}^1(\mathtt{L,H})),\ \mathtt{c(H,q}^1(\mathtt{s}^2(\mathtt{L,H}))))$
3.	$\mathtt{s}^1(\mathtt{nil,Y}) \rightarrow \mathtt{nil}$
3'.	$\mathtt{s}^2(\mathtt{nil,Y}) \rightarrow \mathtt{nil}$
4.	$\mathtt{s}^1(\mathtt{c(X,Xs),Y}) \rightarrow \mathtt{c(X,s}^1(\mathtt{Xs,Y}))$
4'.	$\mathtt{s}^2(\mathtt{c(X,Xs),Y}) \rightarrow \mathtt{s}^2(\mathtt{Xs,Y})$
5.	$\mathtt{s}^1(\mathtt{c(X,Xs),Y}) \rightarrow \mathtt{s}^1(\mathtt{Xs,Y})$
5'.	$\mathtt{s}^2(\mathtt{c(X,Xs),Y}) \rightarrow \mathtt{c(X,s}^2(\mathtt{Xs,Y}))$
6.	$\mathtt{a}^1(\mathtt{nil,X}) \rightarrow \mathtt{X}$
7.	$\mathtt{a}^1(\mathtt{c(H,X),Y}) \rightarrow \mathtt{c(H,a}^1(\mathtt{X,Y}))$

Here, we explain how rule 2 is derived from the second clause. The other rules can be derived in a similar fashion. The head $\mathtt{q(c(H,L),S)}$ contains $\mathtt{c(H,L)}$ in the input position and variable $\mathtt{S}$ in the output position hence the left-hand-side of the rewrite rule is $\mathtt{q}^1(\mathtt{c(H,L)})$. Producers of variables not occurring in the input positions of the head are as follows: $Prod(\mathtt{A}) = \{\ \langle \mathtt{s}^1(\mathtt{L,H}),\mathtt{A}\rangle\ \}$, $Prod(\mathtt{B}) = \{\ \langle \mathtt{s}^2(\mathtt{L,H}),\mathtt{B}\rangle\ \}$,
$Prod(\mathtt{A1}) = \{\ \langle \mathtt{q}^1(\mathtt{A}),\mathtt{A1}\rangle\ \}$, $Prod(\mathtt{B1}) = \{\ \langle \mathtt{q}^1(\mathtt{B}),\mathtt{B1}\rangle\ \}$, $Prod(\mathtt{S}) = \{\ \langle \mathtt{a}^1(\mathtt{A1,c(H,B1)}),\mathtt{S}\rangle\ \}$.
To construct the right-hand-side term, algorithm **TRANSFORM** calls function **ELIMINATE-LOC-VARS** with argument $T = \{\mathtt{S}\}$. Values of T at the end of various iterations of the **while** loop in **ELIMINATE-LOC-VARS** are given below.
Iteration 1 $T = \{\ \mathtt{a}^1(\mathtt{A1,c(H,B1)})\ \}$ *variable* S *is replaced by its producer.*
Iteration 2 $T = \{\ \mathtt{a}^1(\mathtt{q}^1(\mathtt{A}),\mathtt{c(H,q}^1(\mathtt{B})))\ \}$ *local variables* A1 *and* B1 *are replaced by their producers.*
Iteration 3 $T = \{\ \mathtt{a}^1(\mathtt{q}^1(\mathtt{s}^1(\mathtt{L,H})),\mathtt{c(H,(q}^1(\mathtt{s}^2(\mathtt{L,H}))))\ \}$ *local variables* A *and* B *are replaced.*
Since there are no local variables in T after 3^{rd} iteration, **ELIMINATE-LOC-VARS** returns this T to **TRANSFORM** which produces the rewrite rule 2 given above. □

Example 10. Let us consider the third clause of the program given in Example 8.

$$\mathtt{c(X,\ Y)} \leftarrow \mathtt{a(X,\ Z),\ r(X,\ Z,\ Y,\ Z1)}$$

The following producers of the variables are computed.
$prod(\mathtt{Z}) = \{\ \langle \mathtt{a}^1(\mathtt{X}),\ \mathtt{Z}\rangle\ \}$,
$prod(\mathtt{Y}) = \{\ \langle \mathtt{r}^1(\mathtt{X,\ Z}),\ \mathtt{Y}\rangle\ \}$,
$prod(\mathtt{Z1}) = \{\ \langle \mathtt{r}^2(\mathtt{X,\ Z}),\ \mathtt{Z1}\rangle\ \}$,
and $Unsry = \{\mathtt{r}^2(\mathtt{X,Z})\}$.

The function **ELIMINATE-LOC-VARS** is first invoked with input $T = \{\mathtt{Y}\}$. The values of T at the end of various iterations of the **while** loop are $\{\ \mathtt{Y}\ \}$, $\{\ \mathtt{r}^1(\mathtt{X,\ Z})\ \}$, $\{\ \mathtt{r}^1(\mathtt{X,\ a}^1(\mathtt{X}))\ \}$ and we get a rewrite rule $\mathtt{c}^1(\mathtt{X}) \rightarrow \mathtt{r}^1(\mathtt{X,\ a}^1(\mathtt{X}))$.

The function **ELIMINATE-LOC-VARS** is then invoked with input $T = Unsry = \{\mathtt{r}^2(\mathtt{X,Z})\}$. The value of T after one iteration of the **while** loop is $\{\ \mathtt{r}^2(\mathtt{X,\ a}^1(\mathtt{X}))\ \}$, which does not have any local variables. Corresponding to this set, algorithm **TRANSFORM** adds the rewrite rule $\mathtt{c}^1(\mathtt{X}) \rightarrow \#(\mathtt{r}^2(\mathtt{X,\ a}^1(\mathtt{X})))$. □

Example 11. Let us consider the following program.

```
p ← q, p
```

Here, we have propositions (no in/out arguments). So, the associated function symbols $\{\ \mathrm{p}^0, \mathrm{q}^0\ \}$ are of arity zero (i.e., *constants*). The value of $Unsry$ is $\{\ \mathrm{p}^0, \mathrm{q}^0\ \}$. The algorithm TRANSFORM derives R_P with the following two rewrite rules: $\mathrm{p}^0 \to \#(\mathrm{q}^0)$ and $\mathrm{p}^0 \to \#(\mathrm{p}^0)$. □

5. Formal Correctness

In this section, we first prove that the algorithm TRANSFORM terminates for a given input and study some properties of the rewrite system derived by the algorithm TRANSFORM from a given well-moded logic program. Then, we establish that termination of the derived rewriting system implies termination of the logic program for all well-moded queries under all selection rules implied by the moding information. The following picture gives the interdependence of the technical lemmas and theorems leading to the proof of our main result (Theorem 5).

$$\begin{tikzcd}
\text{Lem. 7} \arrow[r] \arrow[rr, bend right] & \text{Lem. 8} \arrow[r] \arrow[rrr, bend left] & \text{Lem. 9} \arrow[r] & \text{Lem. 11} \arrow[r] & \text{Th. 4} \arrow[r] & \text{Th. 5} \arrow[r] & \text{Th. 6} \\
 & & & \text{Lem. 10} \arrow[u] & & &
\end{tikzcd}$$

Road-map of the technical results

Lemma 5. The algorithm TRANSFORM *terminates.*

Proof: Since there are only finitely many clauses in the program and every predicate has finitely many output positions, the number of iterations of the **for**-loops is finite. Since each body has finitely many atoms, $Unsry$ is finite for every clause. Therefore, there are only finitely many calls to the function ELIMINATE-LOC-VARS and it is enough to prove termination of ELIMINATE-LOC-VARS for proving termination of TRANSFORM.

It is easy to see that for proving termination of ELIMINATE-LOC-VARS, it is enough to prove termination of the **while** loop. The main step in the **while**-loop can be abstracted as follows: *application of substitution* $\{X/p^j(\cdots)\}$ *to the terms in* T*, where* $p^j(\cdots)$ *is a producer of* X. That is, an occurrence of a variable in a term corresponding to its consumer is replaced by its producer. Since the producer-consumer relation of a well-moded clause is acyclic, the **while** loop is bound to terminate. □

5.1. *Properties of the Derived TRS*

In this subsection, we study certain properties of the derived term rewriting systems using the transformation procedure from the given well-moded programs. Our basic aim is to establish that the termination of the derived term rewriting system implies the termination of the logic program. Towards such a goal, we show that corresponding to each resolution step in the SLD-derivations, there is at least one reduction step in the rewrite derivations.

The following lemma establishes that there are no extra variables on the right-hand side of the rewrite rules in R_P, and thus, all the termination techniques of rewriting systems can be used in proving the termination of logic programs.

Lemma 6. *$Var(r) \subseteq Var(l)$ for each rewrite rule $l \rightarrow r$ in R_P.*

Proof: Each rule associated with inverse functions is of the form $f_i^{-1}(f(X_1, \ldots, X_n)) \rightarrow X_i$ and hence, the lemma holds for such rewrite rules. Other rewrite rules are of the form $p^j(\overline{t_{in}}) \rightarrow t$ or $p^j(\overline{t_{in}}) \rightarrow \#(t)$, where $\overline{t_{in}}$ is the sequence of input terms of a head and t is a term in the set T returned by ELIMINATE-LOC-VARS. The function ELIMINATE-LOC-VARS terminates with the condition $(Var(T) - INhead) = \phi$. Therefore, $Var(T) \subseteq INhead$ and hence $Var(t) \subseteq INhead = Var(p^j(\overline{t_{in}})) = Var(l)$. Thus, $Var(r) \subseteq Var(l)$ for each rule $l \rightarrow r$ in R_P. □

The basic ingredient in the transformation is the *application of substitution* $\{X/p^j(\cdots)\}$ *to the terms in* T, *where* $p^j(\cdots)$ *is a producer of* X. A study of the derived rewrite rules involves a study of these substitutions and their effect. We need the following notation.

Definition 13. Let c : $head \leftarrow body$ be a well-moded clause and V be the set of variables $Var(c) - invar(head)$. For any variable X, we denote by $ELV(X)$, the set of terms returned by the function ELIMINATE-LOC-VARS for input $\{ X \}$. We denote by Θ, the set of substitutions $\{ \sigma \mid X\sigma \in ELV(X)$ for each variable $X \in V \}$. For any term t, $\Theta(t)$ denotes the set $\{ \sigma \mid X\sigma \in ELV(X)$ for each variable $X \in Var(t) \}$. We call the substitutions in Θ, *Skolem substitutions*.

Remark 4. If X is a variable in a term $s \equiv C[X]$ for some context $C[\ ]$, we can construct an appropriate context $C'[\ ]$ of inverse functions such that $C'[C[X]]$ can be reduced to X by the rewrite rules defining the inverse functions. See Example 7 for such a construction. In the sequel, we use the phrase "context of inverse functions" to mean the context built from the inverse functions as illustrated.

