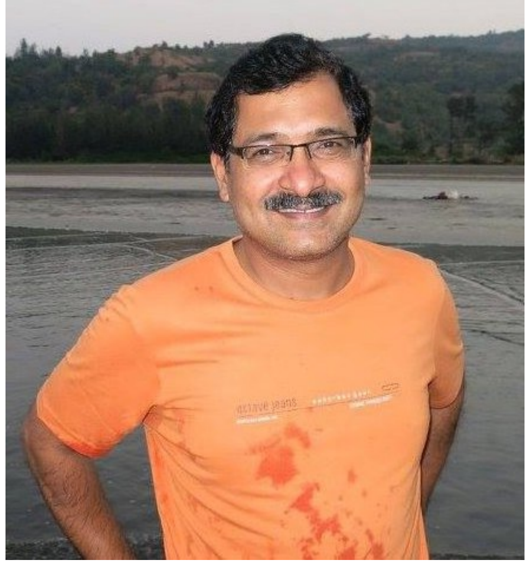
# Block Products for Algebras over Countable words and Applications to Logic

Saptarshi Sarkar, IIT Bombay



Joint work with:
Bharat Adsul, IIT Bombay
&
A.V. Sreejith, IIT Goa

LICS 2019

# This talk is

- in the setting of Countable Linear Orderings

- about finding small objects

  that combine to recognize FO or LTL languages

# Background

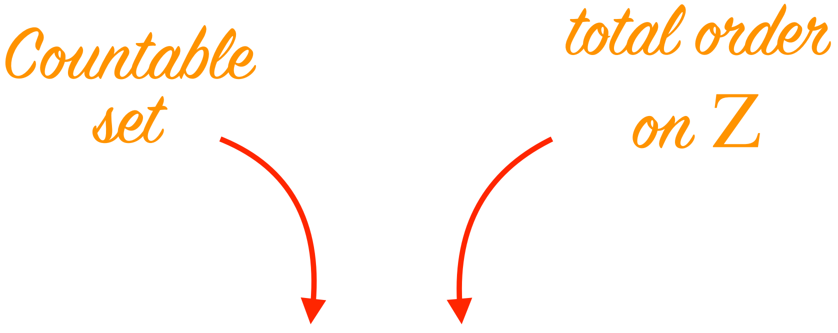
# The Model

Words over countable linear orderings

# Countable Linear Ordering

Countable Linear Ordering - $\left( \overset{\text{Countable set}}{Z} , \overset{\text{total order on Z}}{<} \right)$

# Countable Linear Ordering



*Countable set*

*total order on Z*

Countable Linear Ordering - ( $Z$ , < )

Example 1

({1, 2, 3, 4, 5}, < )

# Countable Linear Ordering

Countable set

total order on Z

Countable Linear Ordering - ( $Z$ , $<$ )

**Example 1**

$(\{1, 2, 3, 4, 5\}, <)$

**Example 2**

$\omega : (\mathbb{N}, <)$

# Countable Linear Ordering

Countable set

total order on Z

Countable Linear Ordering - ( $Z$ , $<$ )

Example 1

$(\{1, 2, 3, 4, 5\}, <)$

Example 2

$\omega : (\mathbb{N}, <)$

Example 3

$\omega^* : (\mathbb{N}^{-}, <)$

# Countable Linear Ordering

Countable set

total order on Z

Countable Linear Ordering - $( Z , < )$

**Example 1**

$(\{1, 2, 3, 4, 5\}, <)$

**Example 2**

$\omega : (\mathbb{N}, <)$

**Example 3**

$\omega^* : (\mathbb{N}^-, <)$

**Example 4**

$\alpha : \omega + \omega^*$

# Countable Linear Ordering

Countable set

total order on Z

Countable Linear Ordering - $( Z , < )$

**Example 1**

$(\{1, 2, 3, 4, 5\}, < )$

**Example 2**

$\omega : (\mathbb{N}, < )$

**Example 3**

$\omega^* : (\mathbb{N}^-, < )$

**Example 4**

$\alpha : \omega + \omega^*$

**Example 5**

$\eta = (\mathbb{Q}, < )$

# Countable Words

countable word $u : \overset{\text{countable linear ordering}}{\alpha} \to \overset{\text{finite alphabet}}{\Sigma}$

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## Examples

# Countable Words

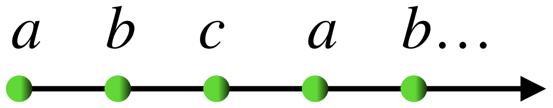
countable linear ordering

finite alphabet

countable word $u : \alpha \to \Sigma$

## Examples

1. $(abc)^\omega$



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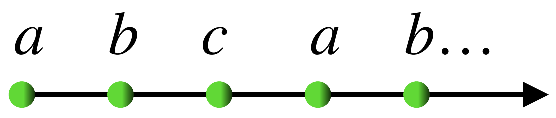
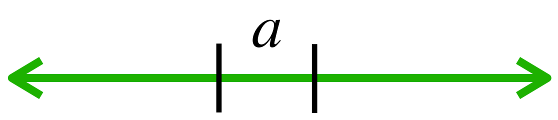


2. $\{a, b, c\}^\eta$



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countable linear ordering

finite alphabet

countable word $u : \alpha \to \Sigma$

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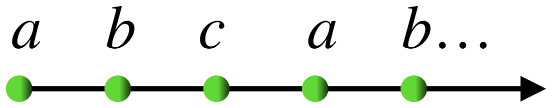

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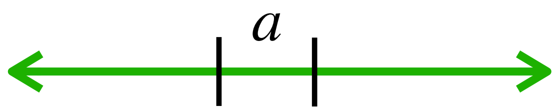

set of all countable words – $\Sigma^{\circledast}$

# Algebra in Formal Language

| | | | |
|---|---|---|---|
| 1960 | Finite Words | Finite Monoids | Schutzenberger<br>McNaughton |
| 1993 | $\omega$ words | Wilke Algebras | Thomas Wilke |
| 2007 | Countable<br>Scattered Words | Finite $\diamond-$Monoid | Bedon<br>Bes<br>Carton<br>Bruyere<br>et . al . |
| 2011 | Countable Words | Finite $\circledast-$Monoid<br>Finite $\circledast-$Algebra | Carton<br>Colcombet<br>Puppis |

# ⊛ – Monoid

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Monoid $(S, \cdot\,)$ $\cdot : S^* \to S$ Associativity

# $\circledast$ – Monoid

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| Monoid | $(S, \cdot\ )$ | $\cdot : S^* \to S$ | Associativity |
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$u$ ——————————— $\pi(u)$

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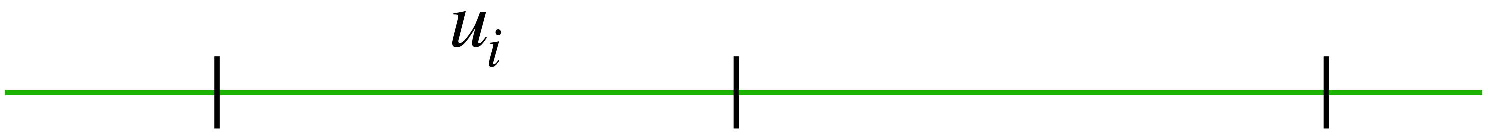

$\pi(u)$

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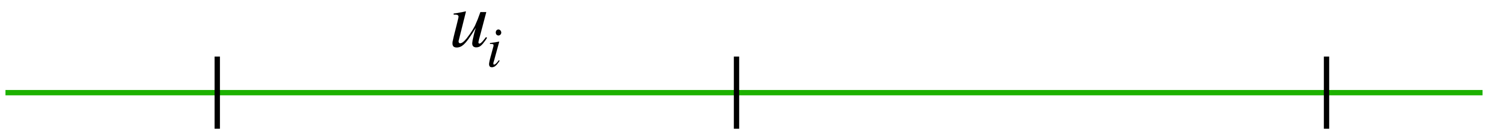

$\pi(\Pi_{i \in \alpha} u_i)$

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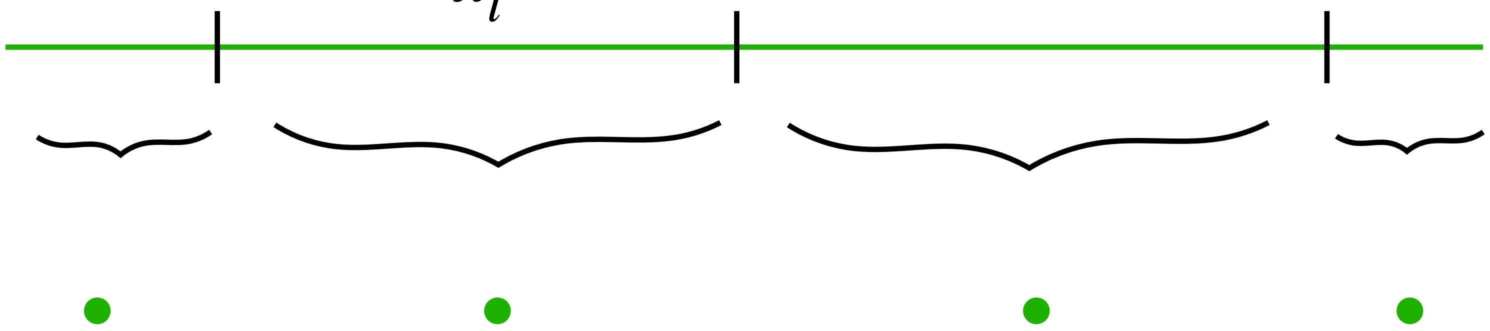

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$u_i$

$\pi(\Pi_{i \in \alpha} u_i)$

$\pi(u_i)$

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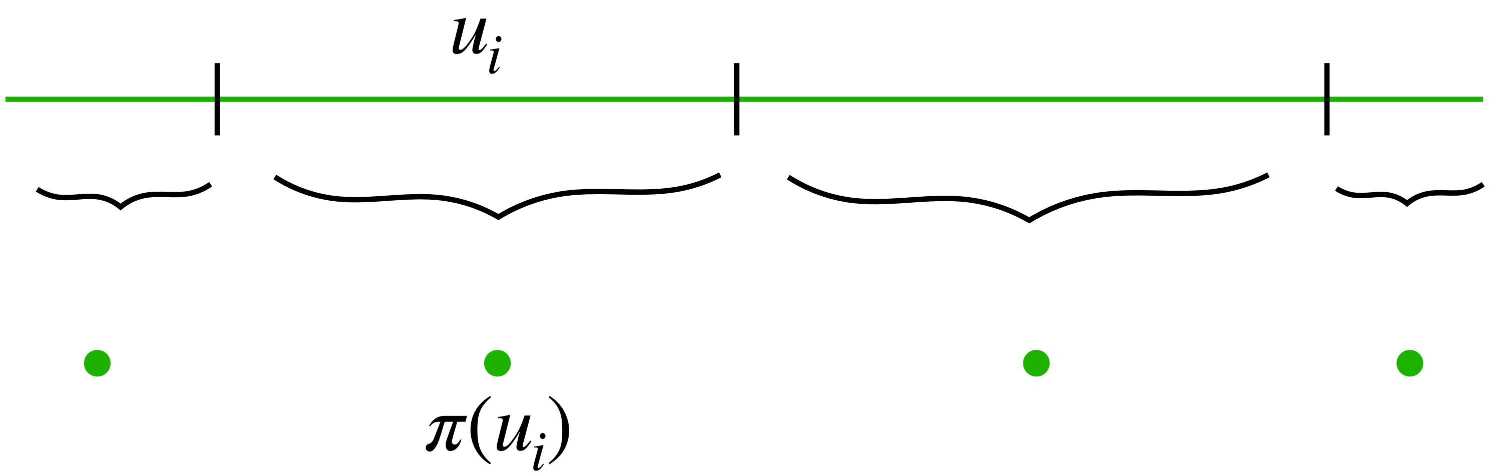

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$u_i$

$\pi(\Pi_{i \in \alpha} u_i)$

$\Pi_{i \in \alpha} \pi(u_i)$

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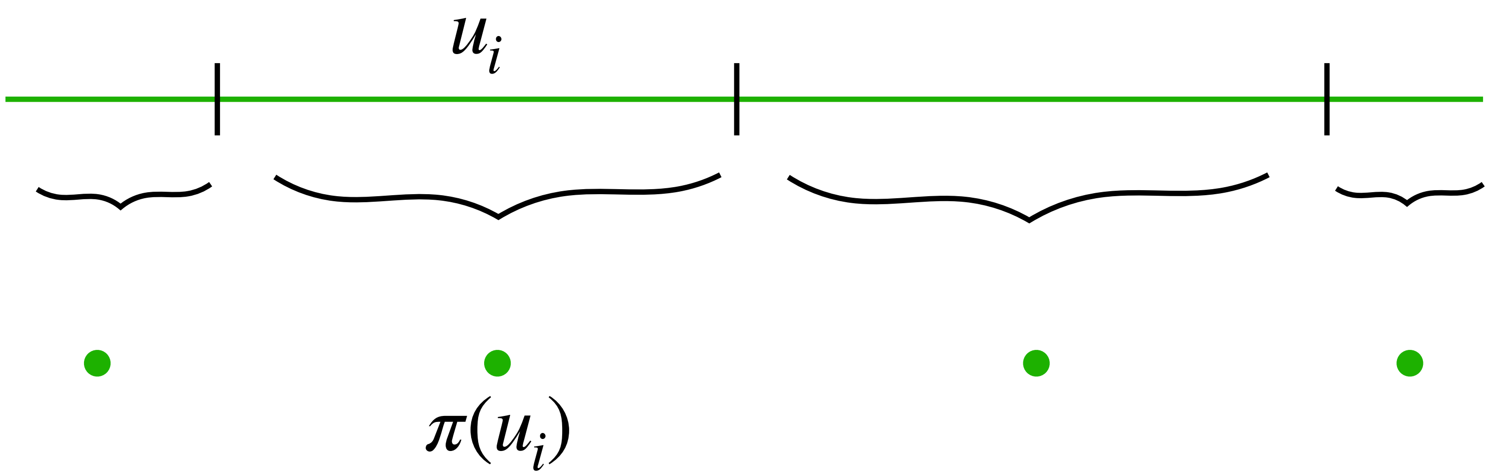

$\pi(\Pi_{i\in\alpha} u_i)$

$\Pi_{i\in\alpha}\pi(u_i)$

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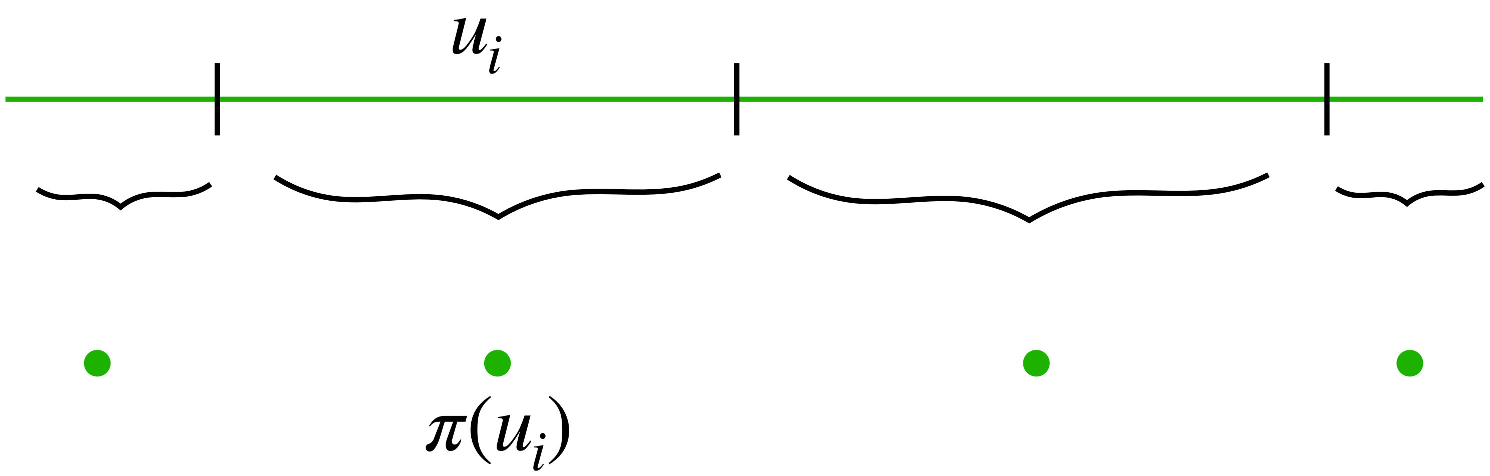

