Green's Relations for o-Algebra

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Abstract

Results concerning green's relations in $\circ$-algebra. Here we only consider finite $\circ$-algebra. Unless stated otherwise, M is a finite $\circ$-algebra.

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▶ **Lemma 1.** *Let $e, f \in E(M)$. Then $e\mathcal{J}f \Rightarrow e^\omega \mathcal{L} f^\omega$. In particular, if $e\mathcal{R}f$ then $e^\omega = f^\omega$.*

Proof. $e\mathcal{J}f$ means there exists two elements $x, y \in M$ such that $xy = e$ and $yx = f$. So $e^\omega = (xy)^\omega = x(yx)^\omega = xf^\omega$. Hence $e^\omega \leq_\mathcal{L} e^\omega$. Similarly we can prove $f^\omega \leq_\mathcal{L} e^\omega$ and thus $e^\omega \mathcal{L} f^\omega$.

If $e\mathcal{R}f$, then $e = fe$ and $f = ef$. So $e^\omega = (fe)^\omega = f(ef)^\omega = ff^\omega = f^\omega$. ◀

▶ **Lemma 2.** *Let $a \in M$. Then $a\mathcal{J}a^\omega$ implies the following things-*

1. *$a\mathcal{R}a^\omega$*
2. *a is an idempotent*
3. *all $\mathcal{H}$ class in $\mathcal{J}(a)$ is singleton*
4. *for any $e \in E(M) \cap \mathcal{J}(a)$, $e\mathcal{J}e^\omega$*
5. *there is a special column in $\mathcal{J}(a)$ whose every element is ω power of some idempotent in $\mathcal{J}(a)$. Also for any $e \in E(M) \cap \mathcal{J}(a)$, e^ω resides in this special column.*

Proof.
1. $a^\omega = aa^\omega$, and so $a^\omega \leq_\mathcal{R} a$. Since M is finite, $a\mathcal{J}a^\omega$ and $a^\omega \leq_\mathcal{R} a$ implies $a\mathcal{R}a^\omega$.
2. Since $a\mathcal{R}a^\omega$ and $\rho_{a^\omega}(a) = a^\omega$ where recall that $\rho_x(y) = yx$, from Green's relations over semigroups, we know that $\rho_{a^\omega} : \mathcal{H}(a) \to \mathcal{H}(a^\omega)$ is a bijection.
   Note that since $a\mathcal{J}a^\omega$, it must be that for all $n \in \mathbb{N}^{\geq 1}$, $a^n\mathcal{J}a$. Also for any $n \geq 2$, we have $a^n = aa^{n-1} = a^{n-1}a$. So $a^n\mathcal{H}a$ for all $n \in \mathbb{N}^{\geq 1}$.
   Suppose a is not an idempotent, then $a^2 \in \mathcal{H}(a)$ and $a^2 \neq a$. But $\rho_{a^\omega}(a) = \rho_{a^\omega}(a^2)$ and so $\rho_{a^\omega} : \mathcal{H}(a) \to \mathcal{H}(a^\omega)$ is not a bijection. Contradiction. Hence a must be an idempotent.
3. Suppose $\mathcal{H}(a)$ is not singleton, and there exists $b \in \mathcal{H}(a)$ where $b \neq a$. Since a is an idempotent, by Green's relations, we know that $\mathcal{H}(a)$ is a group with a as its identity. Hence there exists $n \in \mathbb{N}^{\geq 2}$ such that $b^n = a$. But this means $b^\omega = a^\omega$ which implies $\rho_{a^\omega}(b) = \rho_{a^\omega}(a)$ and so again we get a contradiction. Hence $\mathcal{H}(a)$ must be singleton. Since all $\mathcal{H}$-class in a $\mathcal{J}$ -class are of same cardinality, we get what we wanted to prove.
4. By lemma 1, since $a\mathcal{J}e$ and since a is an idempotent, we know that $e^\omega \mathcal{L} a^\omega$. That in addition to the fact that $a\mathcal{J}a^\omega$ means that $e\mathcal{J}e^\omega$.

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5. Since a is an idempotent, $\mathcal{J}(a)$ is a regular $\mathcal{J}$-class. By Green's relations, every row in $\mathcal{J}(a)$ has an idemptent. By the previous result, the ω-power of all those idempotents are in $\mathcal{J}(a)$. Furthermore by lemma 1, all these ω-power elements are in one column of $\mathcal{J}(a)$. So all elements of this special column, called ω-column, are ω-powers of idempotents from the corresponding row.
   In addition by lemma 1, any idempotent in $\mathcal{J}(a)$ will have its ω-power in this ω-column. ◀

▶ **Lemma 3.** *Let $a \in M$. Then $a\mathcal{J}a^{\omega^*}$ implies the following things-*
1. $a\mathcal{L}a^{\omega^*}$
2. *a is an idempotent*
3. *all $\mathcal{H}$ class in $\mathcal{J}(a)$ is singleton*
4. *for any $e \in E(M) \cap \mathcal{J}(a)$, $e\mathcal{J}e^{\omega^*}$*
5. *there is a special row in $\mathcal{J}(a)$ whose every element is ω^* power of some idempotent in $\mathcal{J}(a)$. Also for any $e \in E(M) \cap \mathcal{J}(a)$, e^{ω^*} resides in this special row.*