Lemma 7. Let c : $head \leftarrow body$ be a well-moded clause, Θ be the set of its Skolem substitutions and X be a variable in $Var(c) - invar(head)$ such that $\langle p^l(t_{i_1}, \cdots, t_{i_n}),\ t_{o_l} \rangle$ is an element in $Prod(X)$ and $q(s_{i_1}, \cdots, s_{i_k},\ s_{o_1}, \cdots, s_{o_{k'}})$

is a consumer of X. Then,

1. There exists a Skolem substitution σ in Θ such that $X\sigma \equiv C[p^l(t_{i_1}, \ldots, t_{i_n})\sigma]$ where C is a context of inverse functions such that $C[t_{o_l}] \Rightarrow^* X$.
2. Corresponding to every Skolem substitution $\sigma \in \Theta(p^l(t_{i_1}, \cdots, t_{i_n}))$ there is a substitution $\sigma' \in \Theta(q^m(t_{i_1}, \cdots, t_{i_n}))$ such that $X\sigma' \equiv C[p^l(t_{i_1}, \ldots, t_{i_n})\sigma'] \equiv C[p^l(t_{i_1}, \ldots, t_{i_n})\sigma]$ for each $m \in [1, k']$.
3. The output of the function ELIMINATE-LOC-VARS for input $\{t\}$ is the set $\{\ t\sigma \ \mid\ \sigma \in \Theta\ \} = \{\ t\gamma \ \mid\ \gamma \in \Theta(t)\ \}$.

Proof (sketch) : In the function ELIMINATE-LOC-VARS, a local variable is replaced by the terms corresponding to its producers. If a local variable (say, Y) occurs in a non-variable output term (say, t), the term corresponding to the producer is adorned with a context of inverse functions $C1$ such that $C1[t]$ is reducible to Y by the rewrite rules defining the inverse functions. It can be easily seen that statement (1) of the lemma follows easily. Since the local variable X (with a producer $\langle p^l(t_{i_1}, \cdots, t_{i_n}),\ t_{o_l}\rangle$) occurs in the input terms of the atom $q(\cdots)$, the variable X in any term $q^m(t_{i_1}, \cdots, t_{i_n})$) is replaced with $C[p^l(t_{i_1}, \ldots, t_{i_n})]$ by the function ELIMINATE-LOC-VARS. Statement 2 of the lemma now follows. The statement 3 of the lemma follows from the above definition of Θ. □

The following example illustrates this lemma.

Example 12. Consider the following clause:

`moding: d (in, in, out); a (in, out, out);` and `q,b,c (in, out)`

$\mathtt{q(X,W)} \leftarrow \mathtt{a(X,O,U), b(X,U), c(O,V), d(U,V,W)}$

For this clause, $ELV(\mathtt{O}) = \{\mathtt{a^1(X)}\}$, $ELV(\mathtt{U}) = \{\mathtt{a^2(X)},\ \mathtt{b^1(X)}\}$, $ELV(\mathtt{V}) = \{\mathtt{c^1(a^1(X))}\}$, and $ELV(\mathtt{W}) = \{\mathtt{d^1(a^2(X), c^1(a^1(X)))},\quad \mathtt{d^1(b^1(X), c^1(a^1(X)))}\}$.

The value of Θ is $\{\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4\}$, where

$$\sigma_1 = \{\ \mathtt{O/a^1(X)},\ \mathtt{U/a^2(X)},\ \mathtt{V/c^1(a^1(X))},\ \mathtt{W/d^1(a^2(X),\ c^1(a^1(X)))}\ \},$$
$$\sigma_2 = \{\ \mathtt{O/a^1(X)},\ \mathtt{U/b^1(X)},\ \mathtt{V/c^1(a^1(X))},\ \mathtt{W/d^1(a^2(X),\ c^1(a^1(X)))}\ \},$$
$$\sigma_3 = \{\ \mathtt{O/a^1(X)},\ \mathtt{U/a^2(X)},\ \mathtt{V/c^1(a^1(X))},\ \mathtt{W/d^1(b^1(X),\ c^1(a^1(X)))}\ \},$$
$$\sigma_4 = \{\ \mathtt{O/a^1(X)},\ \mathtt{U/b^1(X)},\ \mathtt{V/c^1(a^1(X))},\ \mathtt{W/d^1(b^1(X),\ c^1(a^1(X)))}\ \}.$$

The output of ELIMINATE-LOC-VARS for input $\{\mathtt{d^1(U,V)}\}$ is the set $\{\mathtt{d^1(a^2(X), c^1(a^1(X)))}, \mathtt{d^1(b^1(X), c^1(a^1(X)))}\}$ which is equal to $\{\mathtt{d^1(U,V)}\sigma \mid \sigma \in \Theta\}$. The term $\mathtt{d^1(U,V)}$ is an element in $Prod(\mathtt{W})$ and substitutions σ_1 and σ_4 satisfy the equation $\mathtt{W}\sigma \equiv \mathtt{d^1(U,V)}\sigma$. □

The following lemma plays a crucial role in proving our main result. It shows that corresponding to each atom in the body of a clause, there are some (sub)terms in the derived rewrite rules.

Lemma 8. Let $c : head \leftarrow body$ be a well-moded clause and Θ be the set of Skolem substitutions. Then, for every atom $p(t_{i_1}, \cdots, t_{i_k},\ t_{o_1}, \cdots, t_{o_{k'}}) \in body$, for each $\sigma \in \Theta$ and for each j such that $1 \le j \le k' \ne 0$ or $j = k' = 0$, the term $p^j(t_{i_1}, \cdots, t_{i_k})\sigma$ occurs as a subterm of the right-hand side of a rewrite rule derived from clause c.

Proof : There are two cases:

($k' = 0$) In this case, the term $p^0(t_{i_1}, \cdots, t_{i_k})$ is included in the set $Unsry$ and the function ELIMINATE-LOC-VARS is called with input $Unsry$. Rewrite rules are constructed with terms from the output of ELIMINATE-LOC-VARS on the right-hand sides. It follows from Lemma 7 that the Lemma holds in this case.

($k' > 0$) The lemma is proved in this case using Noetherian induction with the following Noetherian relation $\prec$. The relation $\prec$ is defined over the terms of the form $p^j(t_{i_1}, \cdots, t_{i_k})$, where $p(t_{i_1}, \cdots, t_{i_k}, t_{o_1}, \cdots, t_{o_{k'}})$ is an atom in the body of c and $1 \le j \le k'$, as follows:

$$q^l(s_{i_1}, \cdots, s_{i_n}) \prec p^j(t_{i_1}, \cdots, t_{i_k})$$

if and only if $\langle p^j(t_{i_1}, \cdots, t_{i_k}),\ t_{o_j}\rangle$ is a producer of a variable in $Var(q^l(s_{i_1}, \cdots, s_{i_n})) - invar(head)$. Since the producer-consumer relation of a well-moded clause is acyclic, the relation $\prec$ is Noetherian.

Minimal elements: A minimal element (say, $r^k(u_{i_1}, \cdots, u_{i_m})$) in this relation is either a producer of a variable in $(outvar(head) - invar(head))$ or is a member of $Unsry$.

- If it is a member of $Unsry$, the lemma holds for this element because the function ELIMINATE-LOC-VARS is called with $unsry$ as the input and the output of this function is $\{\ t\sigma \mid t \in Unsry,\ \ \sigma \in \Theta\ \}$. Each term in this set is a subterm of the right-hand side term of a rewrite rule.
- If it is not a member of $Unsry$, it is a producer of a variable (say, X) in an output term (say, t) of the head. By Lemma 7, there is a substitution $\sigma \in \Theta$ such that $X\sigma \equiv C[r^k(u_{i_1}, \cdots, u_{i_m})\sigma]$, where C is a (possibly empty) context of inverse function symbols such that $C[u_{o_k}] \Rightarrow^* X$. The output of the function ELIMINATE-LOC-VARS for input $\{t\}$ is $\{\ t\gamma \mid \gamma \in \Theta\ \}$. Each term in this set is the right-hand side term of a rewrite rule. Since $X \in Var(t)$, $X\sigma \equiv C[r^k(u_{i_1}, \cdots, u_{i_m})\sigma]$ is a subterm of $t\sigma$, the lemma holds.

Non-minimal elements: Now, we prove that the lemma holds for the element $p^j(t_{i_1}, \cdots, t_{i_k})$ if it holds for an element $q^l(s_{i_1}, \cdots, s_{i_n})$ such that $q^l(s_{i_1}, \cdots, s_{i_n}) \prec p^j(t_{i_1}, \cdots, t_{i_k})$. From the definition of the relation $\prec$ it follows that $\langle p^j(t_{i_1}, \cdots, t_{i_k}),\ t_{o_j}\rangle$ is a producer of a variable (say, Y) in $Var(q^l(s_{i_1}, \cdots, s_{i_n})) - invar(head)$. By Lemma 7, corresponding to every substitution $\sigma \in \Theta(p^j(t_{i_1}, \cdots, t_{i_k}))$ there is a substitution $\sigma' \in \Theta(q^l(s_{i_1}, \cdots, s_{i_n}))$ such that

$$Y\sigma' \equiv C[p^j(t_{i_1}, \cdots, t_{i_k})\sigma'] \equiv C[p^j(t_{i_1}, \cdots, t_{i_k})\sigma],$$

for some (possibly empty) context C of inverse function symbols. Hence, $p^j(t_{i_1}, \cdots, t_{i_k})\sigma$ is a subterm of $q^l(s_{i_1}, \cdots, s_{i_n})\sigma'$. By hypothesis, $q^l(s_{i_1}, \cdots, s_{i_n})\sigma'$ occurs as a subterm of the right-hand side of a rewrite rule and hence, the lemma holds for the element $p^j(t_{i_1}, \cdots, t_{i_k})$. $\square$

The following lemma describes the structure of rewrite rules in R_P. It is shown that the function symbols p^i and q^j occur in the right-hand side of rewrite rules in such a way that p^i occurs inside q^j if $q(\cdots)$ is a consumer of a variable (say, X) which occurs in the i^{th} output term of $p(\cdots)$ (i.e., $p(\cdots)$ is a producer of X). In particular, the terms corresponding to the minimal atoms in the body of a clause occur at the innermost level of the right-hand side terms of the rewrite rules.

Lemma 9. Let $p(t_{i_1}, \ldots, t_{i_k},\ t_{o_1}, \ldots, t_{o_{k'}})$, $q(s_{i_1}, \ldots, s_{i_n},\ s_{o_1}, \ldots, s_{o_{n'}})$ be two atoms in the body of a well-moded clause c such that $q(\cdots)$ is a consumer of a variable X and $\langle p^l(t_{i_1}, \ldots, t_{i_k}),\ t_{o_l}\rangle$ is an element in *Prod*(X). Then, *for each $1 \leq j \leq n'$, the term $q^j(s_{i_1}, \ldots, s_{i_n})\sigma$ occurs as a subterm of the right-hand side of a rewrite rule derived from c, where σ is a substitution in Θ such that $X\sigma \equiv C[p^l(t_{i_1}, \ldots, t_{i_k})\sigma]$ and C is a (possibly empty) context of inverse functions such that $C[t_{o_l}] \Rightarrow^* X$.*

Proof : Follows from Lemma 7 (part 2) and Lemma 8. $\square$

The following lemma establishes a correspondence between the computations of logic programs and the derivations of the derived term rewriting systems.

Lemma 10. Let P be a well-moded program, $Q = \leftarrow q(s_{i_1}, \cdots, s_{i_m},\ s_{o_1}, \cdots, s_{o_{m'}})$, be a well-moded query and R_P be the term rewriting system derived from P. *If θ is a computed answer substitution of $P \cup \{Q\}$ then $q^j(s_{i_1}, \cdots, s_{i_m}) \Rightarrow_{R_P} s_{o_j}\theta$, for each $j \in [1, m']$.*

Proof : Induction on the length l of the SLD-refutation of $P \cup \{\leftarrow q(\cdots)\}$. $\square$

5.2. Termination of a given Logic Program and the Derived TRS

Now, we prove that the termination of R_P implies termination of the given program P for all well-moded queries under all selection rules implied by the moding information. For this purpose, we introduce the notion of *rewrite-tree* of a well-moded query and establish the relationship between the SLD-tree starting with a query and the rewrite-tree of that query.