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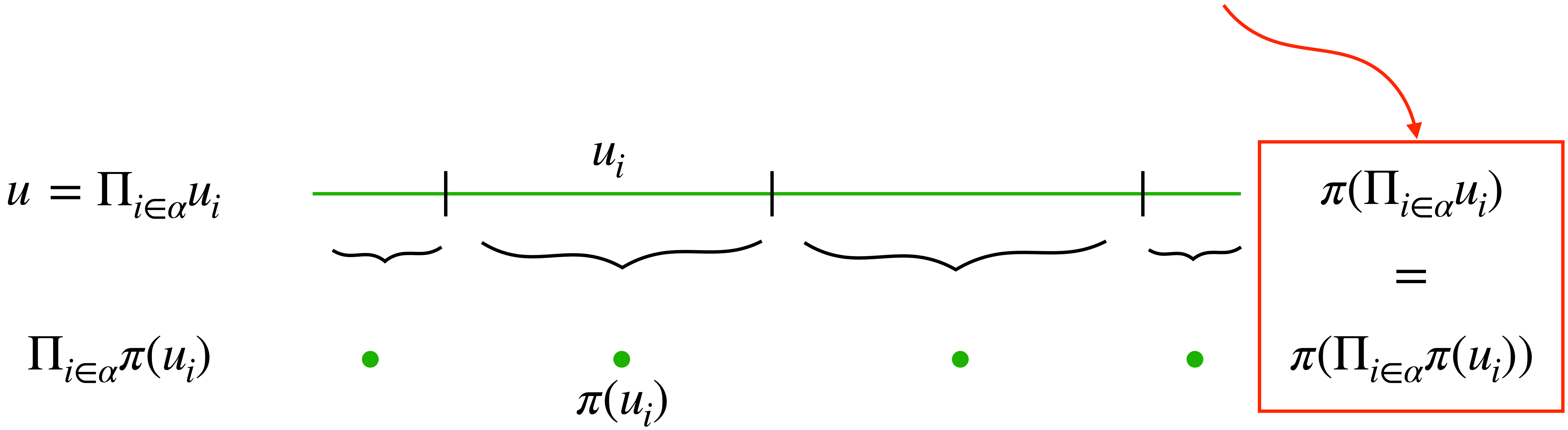

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# $\circledast$ – Monoid examples

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Example 1 $(\Sigma^{\circledast}, \Pi)$

*string concatenation*

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Example 2 $U_1 = (\{0, 1\}, \pi)$

$$\pi(u) = \begin{cases} 1 & \text{if } u \text{ has no } 0 \\ 0 & \text{if } u \text{ has at least one } 0 \end{cases}$$

# ⊛ – Monoid examples

**Example 1** $(\Sigma^{\circledast}, \Pi)$

string concatenation

**Example 2** $U_1 = (\{0,\ 1\}, \pi)$

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**Example 3** $Sing = (\{1,\ s,\ 0\}, \pi)$

$$\pi(u) = \begin{cases} 1 & \text{if } u \text{ has no } s \text{ or } 0 \\ s & \text{if } u \text{ has exactly one } s \\ & \quad \text{and no } 0 \\ 0 & \text{if } u \text{ has at least one } 0 \\ & \quad \text{or more than one s} \end{cases}$$

# Morphism

# Morphism

$h : (S, \pi) \rightarrow (T, \pi')$

$$\begin{tikzcd}
S^{\circledast} \arrow[r, "\pi"] \arrow[d, "\bar{h}"'] & S \arrow[d, "h"] \\
T^{\circledast} \arrow[r, "\pi'"] & T
\end{tikzcd}$$

$h(\pi(u)) = \pi'(\bar{h}(u))$ where $u \in S^{\circledast}$

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$h(\pi(u)) = \pi'(\bar{h}(u))$ where $u \in S^{\circledast}$

$(\Sigma^{\circledast}, \Pi)$ is a free $\circledast -$ monoid

- Any morphism $h : \Sigma^{\circledast} \to S$ is completely determined by $h$ restricted to $\Sigma$
- Conversely, any function $h : \Sigma \to S$ uniquely extends to a morphism

# Language Recognition

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morphism

language recognized by $h$

subset of $S$

$$h : \Sigma^{\circledast} \to S \qquad L = h^{-1}(X)$$

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Example

$$h : \{a, b\}^{\circledast} \to U_1$$

$$a \mapsto 0$$

$$b \mapsto 1$$

$$h^{-1}(0) = \{u \mid u \text{ contains letter } a\}$$

$$U_1 = (\{0,1\}, \pi) \qquad \pi(u) = \begin{cases} 1 & \text{if } u \text{ has no } 0 \\ 0 & \text{if } u \text{ has at least one } 0 \end{cases}$$

# Language Recognition

morphism

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subset of $S$

$$h : \Sigma^{\circledast} \to S \qquad L = h^{-1}(X)$$

Example

$$h : \{a, b\}^{\circledast} \to Sing$$
$$a \mapsto s$$
$$b \mapsto 1$$

$$h^{-1}(s) =$$
$$\{u \,|\, u \text{ contains exactly one position labelled } a\}$$

$$Sing = (\{1,\ s,\ 0\}, \pi) \qquad \pi(u) = \begin{cases} 1 & \text{if } u \text{ has no 0 or } s \\ 0 & \text{if } u \text{ has a 0 or multiple } s \\ s & \text{otherwise} \end{cases}$$

# MSO Logic Definability

$$\phi = a(x) \mid x < y \mid \neg\phi \mid \phi \vee \phi \mid \exists x\ \phi \mid \exists X\ \phi$$

The following two statements are equivalent [CCP2011]:

1) A language of countable words is MSO logic definable

2) A language of countable words is recognized by some morphism to a finite $\circledast-$Monoid

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MSO logic definability is decidable?

# $\circledast$ – Algebra

# ⊛ – Algebra

no finite presentation

$(S, \pi)$

# ⊛ – Algebra

*no finite presentation*

$(S,\ \pi)$

$(S,\ \cdot\ ,\ \tau,\ \tau^*,\ \kappa)$

# ⊛ – Algebra

no finite presentation

$$(S, \pi)$$

$$\cdot : S \times S \to S$$

$$\tau : S \to S$$

$$\tau^* : S \to S$$

$$\kappa : \mathscr{P}(S) \backslash \varnothing \to S$$

$$(S, \cdot\, , \tau, \tau^*, \kappa)$$

$$a \cdot b = \pi(ab)$$

$$a^\tau = \pi(a^\omega)$$

$$a^{\tau^*} = \pi(a^{\omega^*})$$

$$\{a, b, c\}^\kappa = \pi(\{a, b, c\}^\eta)$$

# ⊛ – Algebra

no finite presentation

finite $S$

$$\boxed{(S,\ \pi) \qquad \equiv \qquad (S,\ \cdot\ ,\ \tau,\ \tau^*,\ \kappa)}$$

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$\kappa : \mathscr{P}(S) \backslash \varnothing \to S$ $\qquad$ $\{a,\ b,\ c\}^\kappa = \pi(\{a,\ b,\ c\}^\eta)$

# $\circledast$ – Algebra examples

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Example 1 $\quad Sing = (\{1,\ s,\ 0\}, \pi)$

$$\pi(u) = \begin{cases} 1 & \text{if } u \text{ has no } s \text{ or } 0 \\ s & \text{if } u \text{ has exactly one } s \\ & \quad \text{and no } 0 \\ 0 & \text{otherwise} \end{cases}$$

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$Sing = (\{1, s, 0\}, \cdot\ , \tau, \tau^*, \kappa)$

$s \cdot s = 0 \qquad s^\tau = s^{\tau^*} = 0$

$$P^\kappa = \begin{cases} 1 & \text{if } P = \{1\} \\ 0 & \text{otherwise} \end{cases}$$

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# Block Product

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$(M, \cdot\,)$ $(N, +\,)$ $M \,\square\, N$

# Block Product (operational view)

$(M, \cdot\,)$ $(N, +\,)$ $M \,\Box\, N$

$h : \Sigma \to M \,\Box\, N$

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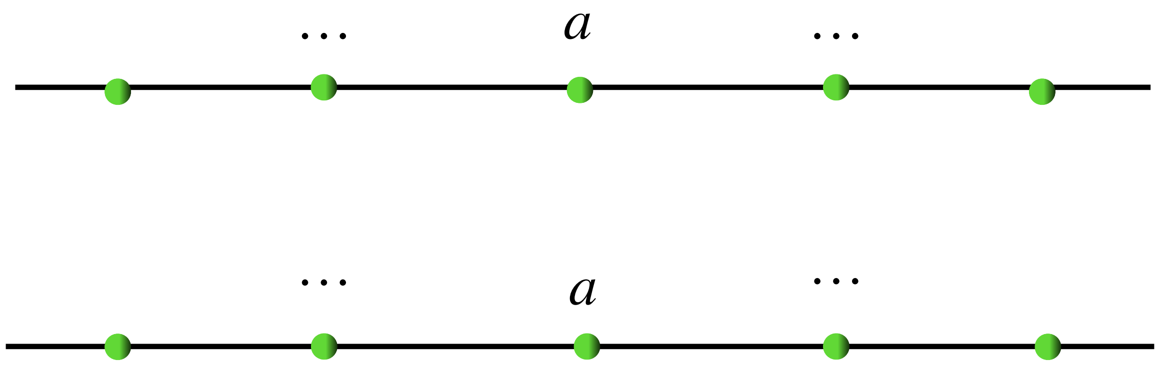
$(M, \cdot\,)$ $(N, +\,)$ $M \Box N$

$h : \Sigma \to M \Box N$

$h_1 : \Sigma \to M$



$u$

$u$

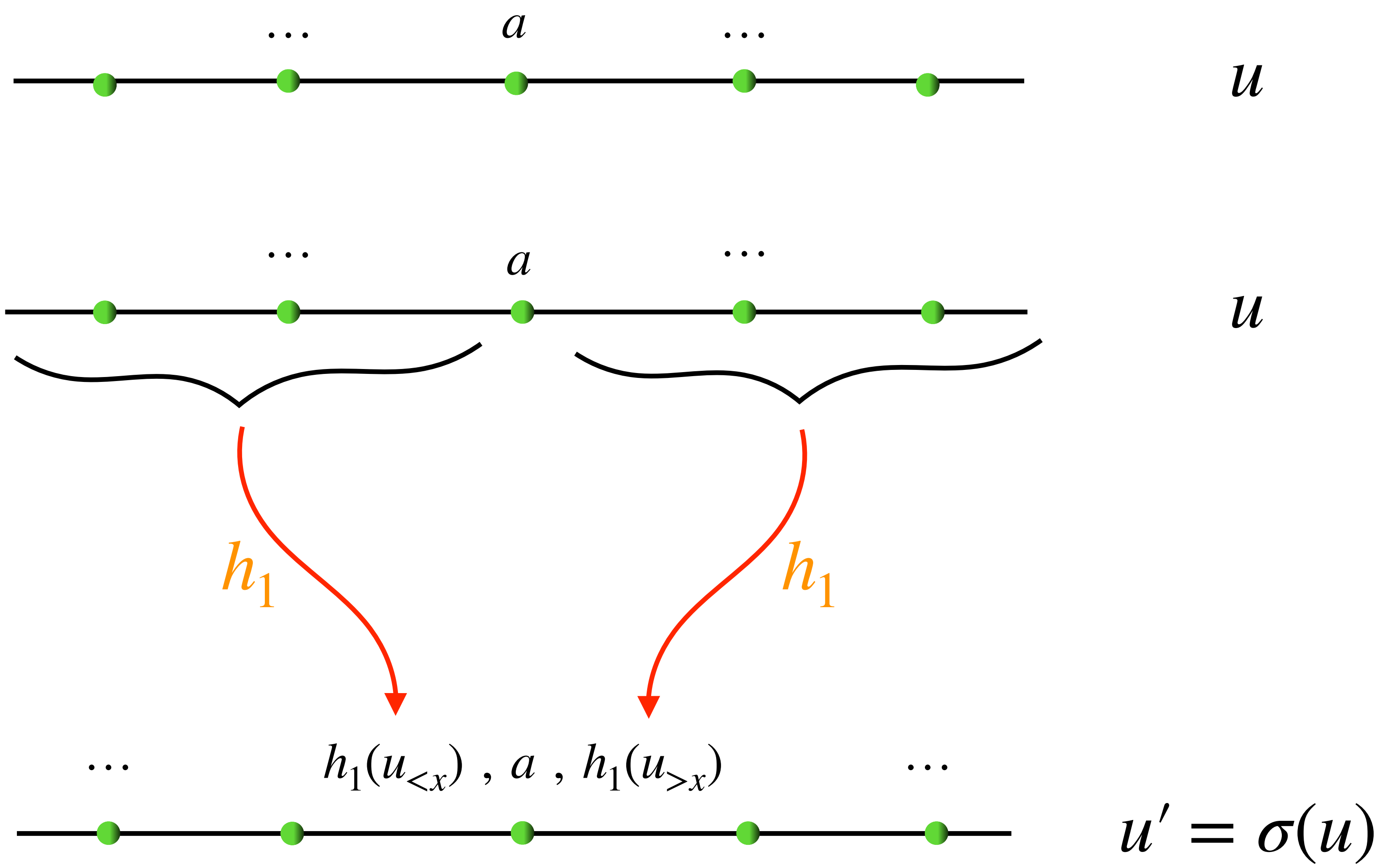
# Block Product (operational view)