▶ **Lemma 4.** *Consider $R, S \in 2^M \setminus \emptyset$. Then $S^\eta \mathcal{J} R^\eta$ implies $S^\eta = R^\eta$*

Proof. Note that S^η is a ω-idempotent, ω^*-idempotent and η-idempotent. So $\mathcal{J}(S^\eta)$ has a ω-column and a ω^*-row and S^η is in the intersection of these two. Similarly, R^η is also in the same $\mathcal{H}$-class. But all $\mathcal{H}$-classes in this $\mathcal{J}$-class are singleton. Hence $S^\eta = R^\eta$. ◀

▶ **Lemma 5.** *Let J be a regular $\mathcal{J}$-class. Then the following are equivalent.*
1. *J contains an ordinal idempotent*
2. *J contains an idempotent e such that $e^\omega \in J$.*
3. *Every $\mathcal{R}$ class in J contains an idempotent e such that $e^\omega \in J$*

Proof. 1 $\implies$ 2 by definition and 3 $\implies$ 2 is obvious. 2 $\implies$ 3 because if J conains an idempotent e, then every $\mathcal{R}$ class of J contains an idempotent. Furthermore since $e\mathcal{J}e^\omega$, all these idempotents have their ω-power in the ω-column of J. Now 2 $\implies$ 1 because if J conains an idempotent e, then every $\mathcal{L}$ class of J contains an idempotent. In particular, the ω-column of J must have an idempotent and its ω-power must be itself as the $\mathcal{H}$-classes are singleton. ◀

A $\mathcal{J}$-class satisfying one of the clauses of the previous lemma is said ordinal regular. Similarly, a $\mathcal{J}$-class J is called ordinal* regular (resp. shuffle regular and scattered regular) if J contains a ordinal* idempotent (resp. shuffle idempotent and scattered idempotent).

▶ **Lemma 6.** *Let J be a regular $\mathcal{J}$-class. Then the following are equivalent.*
1. *J contains an ordinal* idempotent*
2. *J contains an idempotent e such that $e^{\omega^*} \in J$.*
3. *Every $\mathcal{L}$ class in J contains an idempotent e such that $e^{\omega^*} \in J$*

▶ **Lemma 7.** *Let J be a regular $\mathcal{J}$-class. Then the following are equivalent.*
1. *J is scattered regular*
2. *for all idempotents e in J, $e^\omega.e^{\omega^*} = e$*
3. *there exists an idempotent e in J such that $e^\omega.e^{\omega^*} = e$.*

Proof. 1 $\implies$ 3 and 2 $\implies$ 3 are obvious. We prove 3 $\implies$ 2 and 3 $\implies$ 1. Let 3 hold. Then $e\mathcal{J}e^\omega$ and $e\mathcal{J}e^{\omega^*}$. This means by lemma 2 and lemma 3 that for every idempotent $f \in \mathcal{J}(e)$, $f\mathcal{J}f^\omega$ and $f\mathcal{J}f^{\omega^*}$. Now since $e = e^\omega.e^{\omega^*}$, by Green's relations on semigroups, we know that the element g "in the opposite corner", i.e. in the intersection of the ω-column and the ω^*-row is an idempotent. Clearly g must be a scattered idempotent. Thus 3 $\implies$ 1 is proved. Furthermore since g is an idempotent, for any idempotent f, $f^\omega.f^{\omega^*}$ must be "in the opposite corner" and $\mathcal{H}(f)$ being singleton, this means $f = f^\omega.f^{\omega^*}$. Thus 3 $\implies$ 2. ◀

▶ **Lemma 8.** *Let J be a regular $\mathcal{J}$-class. Then the following are equivalent.*

1. *J is shuffle regular*
2. *for all idempotents e in J, $(e^{\omega^*}.e^\omega)^\eta = e^{\omega^*}.e^\omega$ and $e^{\omega^*}.e^\omega \in \mathcal{J}(e)$*
3. *there exists an idempotent e in J such that $(e^{\omega^*}.e^\omega)^\eta = e^{\omega^*}.e^\omega$ and $e^{\omega^*}.e^\omega \in \mathcal{J}(e)$*

Proof. Again 2 $\implies$ 3 is obvious, and 1 $\implies$ 3 because a shuffle idempotent is also an ordinal idempotent as well as an ordinal* idempotent. Now we prove 3 $\implies$ 1 and 3 $\implies$ 2. Let 3 hold. Then $e^{\omega^*}.e^\omega \in \mathcal{J}(e)$ is a shuffle idempotent. Thus 3 $\implies$ 1 is proved. Let us call this shuffle idempotent g. By previous lemmas, it should be clear that $g = f^{\omega^*}.f^\omega$ for any idempotent $f \in \mathcal{J}(e)$. Hence 3 $\implies$ 2. ◀