Definition 14. (Rewrite Tree)
Let P be a well-moded program, R_P be the term rewrite system derived from P by the transformation, $Q = \leftarrow q_1(\cdots), \ldots, q_n(\cdots)$ be a well-moded query and R_Q be the set of rewrite rules derived from the clause $query \leftarrow q_1(\cdots), \ldots, q_n(\cdots)$,

where *query* is a fresh predicate of arity 0 not occurring in $P \cup \{Q\}$. The *rewrite-tree* RT_{PQ} of P and Q is defined as follows:

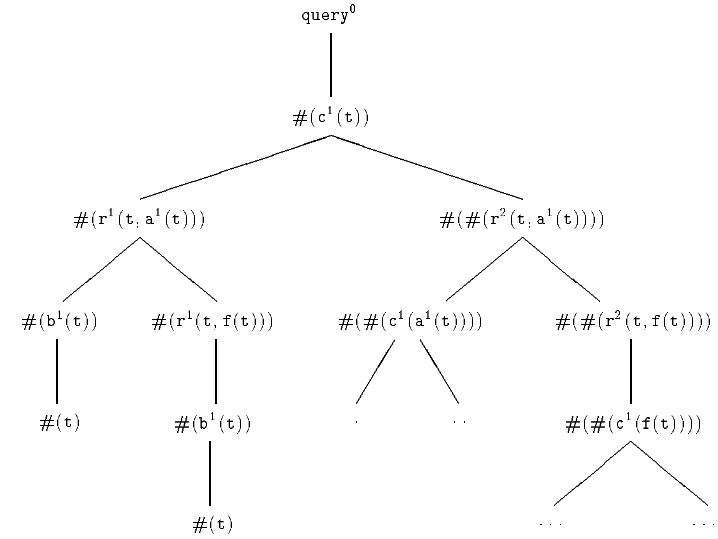
1. Root(RT_{PQ}) $= query^0$
2. Children of a node $t \in RT_{PQ}$ are $\{\ s \mid t \Rightarrow_{R_Q \cup R_P} s\ \}$.

The rewrite-tree RT_{PQ} essentially contains all the (rewriting) derivations of the rewrite system $R_Q \cup R_P$ starting from the initial term $query^0$. The following theorems establish the relationship between RT_{PQ} and the SLD-derivations of $P \cup \{Q\}$. Before giving the formal theorems, we illustrate the relationship between RT_{PQ} and the SLD-derivations through an example.

Example 13. Consider the program given in Example 8 and query $Q = \leftarrow$ c(t,Y), where t is a ground term. Here, we give a possible SLD-derivation and show its correspondence with RT_{PQ}.

$\leftarrow$ c(t,Y)
$\leftarrow$ a(t,Z), r(t,Z,Y,Z1)
$\leftarrow$ r(t,f(t),Y,Z1)
$\leftarrow$ b(t,Y), c(f(t),Z1)
$\leftarrow$ c(f(t),Z1)
$\vdots$

The following diagram shows a (top) portion of the rewrite tree RT_{PQ}. The dots "..." denote a subtree. Observe that corresponding to every selected atom in the SLD-derivation, there is at least one term (with a Skolem function corresponding the predicate of the atom and its input terms) in RT_{PQ}.



In the following, we show that corresponding to every resolution step in SLD-derivations of $P \cup \{Q\}$, there is at least one rewrite step in RT_{PQ}. For establishing this property, we need the following technical lemma.

Lemma 11. Let $G_0, G_1, \cdots, G_n$ be an SLD-derivation of a well-moded program P and a well-moded query Q under a selection rule implied by the moding information of P and Q. Further, assume that $H \leftarrow B_1, \cdots, B_m$ is the input clause and θ_{n_1} is the mgu used in deriving G_{n_1+1} from G_{n_1} and $B_i\theta$ is the selected atom in $G_{n_1+n_2}$, where θ is the composition of mgus used in the derivation $G_{n_1}, \cdots, G_{n_1+n_2}$. Then, $p^j(s_{i_1}, \cdots, s_{i_k})\sigma\theta_{n_1} \Rightarrow^*_{R_P} p^j(s_{i_1}, \cdots, s_{i_k})\theta$ *if* $B_i \equiv p(s_{i_1}, \cdots, s_{i_k},\ s_{o_1}, \cdots, s_{o_{k'}})$ *for each* $\sigma \in \Theta(p^j(s_{i_1}, \cdots, s_{i_k}))$.

Proof : Induction on n_2.

Basis : $n_2 = 1$.
In this case, $\theta = \theta_{n_1}$ and B_i is a minimal element in the producer-consumer relation of the clause $H \leftarrow B_1, \cdots, B_m$. By definition, $domain(\sigma) \cap invar(H) = \phi$. Since B_i is minimal, $Var(p^j(s_{i_1}, \cdots, s_{i_k})) = invar(B_i) \subseteq invar(H)$ and hence $p^j(s_{i_1}, \cdots, s_{i_k})\sigma \equiv p^j(s_{i_1}, \cdots, s_{i_k})$. Since $B_i\theta_{n_1}$ is the selected atom in G_{n_1+1}, its input terms $s_{i_1}\theta_{n_1}, \cdots, s_{i_k}\theta_{n_1}$ are ground. Hence $p^j(s_{i_1}, \cdots, s_{i_k})\sigma\theta_{n_1} \equiv p^j(s_{i_1}, \cdots, s_{i_k})\theta$. The lemma holds.

Induction Hypothesis : Assume that the lemma holds for all $n_2 < l$.

Induction Step : $n_2 = l$.
Let $r^{j1}(t_{i_1}, \cdots t_{i_{k_r}})\sigma\theta_{n_1}$ be a maximal proper subterm (having a Skolem function at

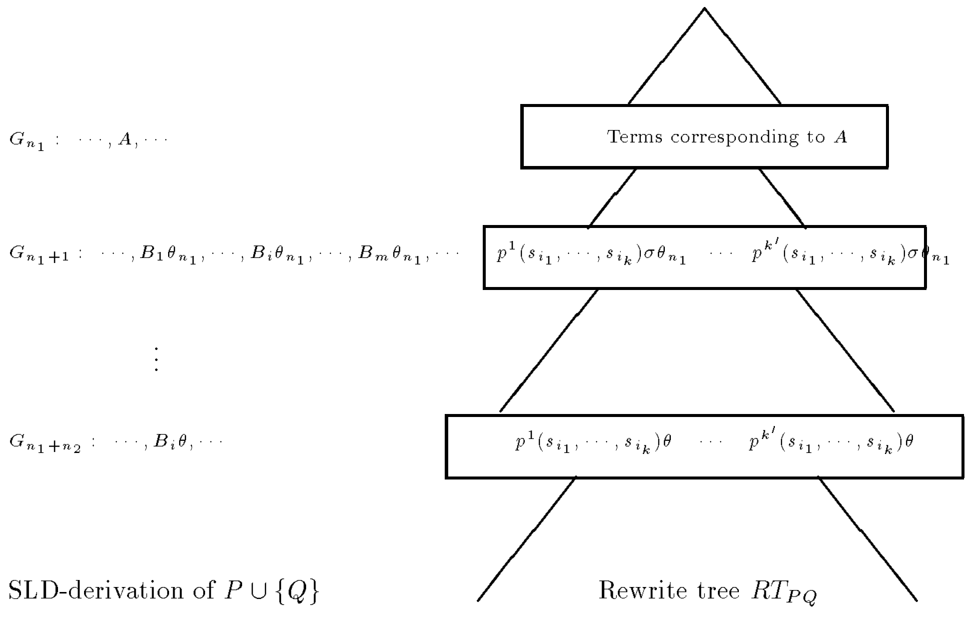
the root) of $p^j(s_{i_1}, \cdots, s_{i_k})\sigma\theta_{n_1}$. Let $G_{n_1+n_r}$ be the goal in which $r(t_{i_1}, \cdots t_{i_{k_r}},\ t_{o_1}, \cdots, t_{o_{k'_r}})\theta'$ is selected and θ' be the composition of mgus used in the derivation $G_{n_1}, \cdots, G_{n_1+n_r}$. It is clear from Lemma 9 that $r(\cdots)$ is a producer of a variable (say, X) in $invar(p(\cdots))$ and $r(\cdots)$ should be selected before $p(\cdots)$ is selected, i.e., $n_r < n_2$. By induction hypothesis,

$$r^{j1}(t_{i_1}, \cdots t_{i_{k_r}})\sigma\theta_{n_1} \Rightarrow^*_{R_P} r^{j1}(t_{i_1}, \cdots t_{i_{k_r}})\theta'.$$

By Lemma 10, $r^{j1}(t_{i_1}, \cdots t_{i_{k_r}})\theta' \Rightarrow^*_{R_P} t_{o_{j1}}\theta'' \equiv t_{o_{j1}}\theta$, where θ'' is the ground computed answer substitution of $r(\cdots)\theta'$. Further, it is easy to see that $C[t_{o_{j1}}\theta''] \Rightarrow^*_{R_P} X\theta''$ if C is the context of inverse function symbols around the subterm $r^{j1}(t_{i_1}, \cdots t_{i_{k_r}})\sigma\theta_{n_1}$ in $p^j(s_{i_1}, \cdots, s_{i_k})\sigma\theta_{n_1}$.

It is easy to see that $p^j(s_{i_1}, \cdots, s_{i_k})\sigma\theta_{n_1}$ can be reduced to $p^j(s_{i_1}, \cdots, s_{i_k})\theta$ by reducing each maximal proper subterm $u^{j2}(s'_{i_1}, \cdots, s'_{i_{k_u}})$ (having a Skolem function at the root) of $p^j(s_{i_1}, \cdots, s_{i_k})\sigma\theta_{n_1}$ to $s'_{o_{j2}}\theta$ and then reducing the contexts of inverse functions. $\Box$

The above lemma establishes a correspondence between SLD-derivations and the derivations in the rewrite tree. The following diagram depicts such a correspondence.



In the following, we prove that corresponding to every selected atom in an SLD-derivation, there is at least one term in RT_{PQ}. This helps us in showing that corresponding to every resolution step in the SLD-derivations of $P \cup \{Q\}$, there is at least one rewrite step in RT_{PQ}.

Theorem 3. If P is a well-moded program, Q is a well-moded query and S is a selection rule implied by $P \cup \{Q\}$, then for each goal G_i in any SLD-derivation $G_1, \cdots, G_n$ of $P \cup \{Q\}$ (under selection rule S), the following holds: *for each minimal (under the evaluation order of G_i) atom $p(t_{i_1}, \cdots, t_{i_k},\ t_{o_1}, \cdots, t_{o_{k'}})$ in G_i and for each j such that $1 \leq j \leq k' \neq 0$ or $j = k' = 0$, the term $p^j(t_{i_1}, \ldots, t_{i_k})$ occurs as a subterm of a node in the rewrite-tree RT_{PQ} of Q.*

Proof : Let $query$ be a fresh predicate symbol of arity 0 and P' be the well-moded program $P \cup \{query \leftarrow Q\}$. It is easy to see that $G_0, G_1, \cdots, G_n$ is a SLD-derivation of $P' \cup \{\leftarrow query\}$ if $G_1, \cdots, G_n$ is a SLD-derivation of $P \cup \{Q\}$ and $G_0 = \leftarrow query$. Now, we prove the theorem using induction on i.