$(M, \cdot\,)$ $(N, +\,)$ $M \square N$

$h : \Sigma \to M \square N$

$h_1 : \Sigma \to M$

$\sigma : \Sigma^* \to (M \times \Sigma \times M)^*$



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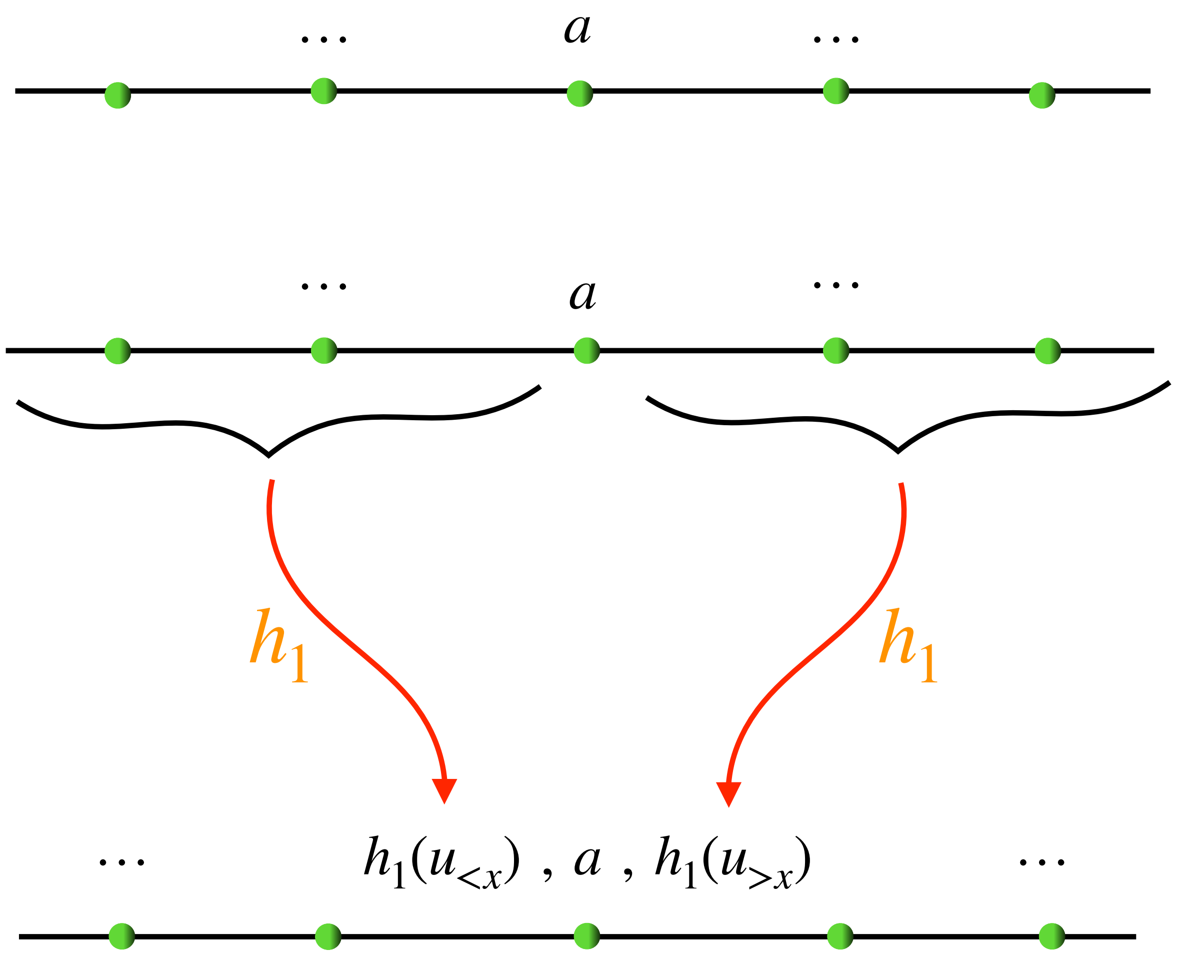
$h : \Sigma \to M \,\square\, N$

$h_1 : \Sigma \to M$

$\sigma : \Sigma^* \to (M \times \Sigma \times M)^*$

$h_2 : M \times \Sigma \times M \to N$



$u$

$u$

$u' = \sigma(u)$

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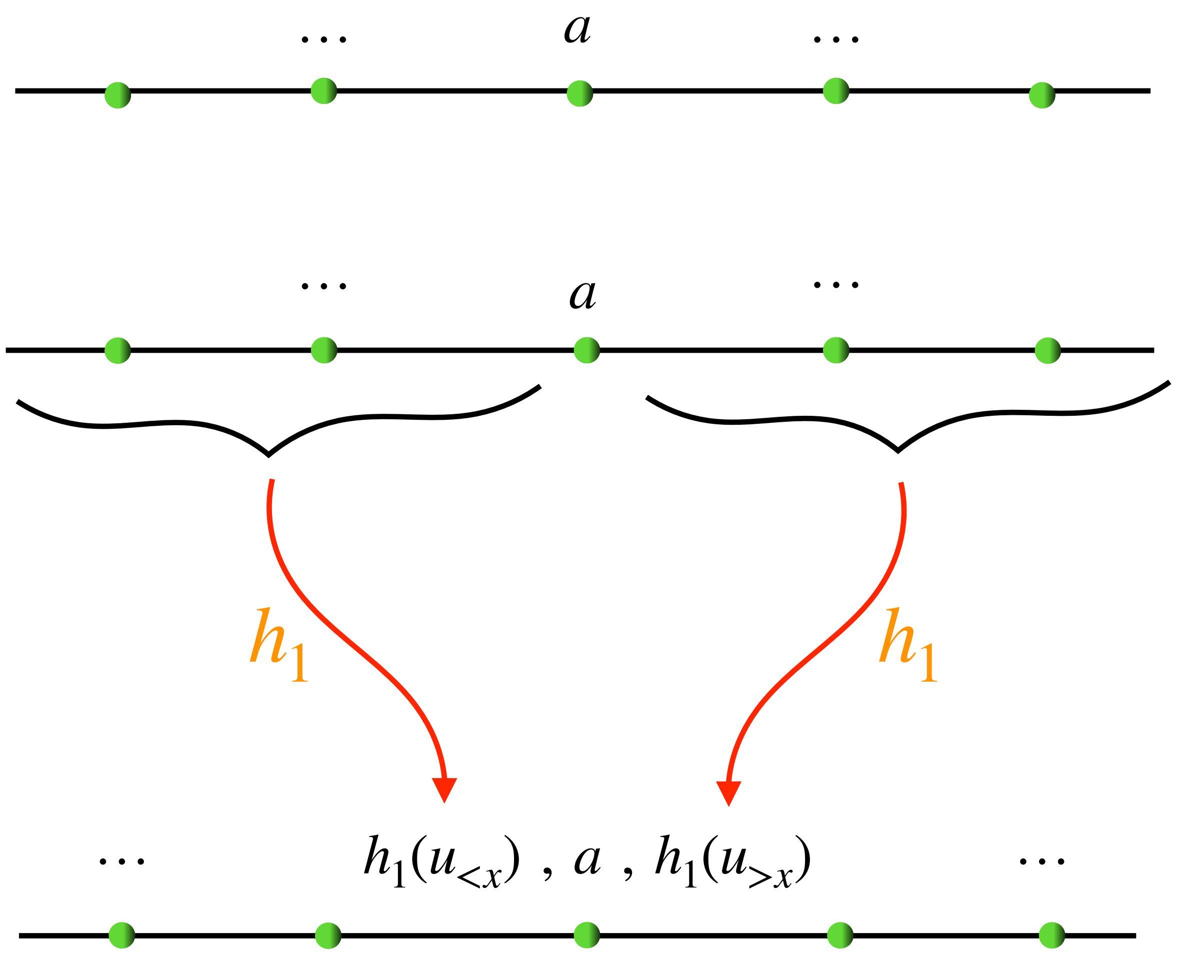
$(M, \cdot\,)$ $(N, +\,)$ $M \,\square\, N$



$h : \Sigma \to M \,\square\, N$ $u$

$h_1 : \Sigma \to M$ $u$

$\sigma : \Sigma^* \to (M \times \Sigma \times M)^*$

$h_2 : M \times \Sigma \times M \to N$ $u' = \sigma(u)$

$h(u)$ is completely determined by $h_1(u)$ and $h_2(u')$

# Block Product example

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$$U_1 = (\{1,0\}, \cdot\,)$$

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$\Sigma = \{a, b\}$ $\quad$ $L = \{u \mid u \text{ contains exactly one } a\}$

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$U_1 = (\{1,0\}, \cdot\,)$ $\qquad U_1 \square U_1$

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# Block Product example

$U_1 = (\{1,0\}, \cdot\,)$

$U_1 \Box U_1$

$\Sigma = \{a, b\}$

$L = \{u \mid u \text{ contains exactly one } a\}$

$$h_1 : \begin{matrix} a \mapsto 0 \\ b \mapsto 1 \end{matrix}$$

# Block Product example

$U_1 = (\{1{,}0\}, \cdot\,)$

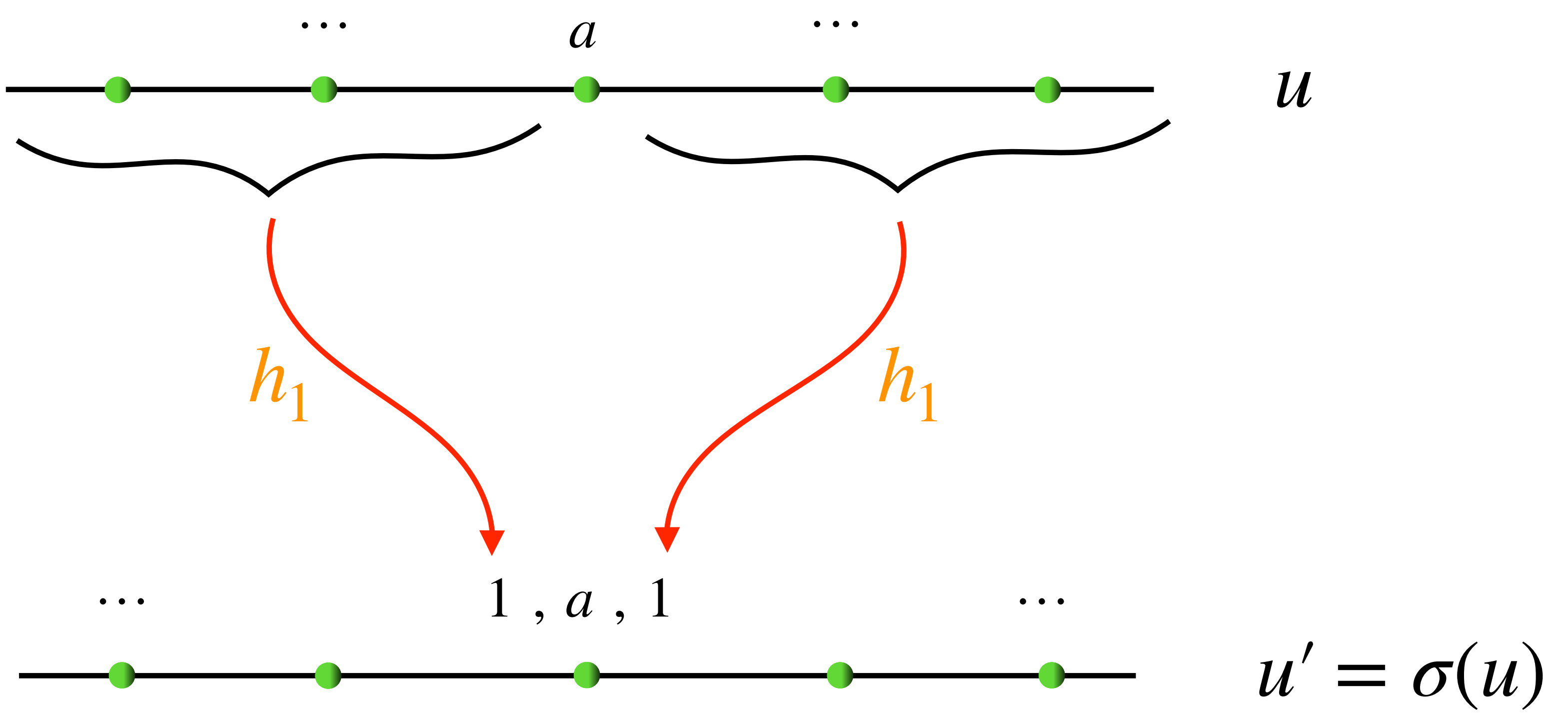
$U_1 \square U_1$

$\Sigma = \{a, b\}$

$L = \{u \mid u \text{ contains exactly one } a\}$

$$h_1 : \begin{array}{l} a \mapsto 0 \\ b \mapsto 1 \end{array}$$

$\sigma : \Sigma^* \to (U_1 \times \Sigma \times U_1)^*$



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$U_1 \,\square\, U_1$

$\Sigma = \{a, b\}$

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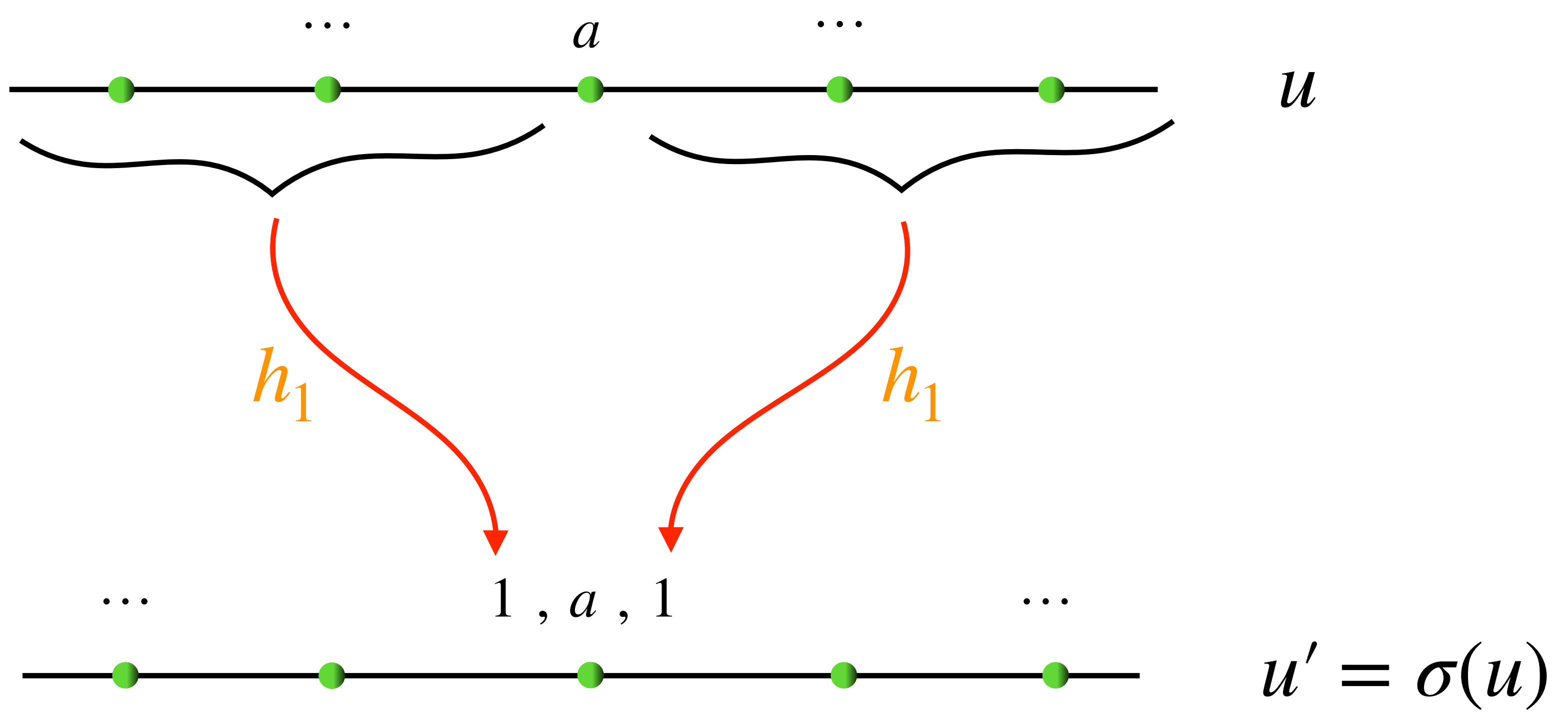
$$h_1 : \begin{array}{l} a \mapsto 0 \\ b \mapsto 1 \end{array}$$