Basis : $i = 1$. The goal G_1 is Q itself. From the validity of Lemma 8 over the rewrite rules in R_Q, it follows that a term $p^j(t_{i_1}, \ldots, t_{i_k})\sigma$ occurs as a subterm of the right-side of rewrite rule in R_Q and hence occurs as a subterm of a node in RT_{PQ}. Since $p(\cdots)$ is a minimal element in the well-moded query, it does not have any variable in input terms and hence, $p^j(t_{i_1}, \ldots, t_{i_k}) \equiv p^j(t_{i_1}, \ldots, t_{i_k})\sigma$ for every substitution σ.

Induction hypothesis : Let assume that the theorem holds for all $i < d$.

Induction step : Now, we prove that the theorem holds for $i = d$.
Let (i) $\leftarrow q_1(\cdots), \ldots, q_{n'}(\cdots)$ be the goal G_{i-1}, (ii) $q_l(\cdots)$ be the selected atom, (iii) $H \leftarrow B_1, \cdots, B_m$ be the input clause and (iv) θ_i be the mgu used in deriving G_i from G_{i-1}. Then G_i is $\leftarrow q_1(\cdots)\theta_i, \ldots, q_{l-1}(\cdots)\theta_i, B_1\theta_i, \ldots, B_m\theta_i, q_{l+1}(\cdots)\theta_i, \ldots, q_{n'}(\cdots)\theta_i$. Now we have two cases: (a) $m \neq 0$, that is, the input clause is not a unit clause and (b) $m = 0$, that is, the input clause is a unit clause.

Case (a) By the assumption on selection rules, minimal elements, $min(G_i) = \{\ q_j(\cdots)\theta_i \mid q_j(\cdots) \in min(G_{i-1})$ and $j \neq l\ \} \cup min(B_1\theta_i, \cdots, B_m\theta_i)$. For those minimal elements which are already in G_{i-1}, the theorem holds by the induction hypothesis. For minimal elements in $B_1\theta_i, \cdots, B_m\theta_i$, the theorem holds because of Lemma 8 (similar argument as in the base case).

Case (b) The set of minimal elements, $min(G_i)$, contains $\{\ q_j(\cdots)\theta_i \mid q_j(\cdots) \in min(G_{i-1})$ and $j \neq l\ \}$ and atoms $q_{l'}(\cdots)\theta_i,\ \ l' \neq l$ such that $q_{l'}(\cdots)$ is only greater than $q_l(\cdots)$ in the evaluation order of G_{i-1}. For minimal elements which are already in $min(G_{i-1})$, the theorem holds by the induction hypothesis.

Now, assume that $p(t_{i_1}, \cdots, t_{i_k},\ t_{o_1}, \cdots, t_{o_{k'}})$ is an element in $min(G_i)$ but not in $min(G_{i-1})$. Since $p(\cdots)$ is an atom in the SLD-derivation, there must be an atom $A \equiv p(s_{i_1}, \cdots, s_{i_k}, s_{o_1}, \cdots, s_{o_{k'}})$ in the body of c' such that $p(\cdots) \equiv A\theta$, where c' is the input clause used in deriving the goal $G_{i'+1}$ from

$G_{i'}$ for some $i' < i$ and θ is the composition of mgus used in $G_{i'}, \cdots, G_i$. Let $r(u_{i_1}, \cdots, u_{i_m},\ u_{o_1}, \cdots, u_{i_{m'}})$ be the selected atom in $G_{i'}$. By the induction hypothesis, $r^j(u_{i_1}, \cdots, u_{i_m})$, $1 \leq j \leq m'$ occurs as a subterm of a node in RT_{PQ}. Now, by Lemma 8 and the construction of RT_{PQ}, $p^j(s_{i_1}, \cdots, s_{i_k})\sigma\theta_{i'}$ occurs as a subterm of a node in RT_{PQ} for each $\sigma \in \Theta(p^j(s_{i_1}, \cdots, s_{i_k}))$ and $1 \leq j \leq k'$, where $\theta_{i'}$ is the mgu used in deriving $G_{i'+1}$ from $G_{i'}$. It follows from Lemma 11 that $p^j(s_{i_1}, \cdots, s_{i_k})\sigma\theta_{i'}$ reduces to $p^j(t_{i_1}, \cdots, t_{i_k})$ by R_P, that is, $p^j(t_{i_1}, \cdots, t_{i_k})$ is a subterm of a node in RT_{PQ}. □

The following theorem establishes that corresponding to every resolution step in SLD-derivations of $P \cup \{Q\}$, there are some rewriting steps in RT_{PQ}.

Theorem 4. If P is a well-moded program, Q is a well-moded query and S is a selection rule implied by $P \cup \{Q\}$, then *corresponding to every resolution step in every SLD-derivation of* $P \cup \{Q\}$ (under selection rule S), *there are reduction steps in the rewrite-tree* RT_{PQ} *of* Q.

Proof : Let us consider a resolution step in which $p(t_{i_1}, \cdots, t_{i_k},\ t_{o_1}, \cdots, t_{o_l})$, $l \neq 0$ is resolved using an input clause c (case $l = 0$ can be handled similarly). By Theorem 3, corresponding to this atom there are terms $p^j(t_{i_1}, \cdots, t_{i_k})$, $1 \leq j \leq l$ occurring as subterms of nodes in the rewrite tree of Q. By Theorem 2, input terms $t_{i_1}, \ldots, t_{i_k}$ of the selected atom $p(\cdots)$ are ground.

Let θ be the mgu and $p(s_{i_1}, \cdots, s_{i_k},\ s_{o_1}, \cdots, s_{o_l})$ be the head of the input clause used in the resolution step. Corresponding to this clause, we have rewrite rules $p^j(s_{i_1}, \cdots, s_{i_k}) \rightarrow r_j$. By definition of the SLD-resolution, $p(s_{i_1}, \cdots, s_{i_k},\ s_{o_1}, \cdots, s_{o_l})\theta \equiv p(t_{i_1}, \cdots, t_{i_k},\ t_{o_1}, \cdots, t_{o_l})\theta$. Therefore, $p^j(s_{i_1}, \cdots, s_{i_k})\theta \equiv p^j(t_{i_1}, \cdots, t_{i_k})\theta \equiv p^j(t_{i_1}, \cdots, t_{i_k})$, $1 \leq j \leq l$ (since $t_{i_1}, \cdots t_{i_k}$ are ground). Hence, the terms $p^j(t_{i_1}, \cdots, t_{i_k})$, $1 \leq j \leq l$, in RT_{PQ} match with the left-hand-sides of rewrite rules derived from clause c and can be reduced. □

The following theorem establishes the relationship between the termination of a given well-moded logic program and that of the derived term rewriting system. Informally, the theorem says that a well-moded logic program terminates for all well-moded queries under any selection rule implied by the moding information, if the derived term rewriting system terminates.

Theorem 5. If P is a well-moded program, Q is a well-moded query, S is a selection rule implied by $P \cup \{Q\}$ and R_P is the term rewriting system derived from P, then *all the SLD-derivations (under* S*) of* $P \cup \{Q\}$ *are of finite length if* R_P *is a terminating system.*

Proof : Follows from Theorem 4 and König's Lemma. □

Before illustrating the application of this theorem in proving termination of logic programs, we briefly review important termination techniques of rewrite systems.

5.3. Termination of term rewriting systems

Though termination of term rewriting systems in general is undecidable – it has been proved that one can simulate Turing machine using a single rewrite rule [11] – several techniques and tools have been proposed in the literature for proving termination of term rewriting systems. One standard technique is to look for a well-founded ordering $\succ$ over $\mathcal{T}$ satisfying a condition: *if* $s \Rightarrow^* t$ *then* $s \succ t$. If the well-founded ordering $\succ$ has the properties of *monotonicity* ($t \succ u$ implies $C[t] \succ C[u]$) and *stability* ($t \succ u$ implies $t\sigma \succ u\sigma$), it is enough to check that $l \succ r$ for each rewrite rule $l \to r$ in the system. Dershowitz [18] has discussed several such well-founded orderings and techniques for proving termination of term rewriting systems. Here we explain two termination techniques, namely, recursive path ordering with status and interpretation based techniques.

Definition 15. A *quasi-ordered* set $(\mathcal{S}, \succsim)$ consists of a set $\mathcal{S}$ and a transitive-reflexive binary relation $\succsim$ defined over $\mathcal{S}$. We define the associated equivalence relation $\approx$ as $s \approx t$ if and only if $s \succsim t$ and $t \succsim s$, and the associated strict partial ordering $\succ$ as $s \succ t$ if and only if $s \succsim t$ but not $t \succsim s$.

We allow an operator to have one of the following three statuses: multiset, lexicographic left-to-right (LR) and lexicographic right-to-left (RL). Multiset status is taken as default status of an operator if it's status is not specified.

Definition 16. (Recursive path ordering with status; [17, 22])
Let $\succsim$ be a quasi-ordering (called *precedence*) over a finite set $\mathcal{F}$ of function symbols. The *recursive path ordering* $\succ_{rpos}$ on the set $\mathcal{T}(\mathcal{F}, \mathcal{X})$ of terms induced by the precedence $\succsim$ is defined recursively as follows:

$s = f(s_1, \ldots, s_m) \succsim_{rpos} g(t_1, \ldots, t_n) = t$ if and only if one of the following is true

— (i) $s_i \succsim_{rpos} t$ for some $i = 1, \ldots, m$ or

— (ii) $f \succ g$ and $s \succ_{rpos} t_j$ for all $j = 1, \ldots, n$ or

— (iii) $f \approx g$, f, g have *multiset* status and $\{s_1, \ldots, s_m\} \succsim\!\!\succsim_{rpos} \{t_1, \ldots, t_n\}$, or

— (iv) $f \approx g$, $s \succ_{rpos} t_j$ for all $j = 1, \ldots, n$ and
(a) f and g have LR status and $(s_1, \ldots, s_m) \succsim^*_{rpos} (t_1, \ldots, t_n)$ or
(b) f and g have RL status and $(s_m, s_{m-1}, \ldots, s_1) \succsim^*_{rpos} (t_n, t_{n-1}, \ldots, t_1)$

where $\succsim\!\!\succsim_{rpos}$ is the multiset ordering induced by $\succsim_{rpos}$ and $\succsim^*_{rpos}$ is the lexicographic ordering induced by $\succsim_{rpos}$.

Theorem 6. For any well-founded quasi-ordering $\succsim$ *and status on a given finite set of function symbols, the recursive path ordering with status* $\succ_{rpos}$ *is a monotonic and stable well-founded ordering* [17, 22].

The above theorem makes the recursive path ordering with status a powerful tool in proving termination of rewrite systems. This ordering has been implemented in theorem provers like RRL, REVE etc. One of the nice properties of the recursive path ordering is that a term is bigger than all its proper subterms. This property is called *subterm property* and we use it frequently in the following.