$\sigma : \Sigma^* \to (U_1 \times \Sigma \times U_1)^*$

$$h_2 : \begin{array}{r} 1,\ a,\ 1 \mapsto 0 \\ \ldots \mapsto 1 \end{array}$$

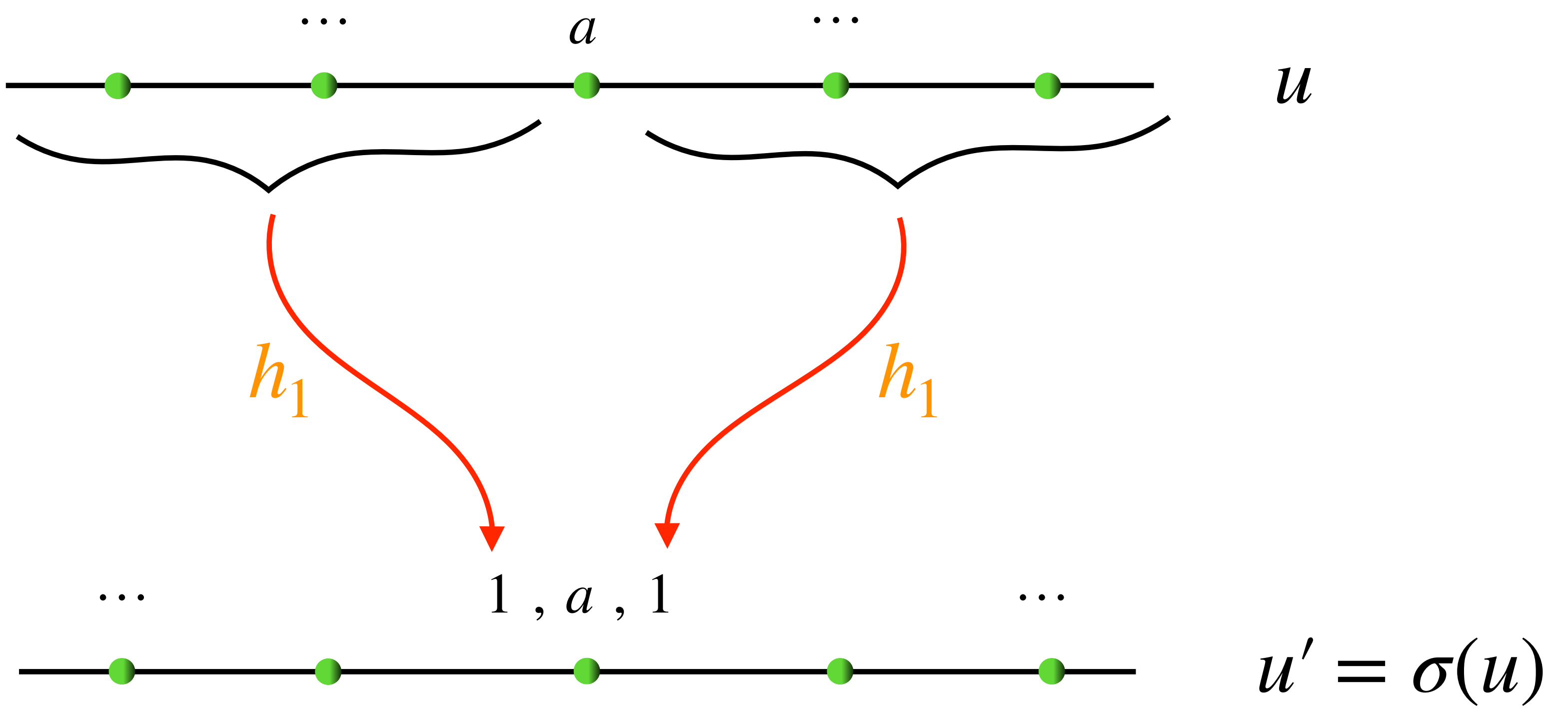


# Block Product example

$U_1 = (\{1{,}0\}, \cdot\,)$ $\quad$ $U_1 \,\square\, U_1$

$\Sigma = \{a, b\}$ $\quad$ $L = \{u \mid u \text{ contains exactly one } a\}$

$$h_1 : \begin{array}{l} a \mapsto 0 \\ b \mapsto 1 \end{array}$$



$$\sigma : \Sigma^* \to (U_1 \times \Sigma \times U_1)^*$$

$$h_2 : \begin{array}{l} 1,\ a,\ 1 \mapsto 0 \\ \quad\ \ldots \mapsto 1 \end{array}$$

$$u \in L \text{ iff } h_1(u) = 0 \ \&\ h_2(u') = 0$$

# Block Product Principle

$$(M, \cdot) \quad \& \quad (N, +) \quad \longrightarrow \quad (M \square N, \tilde{\cdot})$$

Given $h : \Sigma \to M \square N$, let $h_1 : \Sigma \to M$ be the projection of $h$ on $M$

Any language recognized by $M \square N$ is a boolean combination of

1) Languages recognized by $M$

2) $\sigma^{-1}$ of languages recognized by $N$

$\sigma$ is transducer induced by $h_1$

# Block Product for $\circledast$-Monoid

# Block Product for ⊛-Monoid

$(M, \pi)$ $(N, \hat{\pi})$ $(M \,\square\, N\ ,\ \tilde{\pi})$

# Block Product for ⊛-Monoid

$(M, \pi)$

$(N, \hat{\pi})$

$(M \square N \; , \; \tilde{\pi})$

Satisfies Generalized
Associativity

# Block Product for $\circledast$-Algebra

# Block Product for ⊛-Algebra

$$(M, \pi) \qquad \& \qquad (N, \hat{\pi}) \qquad \longrightarrow \qquad (M \,\square\, N \,,\ \tilde{\pi})$$

# Block Product for ⊛-Algebra

$$\begin{tikzcd}
(M,\pi) \arrow[d] & \& & (N,\hat{\pi}) \arrow[rr] \arrow[d] & & (M \square N\ ,\ \tilde{\pi}) \arrow[d] \\
(M,\cdot\ ,\tau,\tau^*,\kappa) & \& & (N,+\ ,\hat{\tau},\hat{\tau}^*,\hat{\kappa}) & & (M \square N\ ,\ \tilde{\cdot}\ ,\ \tilde{\tau},\ \tilde{\tau}^*,\ \tilde{\kappa})
\end{tikzcd}$$

# Block Product for ⊛-Algebra

$$\begin{tikzcd}
(M, \pi) \arrow[d] & \& & (N, \hat{\pi}) \arrow[r] \arrow[d] & (M \square N \, , \, \tilde{\pi}) \arrow[d] \\
(M, \cdot \, , \tau, \tau^{*}, \kappa) & \& & (N, + \, , \hat{\tau}, \hat{\tau}^{*}, \hat{\kappa}) \arrow[r, "\text{Effective}"'] & (M \square N \, , \tilde{\cdot} \, , \, \tilde{\tau}, \, \tilde{\tau}^{*}, \, \tilde{\kappa})
\end{tikzcd}$$

# Block Product (operational view)

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$(M, \cdot\,, \tau, \tau^*, \kappa)$ $(N, +\,, \hat{\tau}, \hat{\tau}^*, \hat{\kappa})$ $(M \square N, \tilde{\cdot}\,, \tilde{\tau}, \tilde{\tau}^*, \tilde{\kappa})$

$h : \Sigma \to M \square N$

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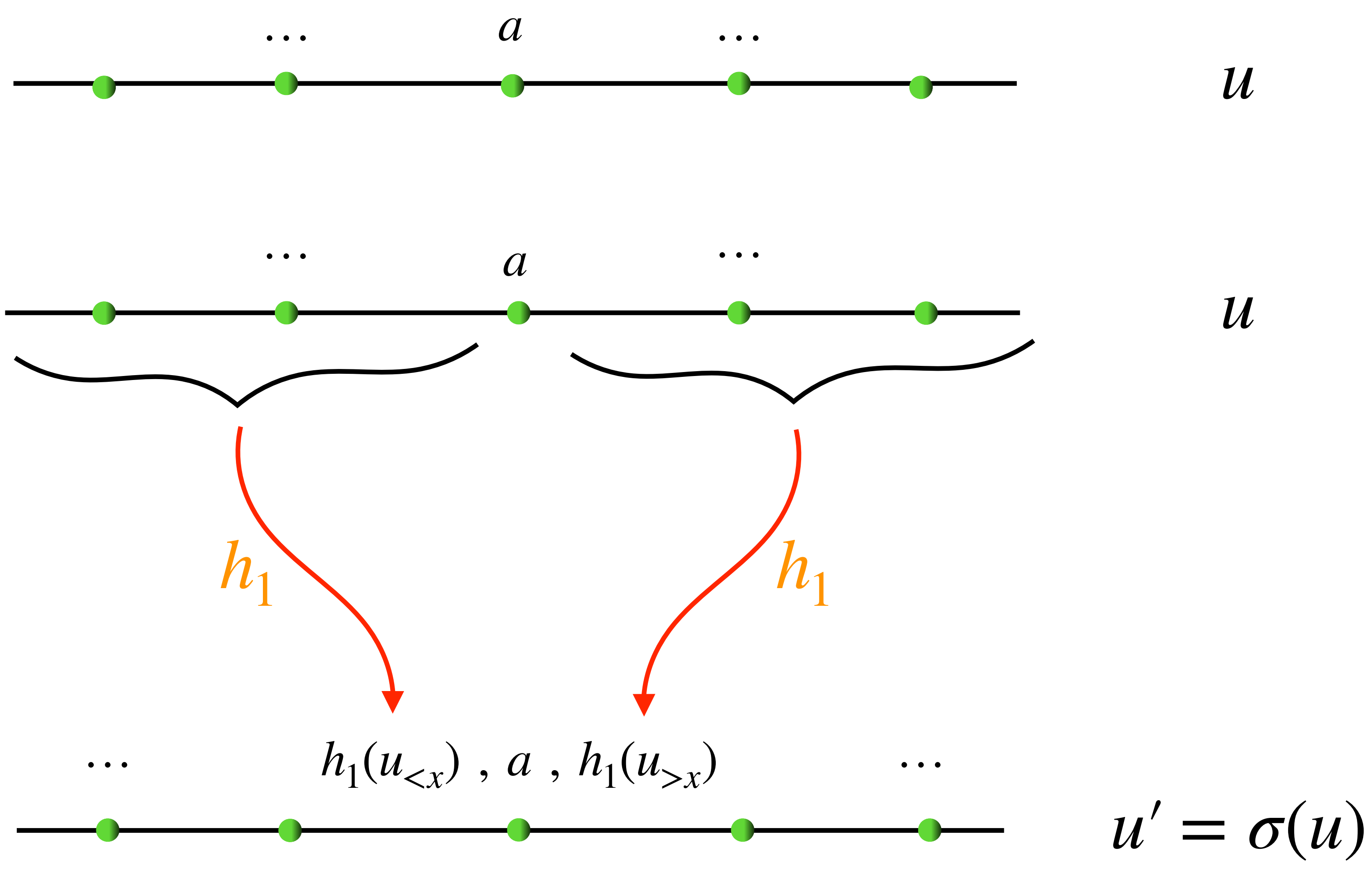
$h : \Sigma \to M \,\square\, N$

$h_1 : \Sigma \to M$

$\sigma : \Sigma^{\oplus} \to (M \times \Sigma \times M)^{\oplus}$



$\dots$ $a$ $\dots$ $u$

$\dots$ $a$ $\dots$ $u$

$h_1$ $h_1$

$\dots$ $h_1(u_{<x})$ , $a$ , $h_1(u_{>x})$ $\dots$

$u' = \sigma(u)$

# Block Product (operational view)

$(M, \cdot\,, \tau, \tau^*, \kappa)$ $(N, +\,, \hat{\tau}, \hat{\tau}^*, \hat{\kappa})$ $(M \,\square\, N, \tilde{\cdot}\,, \tilde{\tau}, \tilde{\tau}^*, \tilde{\kappa})$

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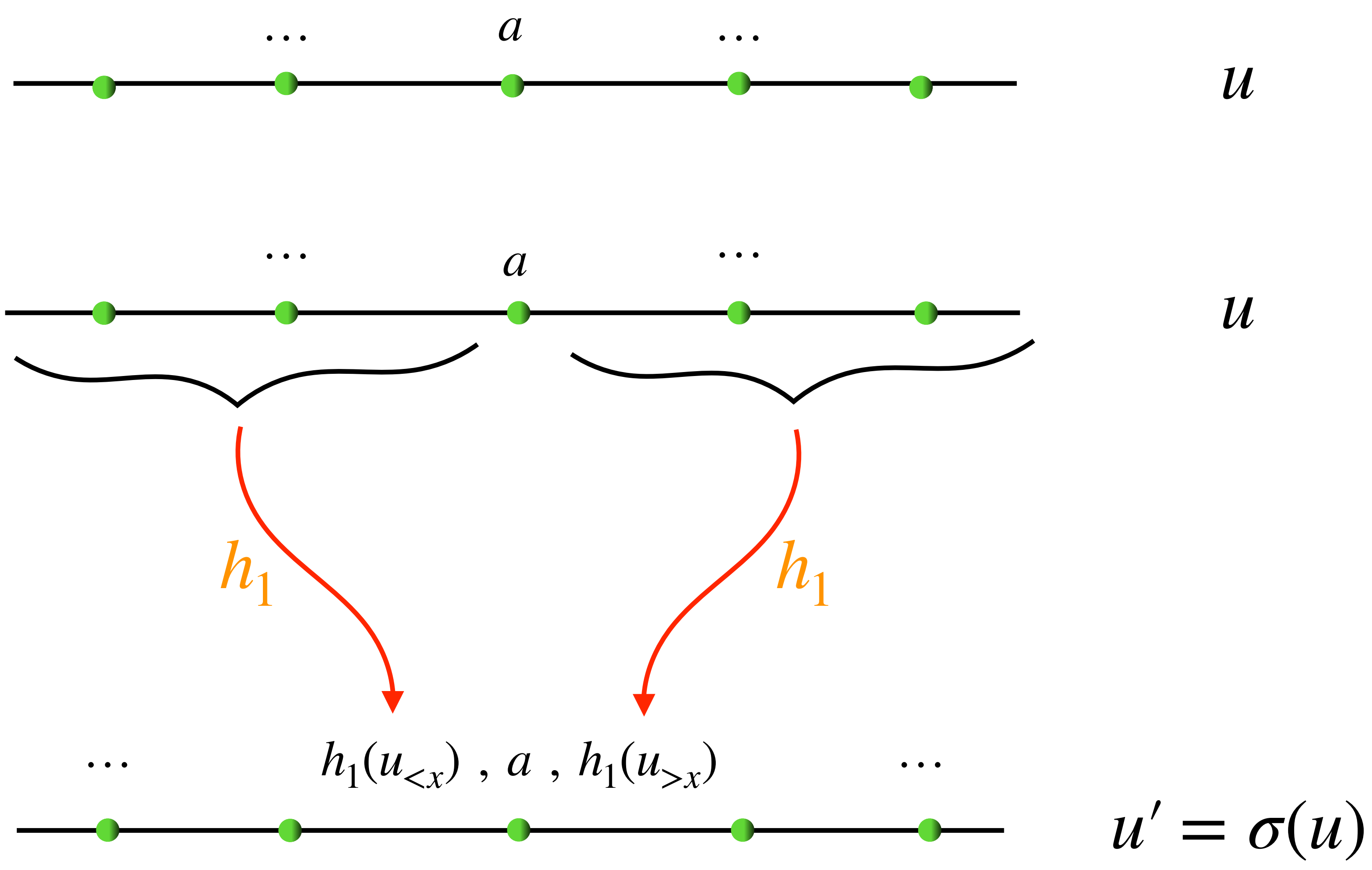
$h_2 : M \times \Sigma \times M \to N$



$h(u)$ is completely determined by $h_1(u)$ and $h_2(u')$

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# Applications to Logic

# Block Product Closure

$((U_1 \,\square\, U_1) \,\square\, U_1) \,\square\, U_1 \ldots$ Iterated Block product

$\ldots U_1 \,\square\, (U_1 \,\square\, (U_1 \,\square\, U_1))$ Weakly Iterated Block product

$\square^* U_1 =$ set of all iterated block products of $U_1$

$\square^*_w U_1 =$ set of all weakly iterated block products of $U_1$

$$U_1 = (\{0,1\}, \pi) \qquad \pi(u) = \begin{cases} 1 & \text{if } u \text{ has no } 0 \\ 0 & \text{if } u \text{ has at least one } 0 \end{cases}$$

# Block Product & Logic (finite words)

$\mathscr{L}(\Box^* U_1)$ $=$ FO definable languages [Krohn, Rhodes]

$=$ LTL definable languages [Kamp]