Example 14. This example illustrates the use of recursive path ordering with status in proving termination of term rewriting systems. To prove termination of the following term rewriting system (derived from the `multiplication` program), take the precedence as $\mathtt{mult}^1 \succ \mathtt{add}^1 \succ \mathtt{s}$.

$$\begin{array}{l}
\mathtt{add}^1(\mathtt{0},\mathtt{Y}) \rightarrow \mathtt{Y} \\
\mathtt{add}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \rightarrow \mathtt{s}(\mathtt{add}^1(\mathtt{X},\mathtt{Y})) \\
\mathtt{mult}^1(\mathtt{0},\mathtt{Y}) \rightarrow \mathtt{0} \\
\mathtt{mult}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \rightarrow \mathtt{add}^1(\mathtt{mult}^1(\mathtt{X},\mathtt{Y}),\mathtt{Y})
\end{array}$$

From the subterm property, it follows that $l \succ_{\mathtt{rpos}} r$ holds for rules 1 and 3. Consider the rule 2. Since $\mathtt{add}^1 \succ \mathtt{s}$, to prove that $\mathtt{add}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \succ_{\mathtt{rpos}} \mathtt{s}(\mathtt{add}^1(\mathtt{X},\mathtt{Y}))$, it is enough to prove that $\mathtt{add}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \succ_{\mathtt{rpos}} \mathtt{add}^1(\mathtt{X},\mathtt{Y})$. To prove $\mathtt{add}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \succ_{\mathtt{rpos}} \mathtt{add}^1(\mathtt{X},\mathtt{Y})$, we need to prove that $\{\ \mathtt{s}(\mathtt{X}),\mathtt{Y}\ \} \succ\!\!\succ_{\mathtt{rpos}} \{\ \mathtt{X},\mathtt{Y}\ \}$, which is indeed the case because $\mathtt{s}(\mathtt{X}) \succ_{\mathtt{rpos}} \mathtt{X}$ by the subterm property.

Now consider rule 4. Since $\mathtt{mult}^1 \succ \mathtt{add}^1$, to prove that $\mathtt{mult}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \succ_{\mathtt{rpos}} \mathtt{add}^1(\mathtt{mult}^1(\mathtt{X},\mathtt{Y}),\mathtt{Y})$, it is enough to prove that $\mathtt{mult}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \succ_{\mathtt{rpos}} \mathtt{mult}^1(\mathtt{X},\mathtt{Y})$ and $\mathtt{mult}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \succ_{\mathtt{rpos}} \mathtt{Y}$. To prove $\mathtt{mult}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \succ_{\mathtt{rpos}} \mathtt{mult}^1(\mathtt{X},\mathtt{Y})$, we need to prove that $\{\ \mathtt{s}(\mathtt{X}),\mathtt{Y}\ \} \succ\!\!\succ_{\mathtt{rpos}} \{\ \mathtt{X},\mathtt{Y}\ \}$, which is indeed the case. Further, $\mathtt{mult}^1(\mathtt{s}(\mathtt{X}),\mathtt{Y}) \succ_{\mathtt{rpos}} \mathtt{Y}$ by the subterm property. This completes the termination proof. $\square$

Example 15. This example illustrates the use of status information. To prove termination of the a term rewriting system with just one rule $(\mathtt{X}+\mathtt{Y})+\mathtt{Z} \rightarrow \mathtt{X}+(\mathtt{Y}+\mathtt{Z})$, we use recursive path ordering with status $\succ_{rpos}$ as follows.

Since both the terms have the same function symbol at the top level, to prove that $(\mathtt{X}+\mathtt{Y})+\mathtt{Z} \succ_{\mathtt{rpos}} \mathtt{X}+(\mathtt{Y}+\mathtt{Z})$, it is enough to prove (i) $(\mathtt{X}+\mathtt{Y},\ \mathtt{Z}) \succ^*_{\mathtt{rpos}} (\mathtt{X},\ \mathtt{Y}+\mathtt{Z})$, (ii) $(\mathtt{X}+\mathtt{Y})+\mathtt{Z} \succ_{\mathtt{rpos}} \mathtt{X}$ and (iii) $(\mathtt{X}+\mathtt{Y})+\mathtt{Z} \succ_{\mathtt{rpos}} (\mathtt{Y}+\mathtt{Z})$. By the subterm property, $\mathtt{X}+\mathtt{Y} \succ_{\mathtt{rpos}} \mathtt{X}$ and hence (i) follows. Again by the subterm property (ii) follows.

To prove (iii), it suffices to prove (a) $(\mathtt{X}+\mathtt{Y},\ \mathtt{Z}) \succ^*_{\mathtt{rpos}} (\mathtt{Y},\ \mathtt{Z})$, (b) $(\mathtt{X}+\mathtt{Y})+\mathtt{Z} \succ_{\mathtt{rpos}} \mathtt{Y}$ and (c) $(\mathtt{X}+\mathtt{Y})+\mathtt{Z} \succ_{\mathtt{rpos}} \mathtt{Z}$. The conditions (b) and (c) follows directly from the subterm property and (a) follows from the fact that $\mathtt{X}+\mathtt{Y} \succ_{\mathtt{rpos}} \mathtt{Y}$. $\square$

Interpretation based termination proofs try to map the set of terms onto a well-founded partially ordered set and prove that left-hand-sides are mapped onto bigger elements than the corresponding right-hand-sides. This is formally described below.

Definition 17. A terminating function τ from a set of terms $\mathcal{T}(\mathcal{F},\mathcal{X})$ to a well-founded partially ordered set $(\mathcal{W},\succ)$ is composed of a set of functions $\{ f_\tau : \mathcal{W}^n \to \mathcal{W}$ — f is a function symbol of arity n in $\mathcal{F}$ $\}$ such that

$$\tau(f(t_1,\cdots,t_n)) = f_\tau(\tau(t_1),\cdots,\tau(t_n)) \text{ for every term } f(t_1,\cdots,t_n) \in \mathcal{T}(\mathcal{F},\mathcal{X})$$

and

$$w \succ w' \text{ implies } f_\tau(\cdots\tau(w)\cdots) \succ f_\tau(\cdots\tau(w')\cdots) \quad \text{for all } w,w' \in \mathcal{W} \text{ and } f \in \mathcal{F}.$$

Basically, terminating function consists of a set of *monotonic* mappings. Polynomial and exponential interpretations are special instances of terminating functions. A terminating function τ from terms to natural numbers, where each f_τ is a polynomial is called a polynomial interpretation. Polynomial interpretations are implemented in REVE [27]. A terminating function τ where each f_τ is a polynomial or an exponential is called an exponential interpretation. Exponential interpretations are implemented in the ORME [28].

Example 16. To prove termination of the following system over set of terms constructed from constants 0 and 1 and the function symbols $+$ and $\times$

$$\begin{array}{l} \mathtt{X} \times (\mathtt{Y} + \mathtt{Z}) \to (\mathtt{X} \times \mathtt{Y}) + (\mathtt{X} \times \mathtt{Z}) \\ (\mathtt{Y} + \mathtt{Z}) \times \mathtt{X} \to (\mathtt{Y} \times \mathtt{X}) + (\mathtt{Z} \times \mathtt{X}) \\ (\mathtt{X} + \mathtt{Y}) + \mathtt{Z} \to \mathtt{X} + (\mathtt{Y} + \mathtt{Z}) \end{array}$$

we can use the following polynomial interpretation:

$$\begin{array}{ll} \tau(0) = 2 & \tau(\mathtt{s} \times \mathtt{t}) = \tau(\mathtt{s}).\tau(\mathtt{t}) \\ \tau(1) = 2 & \tau(\mathtt{s} + \mathtt{t}) = 2\tau(\mathtt{s}) + \tau(\mathtt{t}) + 1. \end{array}$$

□

5.4. *Termination proofs for logic programs*

Now, we illustrate the application of our main theorem in proving termination of logic programs through examples.

Example 17. In Example 14, using recursive path ordering we proved termination of the rewrite system derived from the multiplication program. From Theorem 5, it follows that the `multiplication` program terminates for all well-moded queries. □

Example 18. Termination of the quick-sort program for well-moded queries can be established using the above theorem by proving termination of the derived term rewriting system given in Example 9. Termination of this term rewriting system can be proved using the following elementary interpretation in the ORME system [28].

$$
\begin{array}{ll}
[\![\mathtt{a}^1]\!]((x,\ y),\ (u,\ v)) = (x+u,\ 2y+v) & \\
[\![\mathtt{s}^1]\!]((x,\ y),\ (u,\ v)) = (x+u,\ 2y+v) & [\![\mathtt{q}^1]\!]((x,\ y)) = (2^x,\ y) \\
[\![\mathtt{s}^2]\!]((x,\ y),\ (u,\ v)) = (x+u,\ 2y+v) & [\![\mathtt{nil}]\!] = (0,\ 0) \\
[\![\mathtt{c}]\!]((x,\ y),\ (u,\ v)) = (x+u+2,\ y+v) &
\end{array}
$$

Here, for $f \in \{\mathtt{a}^1, \mathtt{s}^1, \mathtt{s}^2, \mathtt{q}^1, \mathtt{c}, \mathtt{nil}\}$, the function $[\![f]\!]$ denotes the function f_τ given by the elementary interpretation τ. □

5.5. *Converse of the Main Theorem*

The converse of Theorem 5 does not hold as demonstrated by the following example.

Example 19. Let us consider the following well-moded logic program

```
moding: p (in, out)  and  tc (in, out)

p(a,b) ←
p(b,c) ←
tc(X,Y) ← p(X,Z),tc(Z,Y)
```

It is easy to see that this program terminates for all well-moded queries. But the following term rewriting system

$$
\begin{array}{l}
\mathtt{p}^1(\mathtt{a}) \to \mathtt{b} \\
\mathtt{p}^1(\mathtt{b}) \to \mathtt{c} \\
\mathtt{tc}^1(\mathtt{X}) \to \mathtt{tc}^1(\mathtt{p}^1(\mathtt{X}))
\end{array}
$$

derived from the above program has an infinite derivation.

$$\mathtt{tc}^1(\mathtt{a}) \Rightarrow \mathtt{tc}^1(\mathtt{p}^1(\mathtt{a})) \Rightarrow \mathtt{tc}^1(\mathtt{p}^1(\mathtt{p}^1(\mathtt{a}))) \Rightarrow \cdots \Rightarrow \mathtt{tc}^1(\mathtt{p}^1(\cdots \mathtt{p}^1(\mathtt{a}) \cdots)) \Rightarrow \cdots$$ □

In other words, termination of the derived term rewriting system is not a necessary condition for termination of the logic program. Since our aim is to prove termination of logic programs using termination techniques of term rewriting systems, the sufficient conditions for termination of logic programs are more important than the necessary conditions and Theorem 5 provides a sufficient condition for termination of logic programs.

It is interesting to find out classes of programs for which the converse of Theorem 5 holds as well. In the following, we identify two such classes of programs.

Definition 18. A logic program P is a *non-variable-including* program (for short, *nvi*-program) if all the clauses satisfy the property: *all the variables in the body occur in the head.*

The following theorem establishes the converse of Theorem 5 for the class of *nvi*-programs.

Theorem 7. Let P, S, Q and R_P be as in Theorem 5 such that (i) P is an *nvi*-program, (ii) each variable in P has precisely one producer and (iii) the output terms in Q and the bodies of clauses in P are distinct variables. Then, *all the SLD-derivations (under S) of $P \cup \{Q\}$ are of finite length if and only if R_P is terminating.*

Proof : The if-part follows from Theorem 5.