$\mathscr{L}(\Box^*_w U_1)$ $=$ $\mathrm{FO}_2$ definable languages [Straubing, Therien]

$$U_1 = (\{0,1\}, \pi) \qquad \pi(u) = \begin{cases} 1 & \text{if } u \text{ has no } 0 \\ 0 & \text{if } u \text{ has at least one } 0 \end{cases}$$

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$\mathscr{L}(\Box^* U_1) =$ FO definable countable languages

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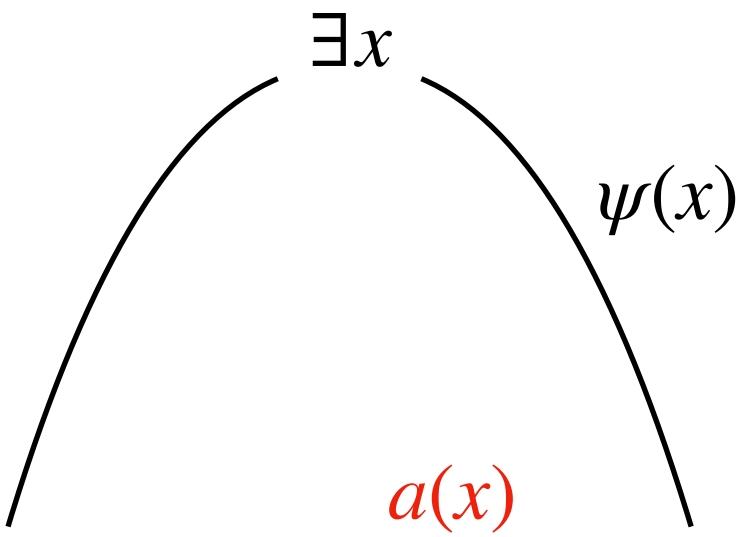
$\phi = \exists x \ \psi(x)$



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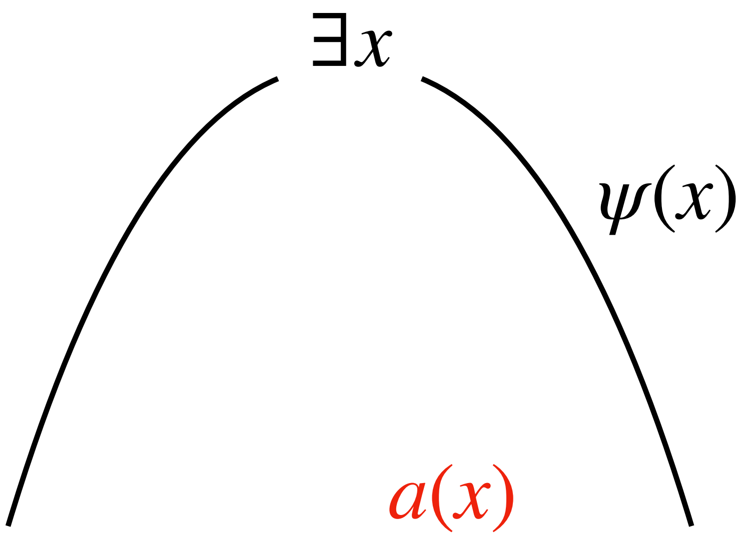

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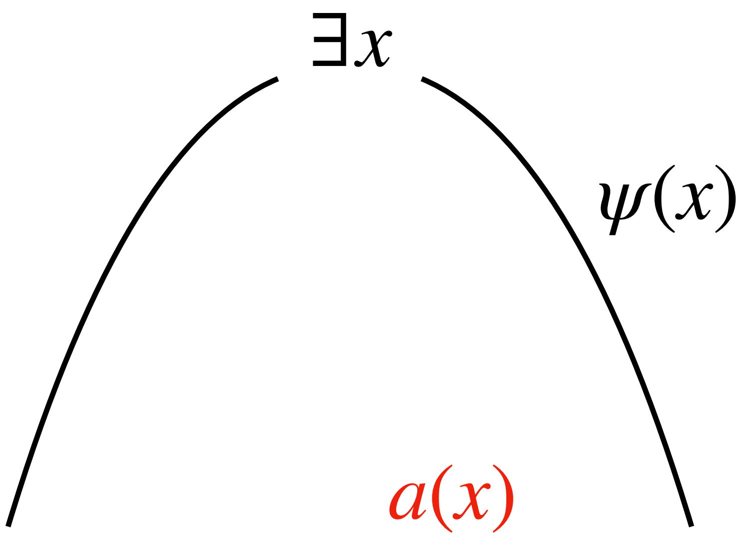

- $\binom{a}{1}$ exists
- exactly one $\binom{*}{1}$

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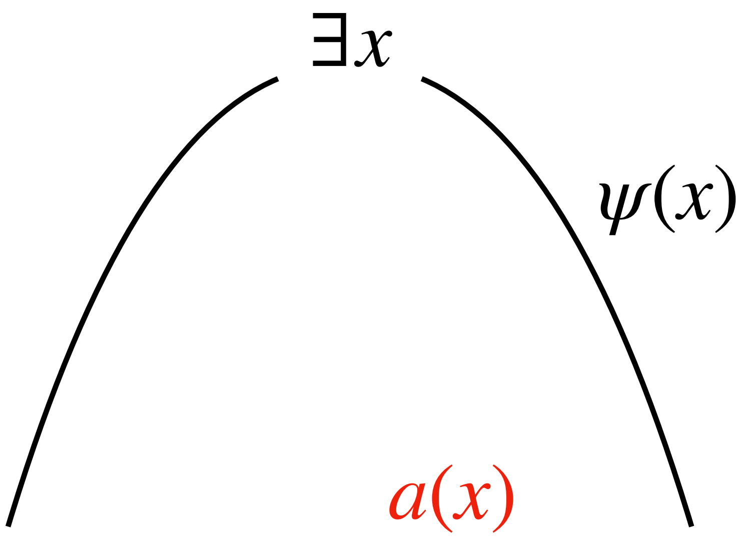

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$U_1$

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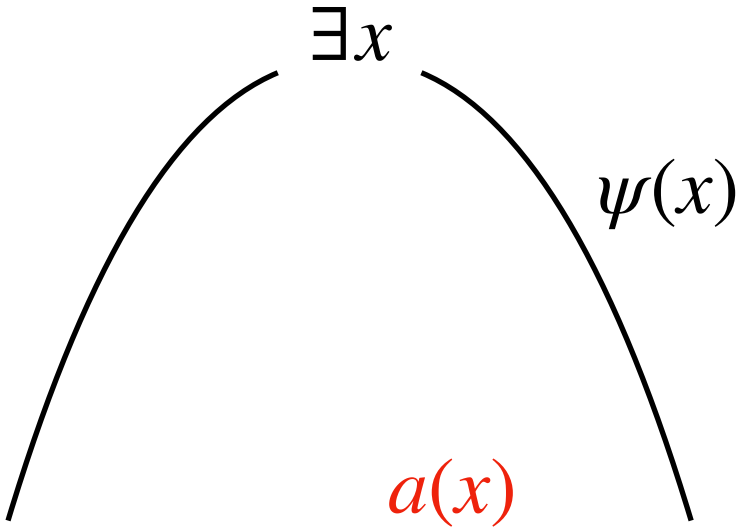

- $\begin{pmatrix} a \\ 1 \end{pmatrix}$ exists
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$U_1$ $\quad (U_1 \square U_1)$

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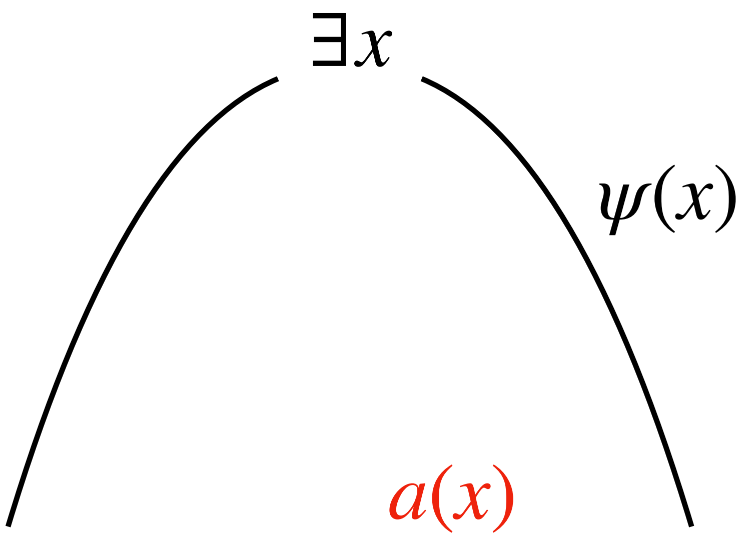

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$U_1 \quad \times \quad (U_1 \square U_1)$

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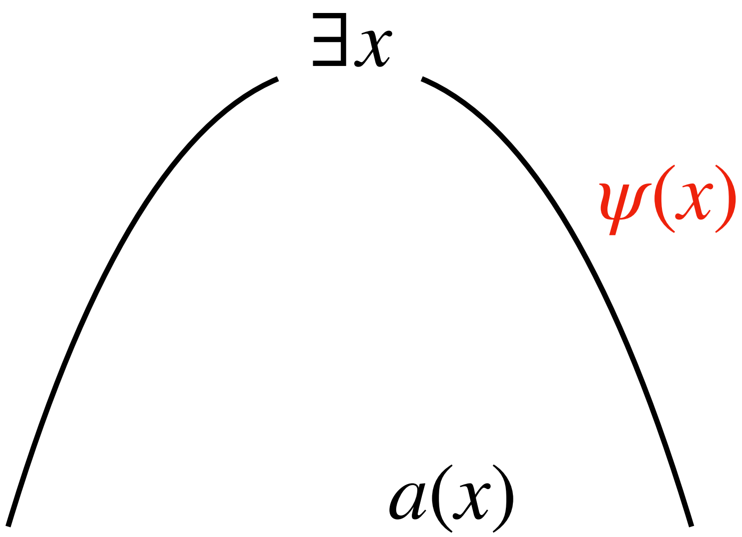

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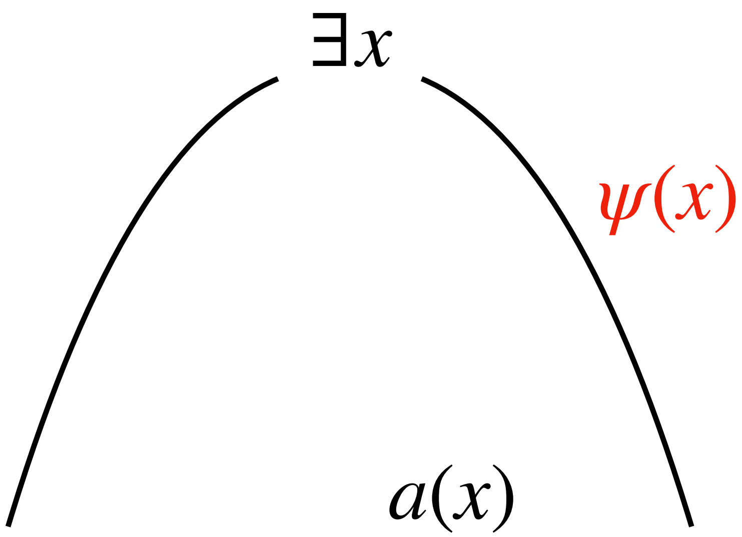

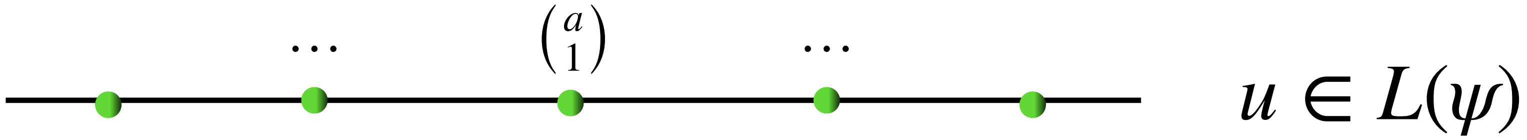

$M$

# First Order Logic (CLO)

$\mathscr{L}(\square^* U_1) \supseteq$ FO definable languages

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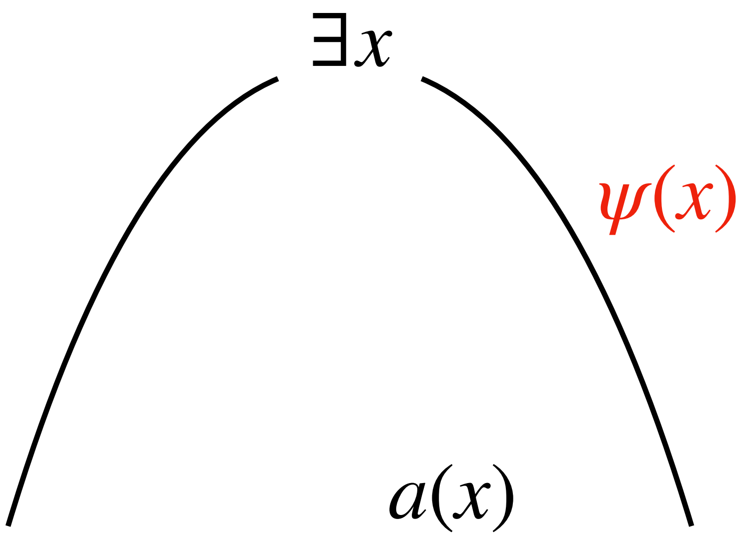

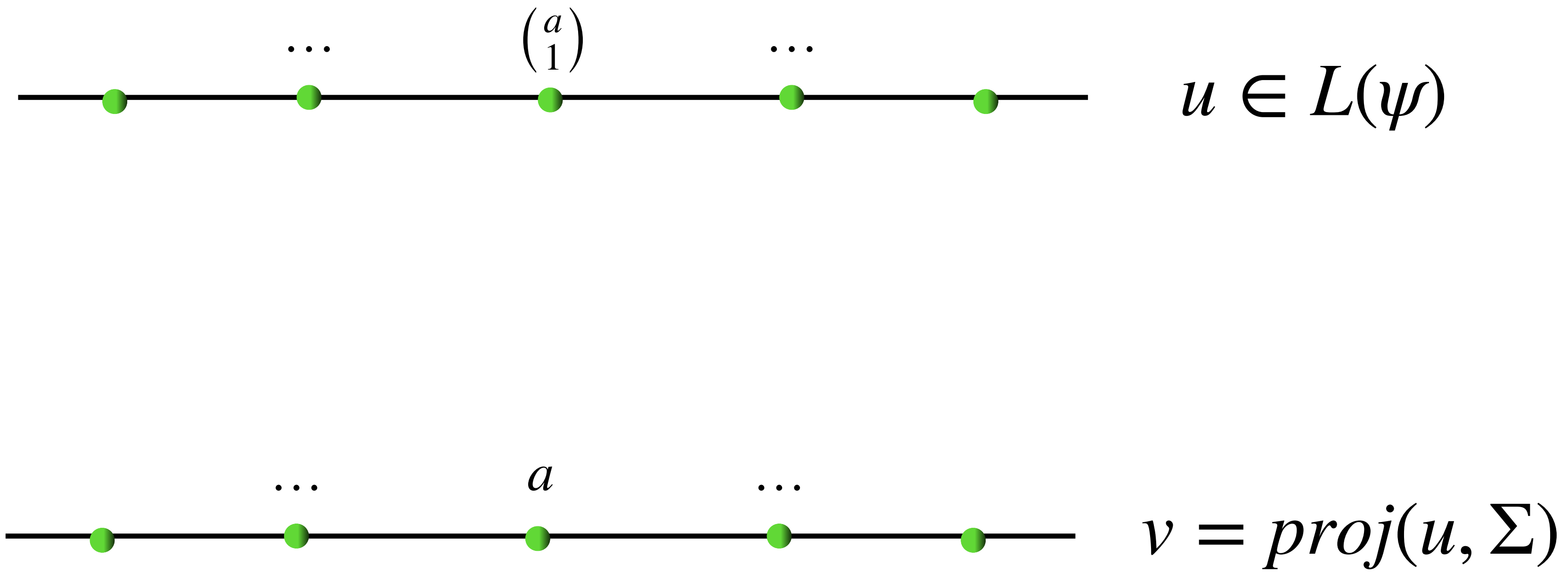

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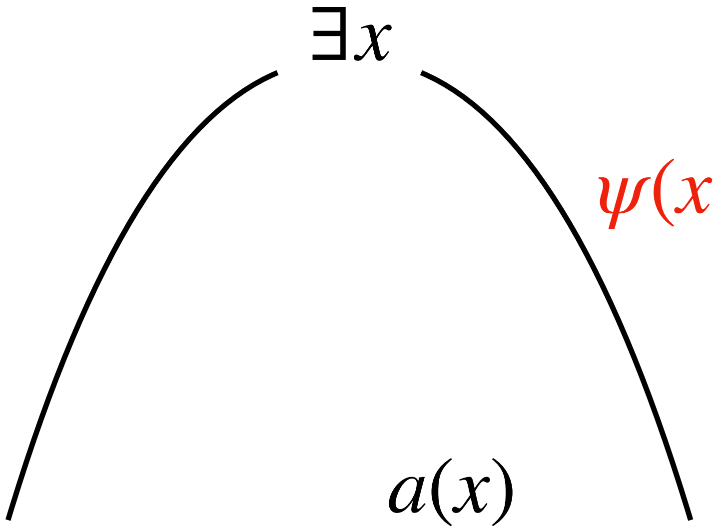

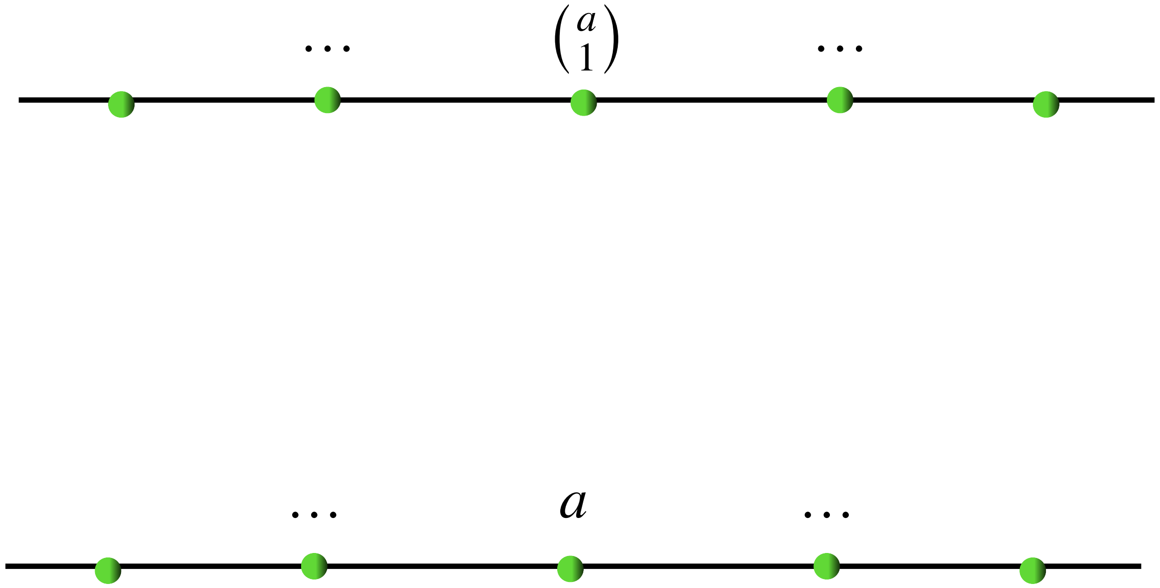

$u \in L(\psi)$

$v = proj(u, \Sigma)$

$M \Box U_1$

# Marked Star Free Expression

# Marked Star Free Expression

$$s = \varnothing \mid a \mid s_1 + s_2 \mid s_1 \cap s_2 \mid \neg s \mid s_1 \,.\, a \,.\, s_2 \qquad a \in \Sigma$$

# Marked Star Free Expression

$$s = \emptyset \mid a \mid s_1 + s_2 \mid s_1 \cap s_2 \mid \neg s \mid s_1 \,.\, a \,.\, s_2 \qquad a \in \Sigma$$

Marked Star Free Expressions $\equiv$ FO $\equiv$ $\Box^* \, U_1$

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# Linear Temporal Logic [S, U]

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$$\phi = p \in P \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \phi_1 \text{ S } \phi_2 \mid \phi_1 \text{ U } \phi_2$$

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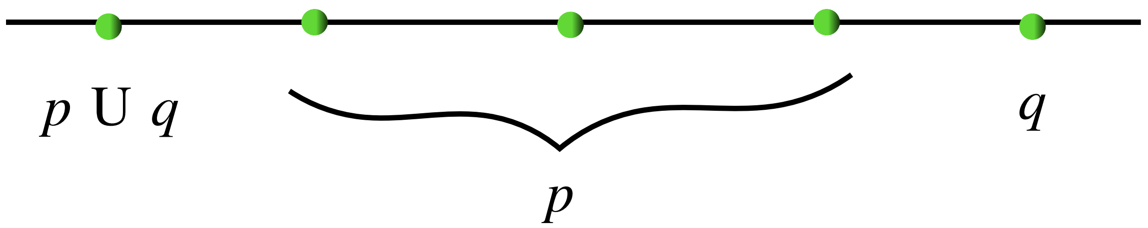

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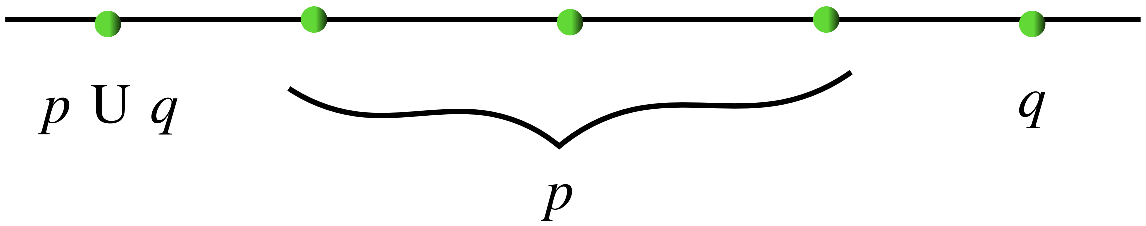


$$\mathscr{L}(\phi) = \{w \in \Sigma\Sigma^{\circledast} \mid w,0 \vDash \phi\}$$

# Linear Temporal Logic [S, U]

$\mathscr{L}(\Box_w^* \{M^r, M^l\}) = \text{LTL[S, U] definable languages}$

$p \ \mathbf{U} \ q$ (finite words)

# $p \text{ U } q$ (finite words)

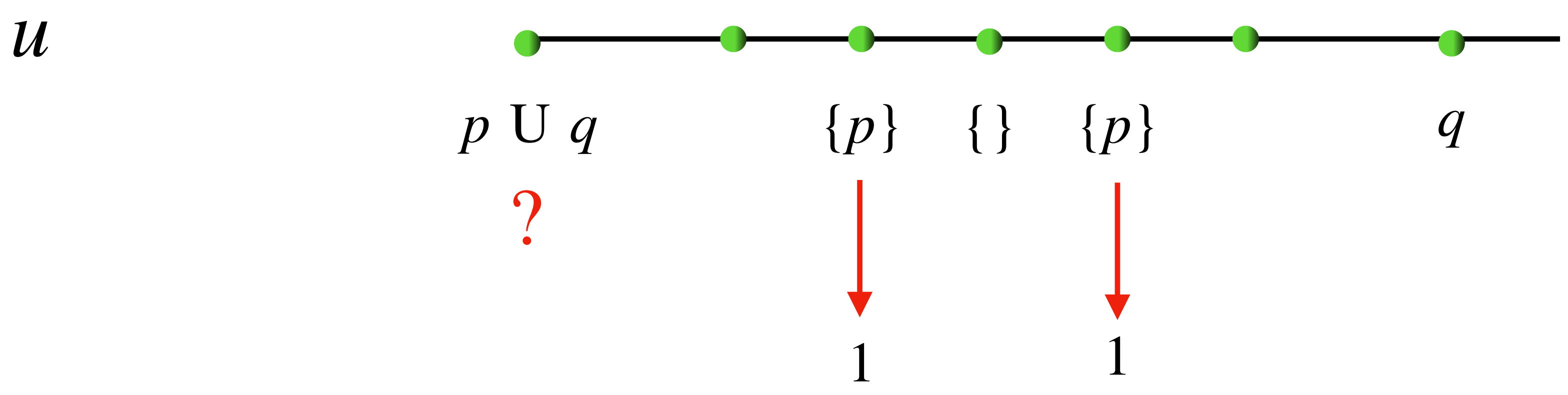
$u$

$p \text{ U } q$ $\{p\}$ $\{\}$ $\{p\}$ $q$

?

$q \equiv \{q\}$ or $\{p, q\}$

# $p \; \mathrm{U} \; q$ (finite words)



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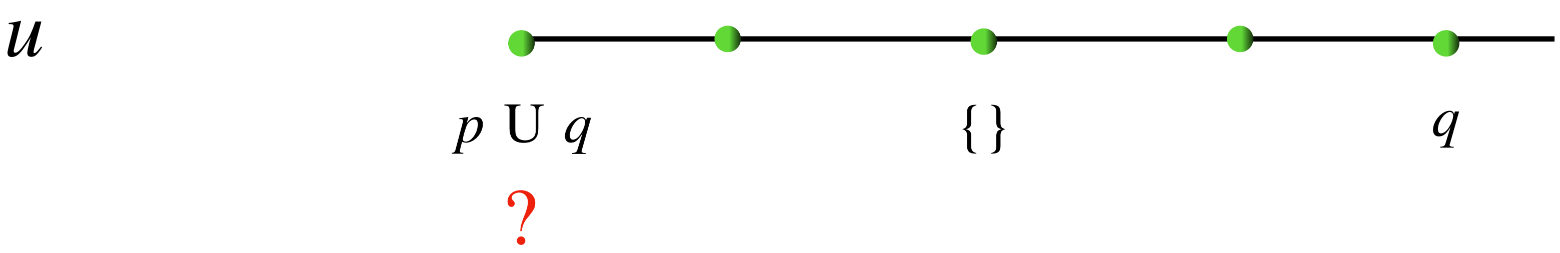
# $p$ U $q$ (finite words)



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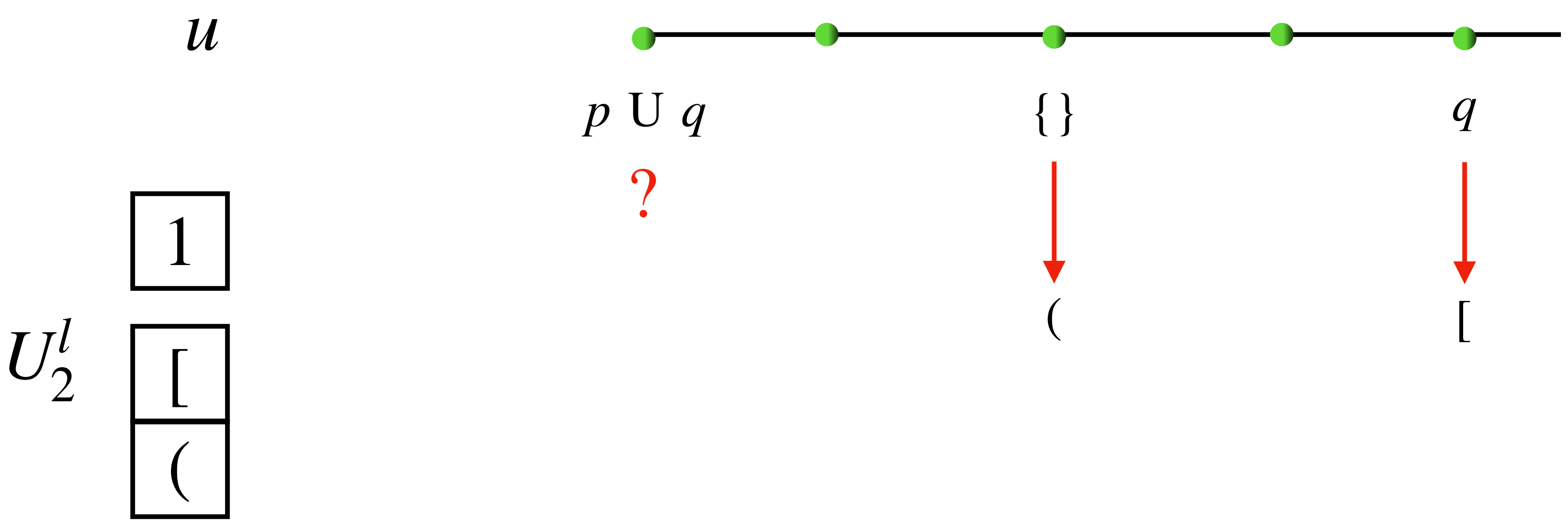

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$( \cdot \star = ($

$[ \cdot \star = [$

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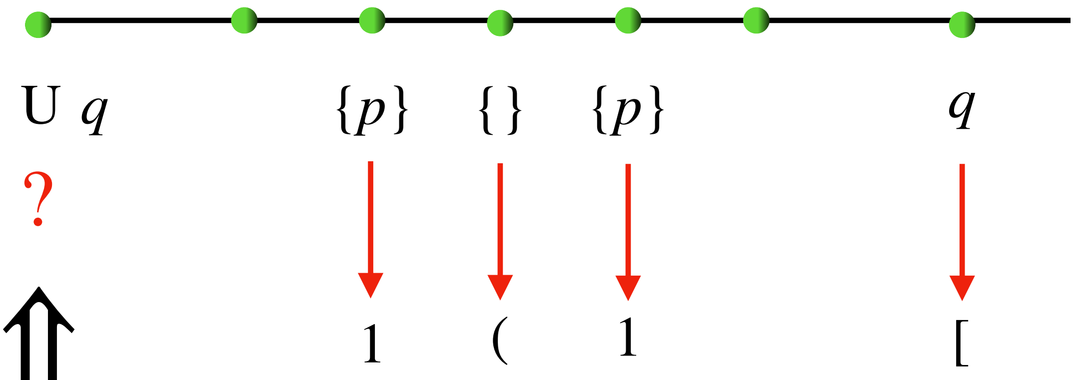
$u$



?

$\Updownarrow$

$h_1(u_{>0}) = [$

or

$h_1(u_{>0}) = ($ ?