Only-if part: A nice property of SLD-derivations of $P \cup \{Q\}$ is that the output terms of each goal are distinct variables. Therefore, a selected atom unifies with head of any clause whose input terms can match with the input terms of the selected atom. A nice property of R_P is that the Skolem functions are not nested on the right-hand sides of the rewrite rules. Therefore, nesting of Skolem functions is forbidden in all the terms of any rewrite derivation of $R_P \cup \mathcal{R}_Q$ starting from Q_0 and it is easy to exhibit an infinite SLD-derivation of $P \cup \{Q\}$ corresponding to an infinite rewrite derivation of $R_P \cup \mathcal{R}_Q$ starting from Q_0. □

Another class of programs for which the converse of Theorem 5 holds is the following.

Definition 19. A logic program P is a *binary program* if no clause in it has more than one atom in the body.

The following theorem establishes the converse of Theorem 5 for the class of binary programs.

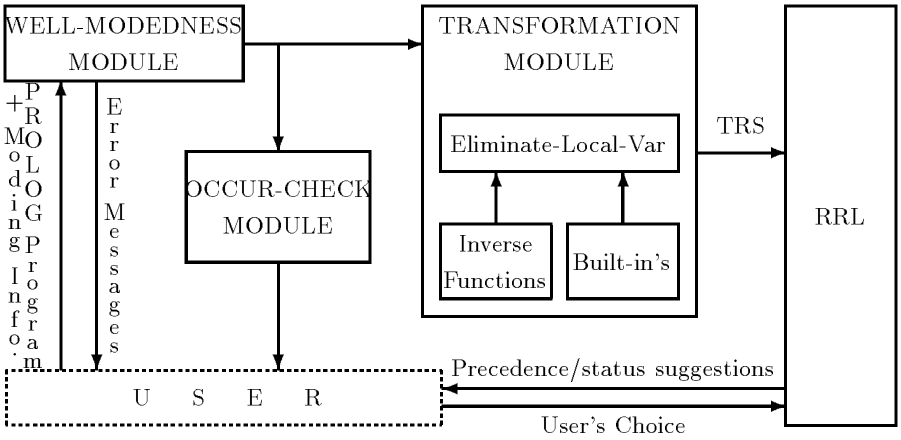
Theorem 8. Let P, S, Q and R_P be as in Theorem 5 such that (i) P is a binary program having just variables in the output positions in the body of each clause, (ii) each variable in P has precisely one producer and (iii) Q has only one atom and has distinct variables in the output positions. Then, *all the SLD-derivations (under S) of $P \cup \{Q\}$ are of finite length if and only if R_P is terminating.*

Proof : Same as that of the above Theorem. □

In fact, there is no sideways information passing in the execution of the above class of programs and every selection rule is implied by the moding information. Therefore, termination of R_P implies termination of P for Q under all selection rules (i.e., strong termination).

6. A tool for proving termination

In the previous sections, we have reduced the termination problem of logic programs to that of term rewriting systems. To obtain a semi-automatic tool for proving termination of logic programs, we have implemented the transformation procedure as a front-end to Rewrite Rule Laboratory (RRL) – a theorem prover based on rewrite techniques developed by Kapur, Zhang and Sivakumar [23]. RRL supports techniques such as recursive path ordering for proving termination of term rewriting systems in an interactive fashion. The following block diagram gives the module structure of the system.



Block-diagram of the modules in the System.

Well-modedness Module: This module checks well-modedness of the given program, applying the following steps on each clause:

1.1 A producer-consumer graph with atoms in the body as nodes is constructed.

1.2 Producer-consumer graph is checked for acyclicity (using depth-first-search).

1.3 If it is not acyclic, a warning is given saying that a particular clause is not well-moded.

2.1 Set of producers of each variable in the clause is computed.

2.2 It is checked whether every variable has at least one producer.

2.3 If any variable has no producer, a warning is given saying that a particular variable has no producer.

Occur-check Module: This module takes well-moded Prolog programs and checks whether they can be executed soundly on Prolog interpreters without occur-check test. This module essentially checks whether any variable occurs more than once in output terms of the head of any clause. If no variable occurs more than once in output terms of heads, the program is declared not subjected to occur-check (NSTO). If there is a variable occurring more than once in output terms of some head, this module notifies the user of the offending variable and the clause.

Transformation Module: This module implements the transformation procedure described in the previous section. For each clause, it computes the sets $Unsry$ and $Consvar$ (as defined in the formal transformation given earlier) and the set of producers of each local variable.[5] The major submodule 'eliminate-local-variable' implements the procedure with the same name described earlier. In addition, there are two more modules; one for generating inverse functions and the other for *built-in's*.

Built-in predicates: Prolog has built-in predicates `=, <, <=, >, >=, \=` with moding `(in, in)` and `is` with moding `(out, in)`. According to the definition of $Unsry$ in the transformation algorithm, for each built-in atom $b(t1, t2)$, $b \in \{$ `=, <, <=, >, >=, \=` $\}$ in a clause, $b^0(t1, t2)$ is included in $Unsry$ and rewrite rules of the form $lhs \rightarrow \#(b^0(t1, t2))$ will be present in R_p. Since we know that the above built-in atoms always terminate, our implementation does not make rules of the above form (this is achieved by removing the above terms from $Unsry$ before making the # rules). An `is`-atom is a producer of the variables occurring in its first argument and the Skolem function is^1 occurs in right-hand-side of the derived rewrite rules. Because of the semantics of `is`, the right-hand-sides of the rules are reduced by the rule $is^1(X) \rightarrow X$ before outputting the rules in R_p.

Krishna Rao, Pandya and Shyamasundar [25] explored the application of our system as a verification tool in the development of provably correct compilers. In particular, they have established the termination of a prototype compiler for **ProCoS** level 0 language PL_0 using the tool. The compiler has been developed using Hoare's refinement algebra approach. The fact that termination of this compiler cannot be shown using other approaches on termination of logic programs (e.g. approaches of Ullman and van Gelder [35], Plümer [32], De Schreye and Verschaetse [12, 13, 37]) demonstrates practicality of the transformational approach.

Consider for instance, the following clause (from the compiler) with moding: `c (in, in, out, out, in, in); ce (in, in, out, out, in, in); mtrans (in, in, out, out); psi (in, in, out)` and `flatten (in, out)`. It is not very difficult to check that this clause does not have an admissible solution graph. Therefore, the approaches of Ullman and van Gelder [35] and Plümer[32] cannot be used

[5] Actually, 'well-modedness' module checks every clause for well-modedness and passes the set of producers of each variable to the 'transformation' module. So there is no repetition of work.

in proving termination of the **ProCoS** compiler.

```
c(Output!E, S, F, M, Psi, Omega) ←
                ce(E, S, L1, M1, Psi, Omega),
                psi(Psi, outputbuf, Psioutputbuf),
                psi(Psi, Output, PsiOutput),
                mtrans(stl(Psioutputbuf), L1, L2, M2),
                mtrans(ldlp(Psioutputbuf), L2, L3, M3),
                mtrans(ldc(PsiOutput), L3, L4, M4),
                mtrans(ldc(4), L4, L5, M5),
                mtrans(out, L5, F, M6),
                flatten([M1, M2, M3, M4, M5, M6], M), !.
```

The following clause (and two more similar clauses) is problematic for the tool built by De Schreye and Verschaetse [12, 13, 37]. However, their tool can handle this clause after unfolding the recursive literal `ce(E1 eql E2, S, L1, M1, Psi, cons(Loc, Omega))` in the body. The problem can be explained as follows. The term size is same for the two terms `E1 <> E2` and `E1 eql E2` which occur in the head and the recursive literal in the body respectively. After applying the unfolding step, the clause contains two recursive literals `ce(E1, ···)` and `ce(E2, ···)` in stead of `ce(E1 eql E2, S, L1, M1, Psi, cons(Loc, Omega))`. The first argument of these atoms are smaller than the first argument of the head. Therefore, the new clause can be handled by the tool built by De Schreye and Verschaetse to handle[6]. In contrast, our tool can handle the original clause, by just taking the precedence `<>` $\succ$ `eql` for the recursive path ordering.

```
ce(E1 <> E2, S, F, M, Psi, cons(Loc, Omega)) ←
                ce(E1 eql E2, S, L1, M1, Psi, cons(Loc, Omega)),
                mtrans(eqc(0), L1, F, M2),
                append(M1, M2, M), !.
```

7. Related Works

In this section, we briefly survey the related works on termination analysis of logic programs. For more details on these works, the reader is referred to a recent survey on termination of logic programs by De Schreye and Decorte [14].

The results on termination of logic programs can be broadly classified into two categories:

1. **Characterizations of terminating programs**: Papers in this category

[6] These comments are based on our e-mail correspondance with Professor De Schreye.

usually present a set of *necessary and sufficient* conditions for the termination of logic programs. Since logic programs have the expressive power of Turing machine, the termination is undecidable and hence these necessary and sufficient conditions are undecidable too. Therefore, the approaches presented in these works may not be easily mechanizable.

2. **Automatic/semi-automatic procedures** for proving termination of logic programs. Papers in this category usually present a set of *sufficient* conditions (which can be easily tested) for the termination of logic programs. Again due to the undecidability of termination of logic programs, none of these sufficient conditions can certify each and every terminating program — for every such sufficient condition there exists a terminating program which does not satisfy it.

In the following, we briefly discuss the works in both these categories.

7.1. Characterizations of terminating programs

Vasak and Potter [36] were the first one (to our knowledge) to attempt giving a mathematical treatment to the termination problem of logic programs. They introduced various notions of termination such as, strong termination, weak termination, universal termination and existential termination. However, there is no attempt towards automatic verification of termination.

Baudinet [6] took a semantic approach towards termination of logic programs. She associated with each program, a system of equations whose least fixpoint is the meaning of the program. By analyzing this fixpoint, various termination properties of the program can be proved. Structural induction is used in her termination proofs. The approach is very general and can be used for studying termination of normal logic programs (i.e., with negation), existential termination (which cannot be studied by most of the other approaches) and the effect of cuts (!) in the programs. Baudinet [6] suggested the use of theorem provers for proving all these properties.

Francez et al [20] took an assertion based approach to termination of logic programs. Proofs are very similar to the termination proofs of imperative logic programs. They give characterizations of both universal and existential termination. There is, however, not much effort towards automatic verification of termination.

Using level mappings, Bezem [7] introduced the class of *recurrent programs* which strongly terminate for a class of queries. He proved that every total recursive function can be computed by some recurrent program. `Append` and `Merge` are examples of recurrent programs, while `Quick-sort, Merge-sort`, etc. do not belong to the class of recurrent programs. Using models in addition to level mappings, Apt and Pedreschi [4] generalized the notion of recurrent programs to *acceptable* programs which terminate for a class of queries under Prolog's left-to-right selection rule.

They called this notion of termination, *left-termination*. Apt and Pedreschi [4] proved that a program terminates for all ground queries under Prolog's selection rule if and only if it is an acceptable program. The main steps in proving a program to be recurrent/acceptable involves finding a suitable level mapping and also a model in the case of acceptability.

Wang and Shyamasundar [38, 39] and Bossi, Cocco and Fabris [8] use graph abstraction to localize the task of finding suitable mappings. These two approaches are very similar. With a given logic program P and a goal G, [38, 39] associated a U-graph and [8] associated a specific graph. The nodes in these graphs are atoms in the program P and goal G, and there are two kinds of edges, namely signed-edges and unification-edges. A signed edge goes from the head of a clause to an atom in the body of the same clause, whereas unification edges go from an atom in the body of a clause/query to the head of a (not necessarily same) clause with which it unifies. The problem of finding a suitable mapping is locally solved by analyzing each strongly connected component (SCC) of the graph. The main steps in proving termination of logic programs using these approaches are construction of the graph, finding a suitable mapping, associating suitable pre/post assertions and proving their correctness and a few simple conditions. The approaches are quite interesting. However, their mechanizability is not very clear.