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$U_2^l \quad \begin{array}{|c|}\hline [ \\ \hline ( \\ \hline \end{array}$$

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$( \cdot \star = ($

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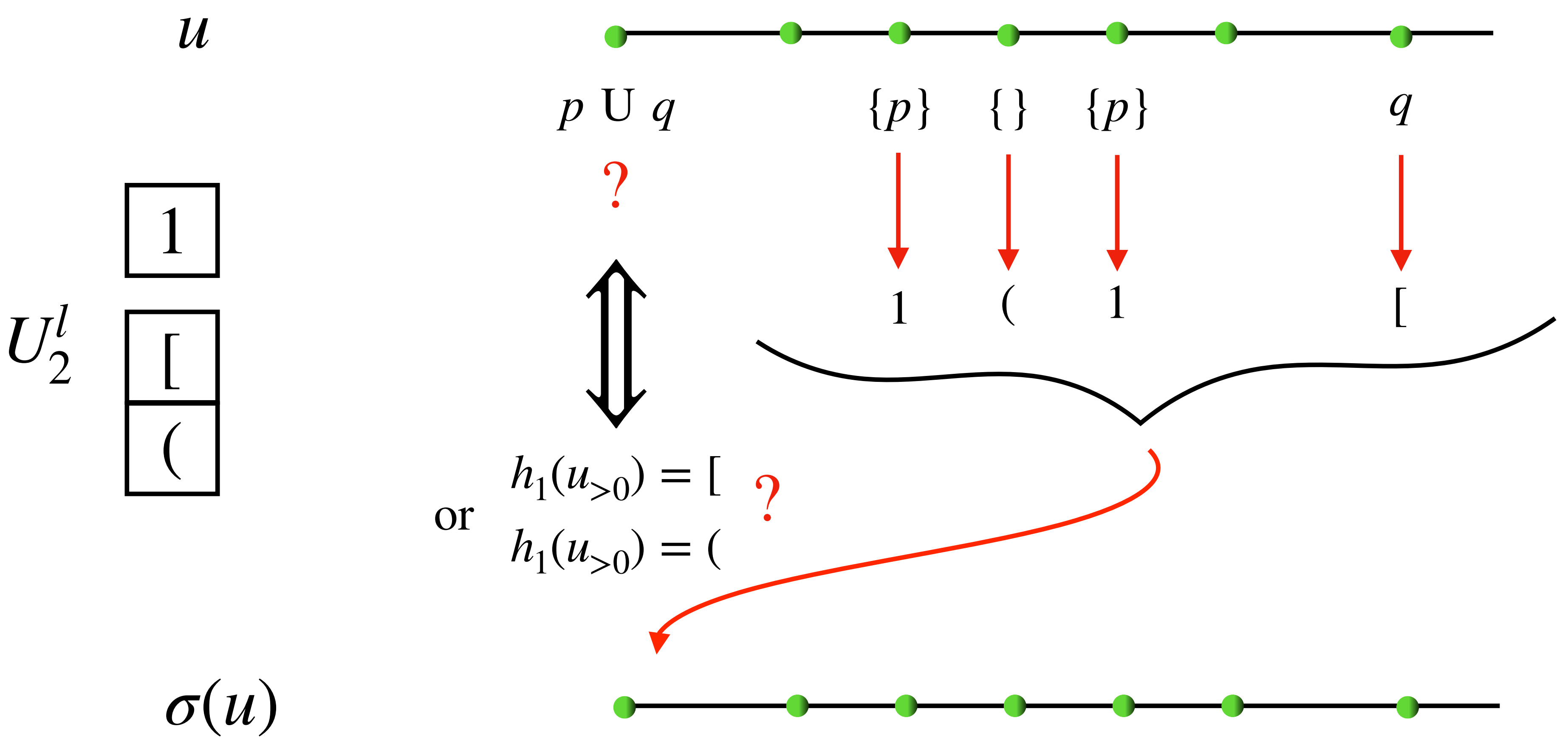
# $p \mathrel{\mathrm{U}} q$ (finite words)



$u$

$p \mathrel{\mathrm{U}} q$ $\{p\}$ $\{\}$ $\{p\}$ $q$

?

1 ( 1 [

$U_2^l$ $\boxed{1}$ $\boxed{[}$ $\boxed{(}$

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$\sigma(u)$

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$p \; \mathbf{U} \; q$ (countable words)

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$\boxed{1}$

$U_2^l$ $\boxed{[}$

$\boxed{(}$

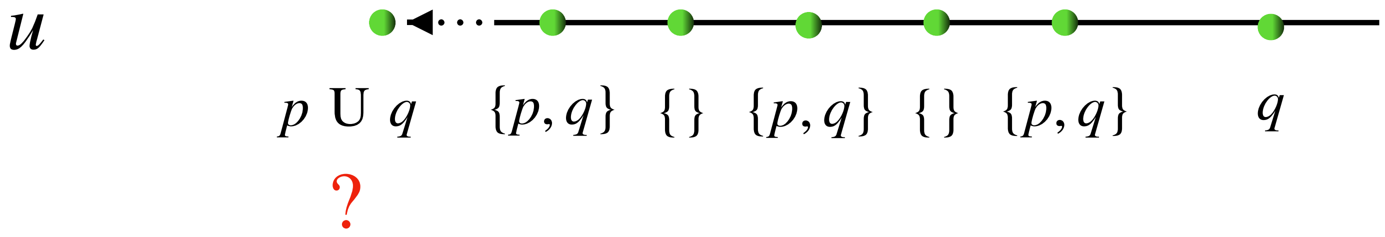
$\{q\}, \{p, q\} \mapsto [$

$h_1 : \quad \{p\} \mapsto 1$

$\{\,\} \mapsto ($

# $p \mathrm{~U~} q$ (countable words)

$$U_2^l \quad \begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad \begin{array}{|c|}\hline [ \\ \hline ( \\ \hline \end{array}$$



$$\{q\},\{p,q\} \mapsto [$$

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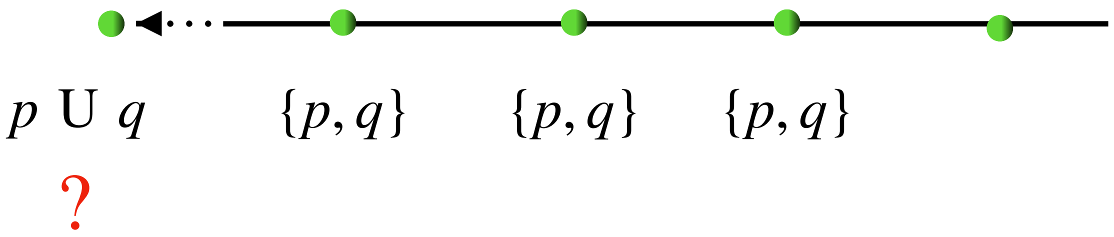
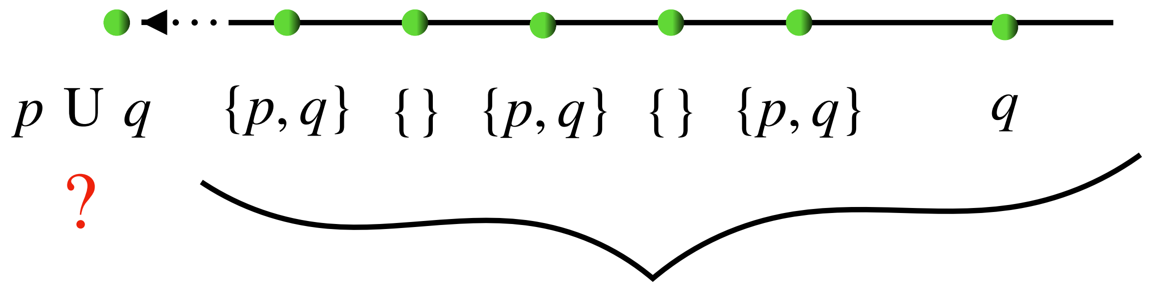


# $p$ U $q$ (countable words)

$U_2^l$ $\boxed{1}$ $\boxed{[}$ $\boxed{(}$

$u$



$h_1(u_{>0}) = ($

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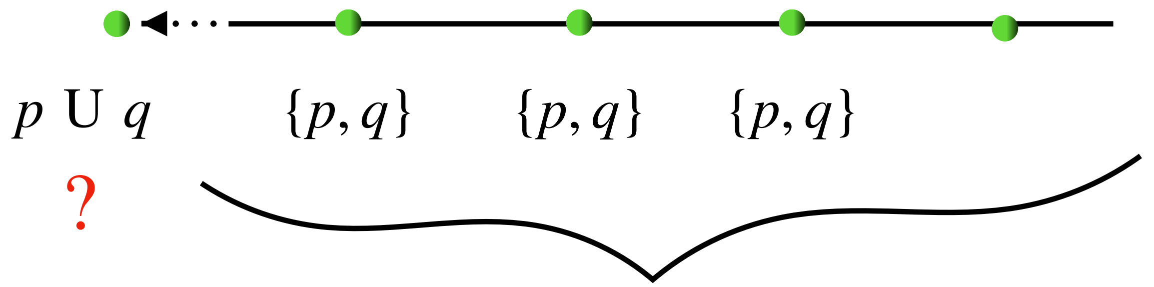

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$p \ \mathbf{U} \ q$ (countable words)

# $p \;\mathrm{U}\; q$ (countable words)

$\boxed{1}$

$M^l$ $\boxed{a}$

$\boxed{ab}$

$\boxed{(ab)^\kappa}$

# $p$ U $q$ (countable words)

$\boxed{1}$

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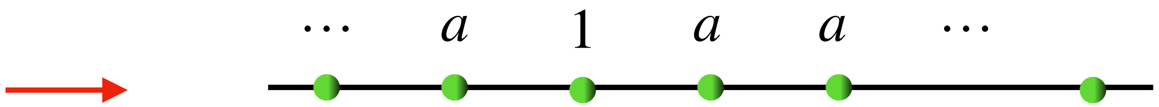
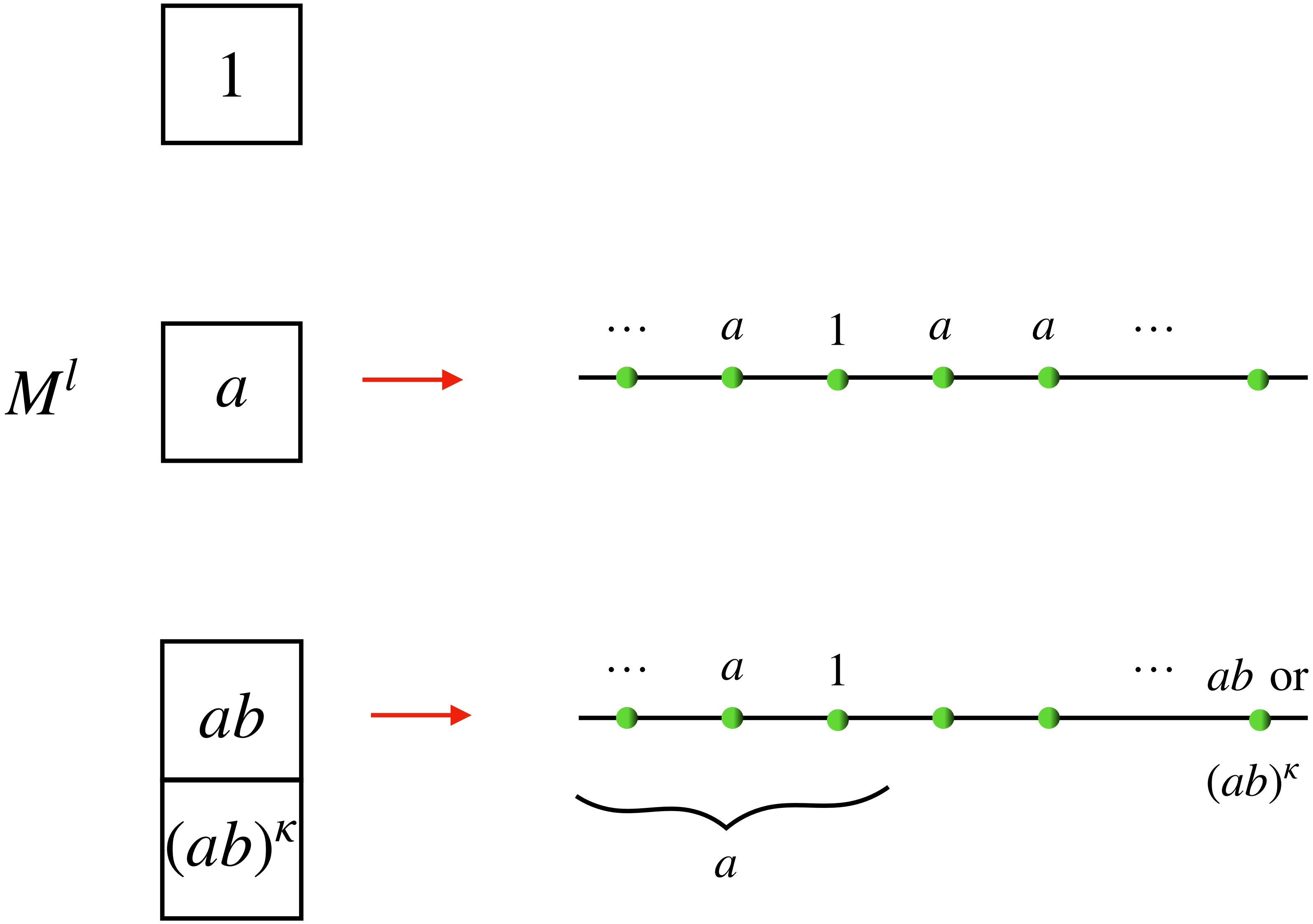