Shyamasundar, Krishna Rao and Kapur [34] characterized strong termination of logic programs using unification closures. The concept of unification closure is closely related to the concepts of forward closure and overlap closure used in term rewriting literature to characterize the termination of linear term rewriting systems. With a given logic program P, they associate a term rewriting system RR_P and define unification closure of P for a given query Q, denoted UC_{P_Q}, in terms of RR_P and Q. Unification closure is a set of pairs of atoms of the form $\langle A,\ B\rangle$ such that corresponding to every atom in SLD-derivations of P starting with Q, there is a pair in the unification closure UC_{P_Q} and vice versa. This relationship between SLD-derivations and the unification closures led to the following characterization of strong termination: a logic program P is strongly terminating for a query Q if and only if UC_{P_Q} is finite and does not contain any pair of the form $\langle a_1,\ a_2\rangle$ such that $a_1 = a_2\sigma$ for some substitution σ.

7.2. Automatic/semi-automatic procedures

We classify approaches proposed in this direction into two categories: (a) approaches based on linear inequalities and (b) transformational approaches.

Ullman and van Gelder's work [35] is the first one to propose an automatic procedure for proving termination of logic programs. Their approach is to generate a set of linear predicate inequalities (of the form $p_i + c \geq p_j$) from a given well-moded program and a goal so that the satisfiability of these inequalities imply the termination of that program for that goal. The intended meaning of the inequality $p_i + c \geq p_j$ is that the size of j^{th} argument of predicate p is greater than the size of i^{th} argument of p by at most c units. The size of a term is defined as the number of function symbols in the term with an assumption that the only function symbol available is '.', the cons operator. Ullman and van Gelder [35] presented an algorithm to generate a set of such inequalities using data flow analysis through variable/argument graphs. For this algorithm to work, all the clauses in the program must satisfy the 'uniqueness' property. A rule violates this property, if a variable occurs more than once in the input positions of the atoms in the body (see [35] for details).

Plümer [32] improved this method by generalizing the form of inequalities to $\Sigma p_i + c \geq \Sigma p_j$ and allowing function symbols other than the cons, '.' operator. This resulted in a more powerful method. The method also uses data flow analysis for deriving the inequalities and some (not all) of the programs violating 'uniqueness' property can be handled and it uses AND/OR data flow graphs to generate the linear predicate inequalities. The algorithm for generating linear predicate inequalities from a given logic program requires that the associated AND/OR data flow graph satisfies the condition that each OR node has exactly one incoming edge. A graph satisfying this condition is called *admissible solution graph*. This condition is in general violated by (not *all*) programs having a variable occurring in input positions of more than one atom in the body of a clause (See [32] for an example program with a variable occurring in input positions of more than one atom in the body of a clause and having an admissible solution graph). Apparently, this requirement is placed to get an efficient algorithm. This approach cannot handle programs with mutual recursion. Termination of `Append, Merge, Quick-sort` and `Permutation` under Prolog's selection rule can be proved using this approach while termination proofs for `multiplication` and **ProCoS** compiler are beyond its scope, as some of the clauses in these programs do not have an admissible solution graphs.

In contrast to our approach, Plümer's approach is automatic. Our approach often needs precedence information or polynomial/elementary interpretations from the user, while Plümer's algorithm derives the linear predicates from the program and moding information. The time complexity of Plümer's algorithm as well as our transformation procedure is *linear* in the number of atoms in the program and both process the given program clause by clause. Transformation procedure has to analyze the literal dependency graph of each clause, which is simpler than the AND/OR data flow graph (of the clause) used by Plümer . Therefore, the running time of the transformation might be shorter than that of the algorithm for deriving linear predicate inequalities.

De Schreye and Verschaetse [12, 13, 37] proposed an amalgamated approach for termination analysis of logic programs using level mappings, linear inequalities and abstract interpretation. This work is motivated by the difficulties involved in deriving automatic proofs of recurrentness [7] and acceptability [4] of logic programs. It is pointed out by the authors that the difficulty is mainly due to the requirement that the level of head is greater than the levels of all the atoms in the body – in the context of acceptability, this is needed when all the predecessors are satisfiable in the model – despite the fact that nontermination is possible only through recursion in logic programming. By relativizing the notions of recurrent and acceptable programs with respect to the set of queries, they refine these notions in such a way that the level of head has to be greater than the levels of (only) those atoms in the body which are in (mutual) recursion with the head. Given a logic program, they try to derive a system of linear inequalities using abstract interpretation so that the *unsatisfiability* of these inequalities implies acceptability of that program. Termination of `Append, Merge, Quick-sort, Multiplication` and `Permutation` under Prolog's selection rule can be proved using this approach. However, termination of **ProCoS** compiler cannot be proved using this approach – the algorithm needs some of the clauses in the compiler to be unfolded to derive a suitable set of linear inequalities.

Related works in the transformational approach

Our main result in the previous sections deals with termination of logic programs under a class of selection rules rather than a particular selection rule. In fact, the notion of *implied selection rules* suggests a class of suitable selection rules for the execution of a given well-moded program; further, the termination of the derived rewrite system implies the termination of the logic program under all these selection rules. In this sense, the result is stronger than the other results on termination of logic program which deals with termination under a particular selection rule (typically, Prolog's *leftmost* selection rule). However, it is also desirable to study termination under a particular selection rule as the program need not be terminating under all the implied selection rules but can be terminating under a particular selection rule (typically, one of the implied selection rules). Our transformation returns a nonterminating rewriting system in that case and we cannot establish termination under the particular selection rule. In the following, we discuss the results obtained using Prolog's *leftmost* selection rule.

As illustrated below, Krishna Rao, Kapur and Shyamasundar [24] proposed a transformation to transform a given logic program into a new program so that only the Prolog's selection rule is implied by the moding information. Then one can use the results presented in the previous sections for proving termination of the transformed program which in turn implies termination of the original program under Prolog's selection rule.

Example 20. Consider the following reachability program. Given a graph $G(V, Ed)$ with set of vertices V and set of edges Ed represented as list of pairs $f(X, Y)$, the problem is to find all the vertices reachable from a given vertex.

```
moding: r (in,out,in,in), e (in,in,out)  and  nm (in, in)

r(X,Y,Ed,V) ← e(X,Ed,Y)
r(X,Y,Ed,V) ← e(X,Ed,Z),  nm(Z,V),  r(Z,Y,Ed,[Z|V])
e(X,[f(X,Y)|T],Y) ←
e(X,[f(X1,Y)|T],Z) ← X ≠ X1,  e(X,T,Z)
nm(X,[]) ←
nm(X,[H|T]) ← X ≠ H,  nm(X,T)
```

This program terminates for the query $\leftarrow r(a, Y, [f(a,b),\ f(b,c),\ f(c,a),\ f(c,d)], [])$ under Prolog's selection rule, even though it has an infinite evaluation under *rightmost atom* selection rule (which also an implied selection rule). In [24], the following program is derived from the above program. The new program essentially encodes Prolog's selection rule into the original program.

$$\mathtt{moding:}\ \ \mathtt{r_{new}}\ \ (\mathtt{in, in, out, in, in}),\ \ \mathtt{e_{new}}\ \ (\mathtt{in, in, in, out})\ \mathtt{and}\ \mathtt{nm_{new}}\ \ (\mathtt{in, in, in, out})$$

$$\begin{array}{l}
\mathtt{r_{new}(true, X, Y, Ed, V)} \leftarrow \mathtt{e_{new}(true, X, Ed, Y)} \\
\mathtt{r_{new}(true, X, Y, Ed, V)} \leftarrow \mathtt{e_{new}(true, X, Ed, Z)},\ \ \mathtt{nm_{new}(true, Z, V, T)}, \\
\qquad\qquad\qquad\qquad\qquad\qquad \mathtt{r_{new}(T, Z, Y, Ed, [Z|V])} \\
\mathtt{e_{new}(true, X, [f(X, Y)|T], Y)} \leftarrow \\
\mathtt{e_{new}(true, X, [f(X1, Y)|T], Z)} \leftarrow \mathtt{noteq(X, X1, S)},\ \ \mathtt{e_{new}(S, X, T, Z)} \\
\mathtt{nm_{new}(true, X, [], true)} \leftarrow \\
\mathtt{nm_{new}(true, X, [H|T], S1)} \leftarrow \mathtt{noteq(X, H, S)},\ \ \mathtt{nm_{new}(S, X, T, S1)}
\end{array}$$

The transformation derives the following rewrite system.

$$\begin{array}{l}
\mathtt{r^1_{new}(true, X, Ed, V)} \rightarrow \mathtt{e^1_{new}(true, X, Ed)} \\
\mathtt{r^1_{new}(true, X, Ed, V)} \rightarrow \mathtt{r^1_{new}(}\ \ \mathtt{nm^1_{new}(true, e^1_{new}(true, X, Ed), V)}, \\
\qquad\qquad\qquad\qquad\qquad \mathtt{e^1_{new}(true, X, Ed), Ed, [e^1_{new}(true, X, Ed)|V])} \\
\mathtt{e^1_{new}(true, X, [f(X, Y)|T])} \rightarrow \mathtt{Y} \\
\mathtt{e^1_{new}(true, X, [f(X1, Y)|T])} \rightarrow \mathtt{e^1_{new}(noteq(X, X1), X, T)} \\
\mathtt{nm^1_{new}(true, X, [])} \rightarrow \mathtt{true} \\
\mathtt{nm^1_{new}(true, X, [H|T])} \rightarrow\ \ \mathtt{nm^1_{new}(noteq(X, H), X, T)}
\end{array}$$

It is easy to observe that this rewrite system terminates because the first two rules can be applied only when the first argument of $\mathtt{r^1_{new}}$ is *true*, which is passed by $\mathtt{nm^1_{new}}$ and the value of $\mathtt{nm^1_{new}}$ is `true` only when `X` is not a member of the visited vertices. So once all the vertices in a cycle are visited, $\mathtt{nm^1_{new}}$ will not pass `true` to the $\mathtt{r^1_{new}}$ preventing it from entering into a cycle. □

Ganzinger and Waldmann [21] proposed to transform a given logic program into a conditional term rewriting system such that termination of the conditional term rewriting system implies termination of the logic program under Prolog's selection rule. The most appealing thing about this transformation is that there is no need to introduce inverse functions. Their transformation can be explained as follows. They introduce two function symbols p^{in} and p^{out} of arities n and m respectively for each predicate symbol p with n input positions and m output positions. If $p(t_{i_1}, \ldots, t_{i_n},\ t_{o_1}, \ldots, t_{o_m})$ is an atom, $p^{in}[t]$ denotes $p^{in}(t_{i_1}, \ldots, t_{i_n})$ and $p^{out}[t]$ denotes $p^{out}(t_{o_1}, \ldots, t_{o_m})$ where t is the tuple of terms $t_{i_1}, \ldots, t_{i_n}, t_{o_1}, \ldots, t_{o_m}$. From each clause $A_0(t_0) \leftarrow A_1(t_1), \cdots, A_k(t_k)$, they derive a conditional rewrite rule

$$A_0^{in}[t_0] \rightarrow A_0^{out}[t_0] \quad \Leftarrow \quad A_1^{in}[t_1] \rightarrow A_1^{out}[t_1], \ldots, A_k^{in}[t_k] \rightarrow A_k^{out}[t_k].$$

Example 21. For the transitive-closure program given in Example 19, the following *ter-*

minating conditional term rewriting system is derived.

$$
\begin{array}{lcl}
\mathtt{p^in(a)} \rightarrow \mathtt{p^{out}(b)} & & \\
\mathtt{p^in(b)} \rightarrow \mathtt{p^{out}(c)} & & \\
\mathtt{tc^{in}(X)} \rightarrow \mathtt{tc^{out}(Y)} & \Leftarrow & \mathtt{p^{in}(X)} \rightarrow \mathtt{p^{out}(Y)} \\
\mathtt{tc^{in}(X)} \rightarrow \mathtt{tc^{out}(Y)} & \Leftarrow & \mathtt{p^{in}(X)} \rightarrow \mathtt{p^{out}(Z)}, \mathtt{tc^{in}(Z)} \rightarrow \mathtt{tc^{out}(Y)}
\end{array}
$$

As demonstrated by the above example, the class of programs for which the transformation of [21] derives a terminating CTRS properly contains the class of programs for which our transformation derives a terminating TRS.

The fact that the termination of general term rewriting systems is well-studied and easier to understand than the termination of conditional term rewriting systems motivated Aguzzi and Modigliani [1] and Chtourou and Rusinowitch [9] to derive a term rewriting system from a given logic program (rather than a conditional term rewriting system) such that its termination implies the termination of the logic program under Prolog's selection rule. Aguzzi and Modigliani [1] and Chtourou and Rusinowitch [9] independently came up with two similar transformations. Basically, [1, 9] derive a term rewriting system from the conditional term rewriting system derived by Ganzinger and Waldmann. From a non-unit clause $A_0(t_0) \leftarrow A_1(t_1), \cdots, A_k(t_k)$, they derive the following set of rewrite rules

$$
\begin{array}{ll}
\{\ A_0^{in}[t_0] \rightarrow f_1(\mathcal{V}_0, A_1^{in}[t_1])\ \} & \\
\bigcup_{j=1}^{k-1} & \{\ f_j(\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_{j-1}, A_j^{out}[t_j]) \rightarrow f_{j+1}(\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_j, A_{j+1}^{in}[t_{j+1}])\ \} \\
\bigcup & \{\ f_k(\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_{k-1}, A_k^{out}[t_k]) \rightarrow A_0^{out}[t_0]\ \},
\end{array}
$$

where $\mathcal{V}_i$ is the sequence (without repetitions) of variables produced by the atom $A_i(t_i)$ and $f_1, \ldots, f_k$ are distinct function symbols not occurring in the program and the rewrite rules derived from other clauses. From a unit clause $A_0(t_0) \leftarrow$, they derive a rewrite rule $A_0^{in}[t_0] \rightarrow A_0^{out}[t_0]$.

Example 22. The following term rewriting system is derived by [1, 9] from the transitive-closure program given in Example 19.

$$
\begin{array}{l}
\mathtt{p^in(a)} \rightarrow \mathtt{p^{out}(b)} \\
\mathtt{p^in(b)} \rightarrow \mathtt{p^{out}(c)} \\
\mathtt{tc^{in}(X)} \rightarrow \mathtt{f_1(X, p^{in}(X))} \\
\mathtt{f_1(X, p^{out}(Y))} \rightarrow \mathtt{tc^{out}(Y)} \\
\mathtt{tc^{in}(X)} \rightarrow \mathtt{g_1(X, p^{in}(X))} \\
\mathtt{g_1(X, p^{out}(Z))} \rightarrow \mathtt{g_2(X, Z, tc^{in}(Z))} \\
\mathtt{g_2(X, Z, tc^{out}(Y))} \rightarrow \mathtt{tc^{out}(Y)}
\end{array}
$$

As the basic idea of [1, 9] is to derive a TRS from the CTRS derived by [21], the class of programs for which the above transformation derives terminating TRSs is same as that of [21].

Aguzzi and Modigliani [1] identified a class of programs, called *input-driven* programs for which the derived TRS terminates if and only if the logic program terminates for well-moded queries. Arts and Zantema [5] worked on this issue further and investigated the classes of programs for which the derived TRS (their transformation is essentially same as that of [1, 9]) can be proved terminating using recursive path ordering and semantic labelling. Using the results of Zantema [40], they show that their transformation derives from any structural recursive program (cf. [32]) a terminating TRS whose termination can be proved using semantic labelling.

Using the results of [31] on unification-freeness, Marchiori [30] presented transformations for two subclasses (called simply-well-moded and flatly-well-moded programs) of the class of well-moded programs such that they derive a terminating TRS from any terminating program in these classes. The added advantage of unification-freeness of these classes of programs helps in proving termination of logic programs which cannot be proved by the other results in the field. The following program with moding $\mathtt{p(in, out)}$ is an example of this.

$$\begin{array}{l} \mathtt{p(X, g(X)} \\ \mathtt{p(X, f(Y))} \leftarrow \mathtt{p(X, g(Y))} \end{array}$$

8. Conclusion

A transformational approach to the termination analysis of logic programs has been presented by reducing the termination problem of logic programs to that of term rewriting system. Unlike the methods of Ullman and van Gelder [35], Plümer [32] and De Schreye and Verschaetse [12, 13, 37], our method does not need any preprocessing and can prove termination of programs, which cannot be handled by them. Mutual recursion does not give any problem for the proposed method. The proposed transformation is purely syntactical and has been implemented as a front-end to RRL. Using this tool, we have successfully proved termination of a prototype compiler for **ProCoS** language PL_0, which cannot be proved by the other mechanizable approaches.

We study a stronger notion of termination, i.e., termination under a class of selection rules, whereas almost all the other approaches available in the literature deal with termination with respect to a particular selection rule, typically Prolog's left-most atom selection rule. In fact, our notion of implied selection rules characterizes the class of selection rules suitable for a given well-moded logic program.

Though, we have only considered well-moded queries in the previous sections, our results hold for a larger class of queries. The well-modedness ensures that the input terms of every selected atom are ground and the unification of input terms is one sided (i.e., matching). This is the main property which enables applicability of termination techniques of rewriting in proving termination of logic programs.

However, unification of input terms is one-sided for many non-well-moded queries. For example, this is true for queries with (not necessarily ground) lists as inputs in the queries to programs such as `append, permutation, reverse`, etc. In fact, our results extend for all unification-free programs and queries, where unification of input terms is always a matching (see [3, 31, 26] for many classes of unification-free programs). This extension of our main result is comparable to the results of Plümer [33].

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Appendix

The following lemma plays a crucial role in proving that the computations of well-moded programs under implied selection rules are data-driven.

Lemma 12. If P is a well-moded program, G_0 is a well-moded query and S is a selection rule implied by $P \cup \{G_0\}$ and $G_0, G_1, \ldots, G_n$ be a SLD-derivation of $P \cup \{G_0\}$, then all the goals (queries) G_i satisfy the following property: *If an atom A in G_i contains a variable X in its input terms then there exists a producer (say B) of X in G_i such that $B \prec_{G_i} A$,* where $\prec_{G_i}$ is the evaluation order of the goal G_i.

Proof : Induction on i.

Basis : $i = 0$. By the hypothesis of the lemma, G_0 is well-moded. By definition 4, variable X has a producer (call it B). It remains to show that $B \prec_{G_i} A$. By the hypothesis of the lemma, the evaluation order of G_0 is an extension of the producer-consumer relation of G_0. Hence $B \prec_{G_i} A$.

Induction hypothesis : Assume that lemma holds for $i = n$.

Induction step : $i = n + 1$. Let G_n be $\leftarrow q_1(\cdots), \ldots, q_m(\cdots)$. Let $q_j(\cdots)$ be the selected atom, $H \leftarrow B_1, \ldots, B_l$ be the input clause and σ be the mgu in the n^{th} resolution step. Goal G_{n+1} will be $\leftarrow q_1(\cdots)\sigma, \ldots, q_{j-1}(\cdots)\sigma,\ B_1\sigma, \ldots, B_l\sigma,$ $q_{j+1}(\cdots)\sigma, \ldots, q_m(\cdots)\sigma$.

We make a few observations before going for the proof. The mgu σ instantiates only those variables in G_n which occur in $q_j(\cdots)$. Selection rule always selects a minimal element, so $q_j(\cdots)$ is minimal and hence *its input terms are ground* by the induction hypothesis. Now we prove the lemma case by case.

Consider an atom $q_k(\cdots)\sigma,\ k \neq j$ containing a variable X in input position. There are two cases: (i) X is occurring in G_n and (ii) X does not occur in G_n. In case (i), by the induction hypothesis there is a producer (say C) of X such that $C \prec_{G_n} q_k(\cdots)$. There are two sub cases: (a) C is $q_j(\cdots)$ or (b) C is $q_{k'}(\cdots),\ k \neq k' \neq j$. In sub case (a), since $q_j(\cdots)\sigma \equiv H\sigma$, X occurs in $H\sigma$ and therefore X occurs in the well-moded clause $H\sigma \leftarrow B_1\sigma, \cdots, B_l\sigma$. By definition 4, X has a producer in the clause. Since input terms of $H\sigma$ ($\equiv q_j(\cdots)\sigma$) are ground, $H\sigma$ is not a producer of X. So some atom in the body (say $B_{l'}\sigma$) is a producer of X, i.e., X occurs in output positions of $B_{l'}\sigma$ and by the above construction, $B_{l'}\sigma \prec_{G_{n+1}} q_k(\cdots)\sigma$. In sub case (b), $q_{k'}(\cdots)\sigma$ is a producer of X and the lemma holds. This completes case (i).

Case (ii): X does not occur in G_n. That is, σ replaces a variable (say Y) in $q_k(\cdots)$ by a term containing X. By the above observation, variable Y occurs in output terms of $q_j(\cdots)$ (note that input terms are ground). Proof of the lemma in this case is similar to that of sub case (a) of case (i).

Consider an atom $B_m\sigma$, $1 \leq m \leq l$ containing a variable X in input terms. It is obvious that there is a variable Y in the input clause $H \leftarrow B_1, \ldots, B_l$ such that $X \in Var(Y\sigma)$. By definition 4, Y has a producer in well-moded clause $H \leftarrow B_1, \ldots, B_l$. Since input terms of $H\sigma$ are ground, Y does not occur in input terms of H and hence H is not producer of Y. Therefore, some atom (say $B_{m'}$) in the body is a producer of Y and hence $B_{m'}\sigma$ is a producer of every variable in $Y\sigma$, in particular it is a producer of X. By the definition of implied selection rule, $B_{m'}\sigma \prec_{G_{n+1}} B_m\sigma$. This completes the proof of the lemma. □

Now, we can prove Theorem 2.

Theorem 2 If P is a well-moded program, Q is a well-moded query and S is a selection rule implied by $P \cup \{Q\}$, then *every SLD-derivation of* $P \cup \{Q\}$ *is a data-driven evaluation.*

Proof: (By Contradiction). Assume that the selected atom at a resolution step has variables in its input positions. By the above lemma, there are producers which are smaller (under the evaluation order) than the selected atom contradicting the minimality of the selected atom. Therefore, input terms of the selected atom are ground at every resolution step. Hence every SLD-derivation starting with a well-moded query is a *data-driven* evaluation. □