$\boxed{ab}$

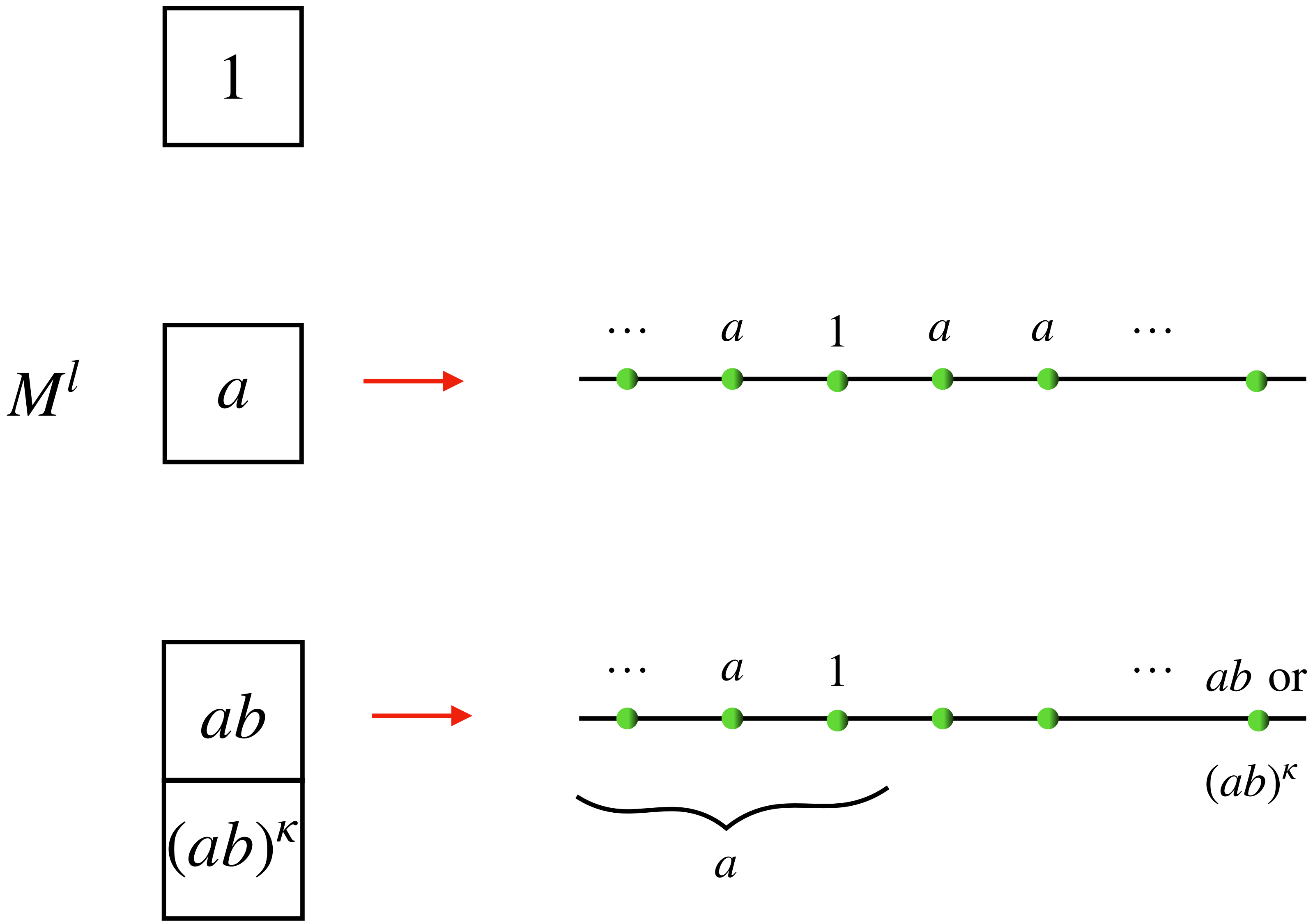
$\boxed{(ab)^\kappa}$

# $p$ U $q$ (countable words)



- contains $ab$ or $(ab)^\kappa$
- prefix of $a$

# $p \mathrm{U} q$ (countable words)



- contains $ab$ or $(ab)^\kappa$
- prefix of $a$

$$a \,.\, (ab)^\kappa = ab$$

# $p \ \mathbf{U} \ q$ (countable words)

# $p$ U $q$ (countable words)

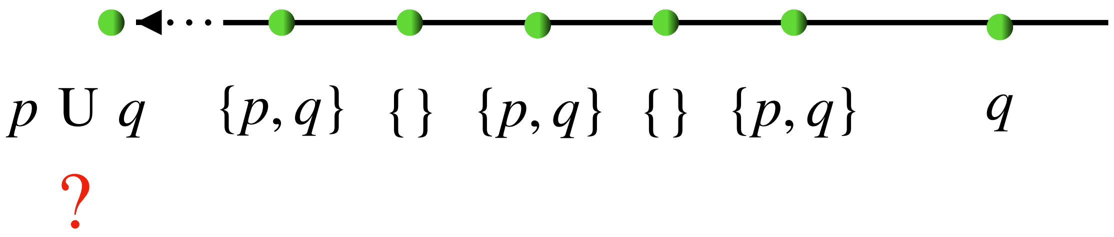
$M^l$

$\boxed{1}$

$\boxed{a}$

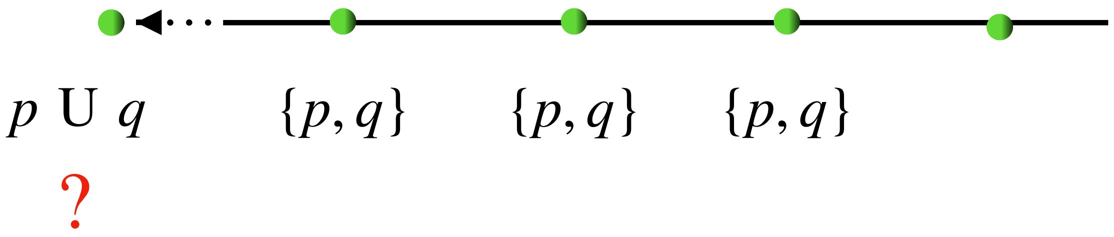
$\boxed{ab}$

$\boxed{(ab)^\kappa}$



$p$ U $q$ $\{p,q\}$ $\{\}$ $\{p,q\}$ $\{\}$ $\{p,q\}$ $q$

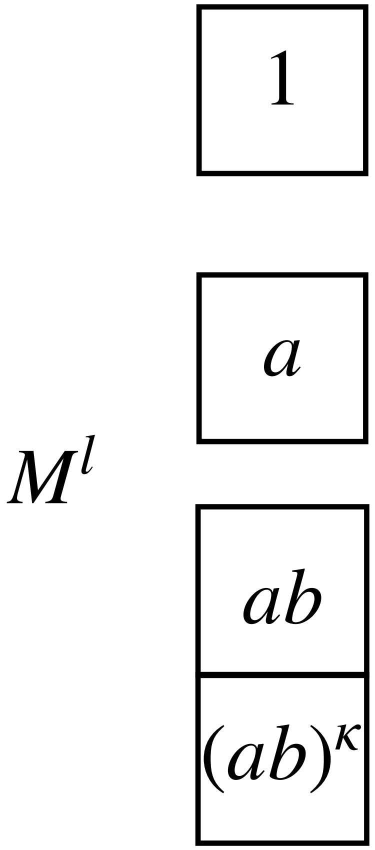
?

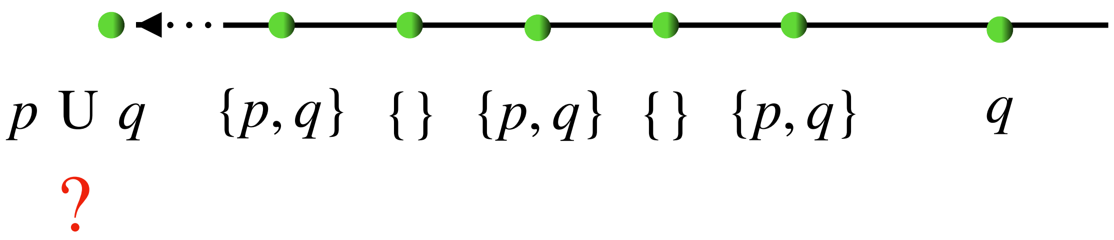


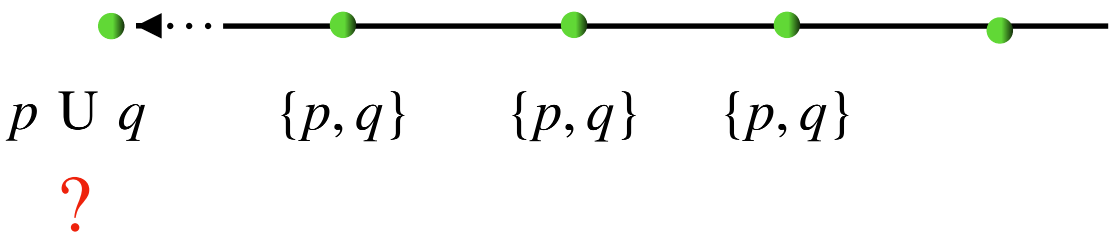
$p$ U $q$ $\{p,q\}$ $\{p,q\}$ $\{p,q\}$

?

# $p \,\mathrm{U}\, q$ (countable words)







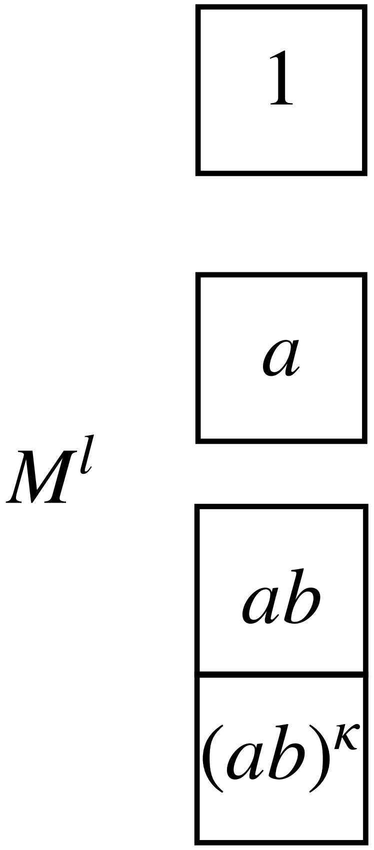
$\{p,q\} \mapsto a$

$h_1 : \quad \{p\} \mapsto 1$

$\{q\} \mapsto ab$

$\{\} \mapsto (ab)^\kappa$

# $p \mathrm{U} q$ (countable words)



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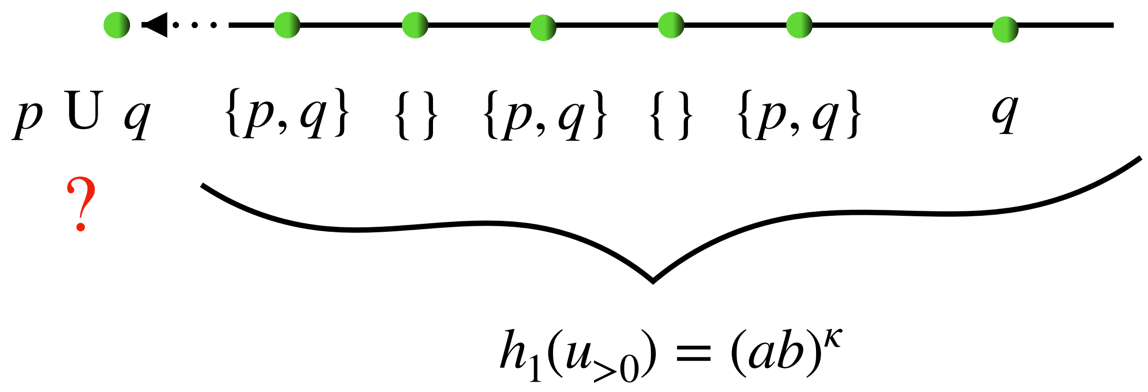
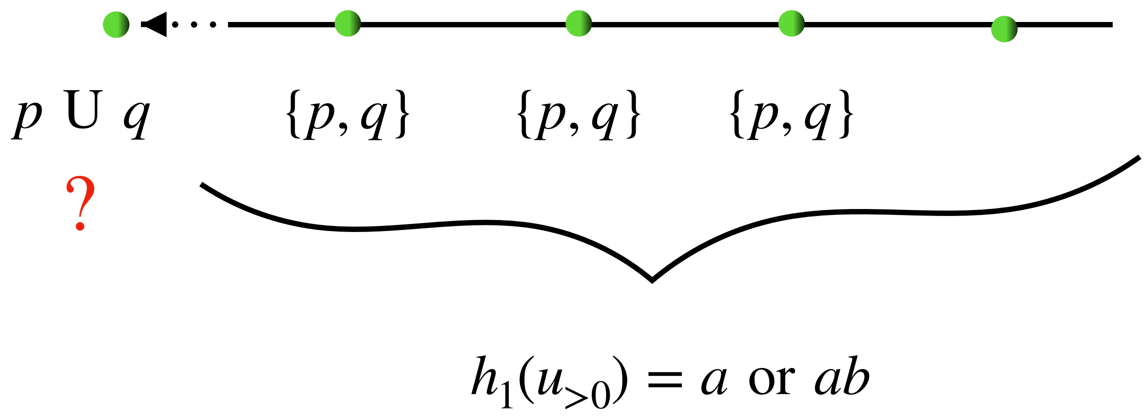
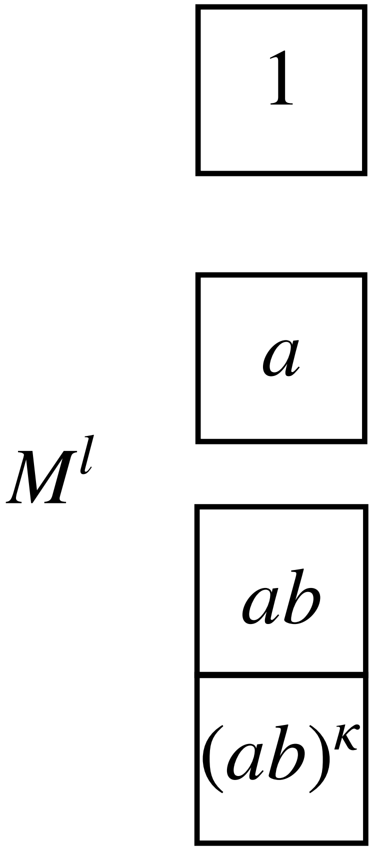




# $p \; \mathrm{U} \; q$ (countable words)



$h_1 :$

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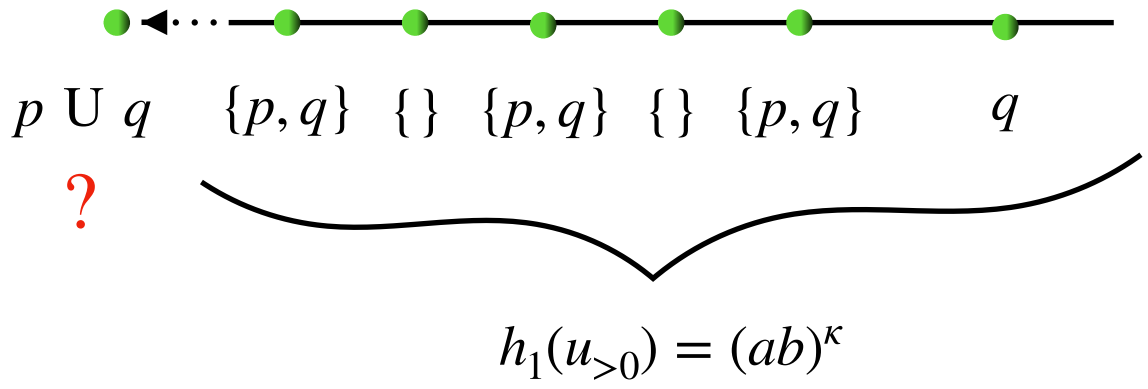
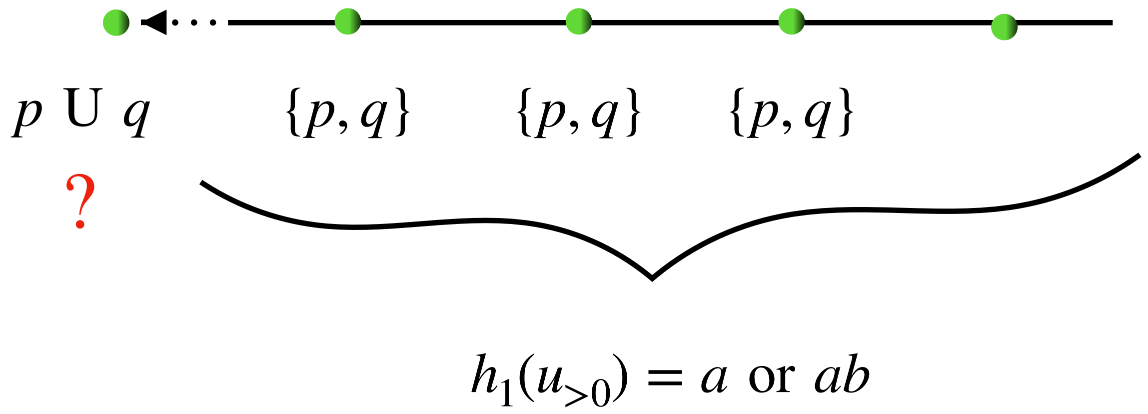


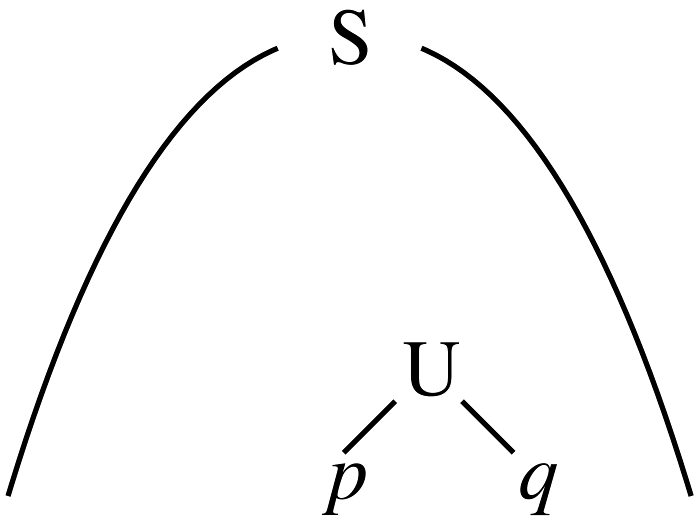


$u,0 \vDash p \; \mathrm{U} \; q$ iff $h_1(u_{>0}) = a$ or $ab$

# $\mathrm{LTL}[\mathrm{S}, \mathrm{U}]$ (countable words)

# LTL[S, U] (countable words)

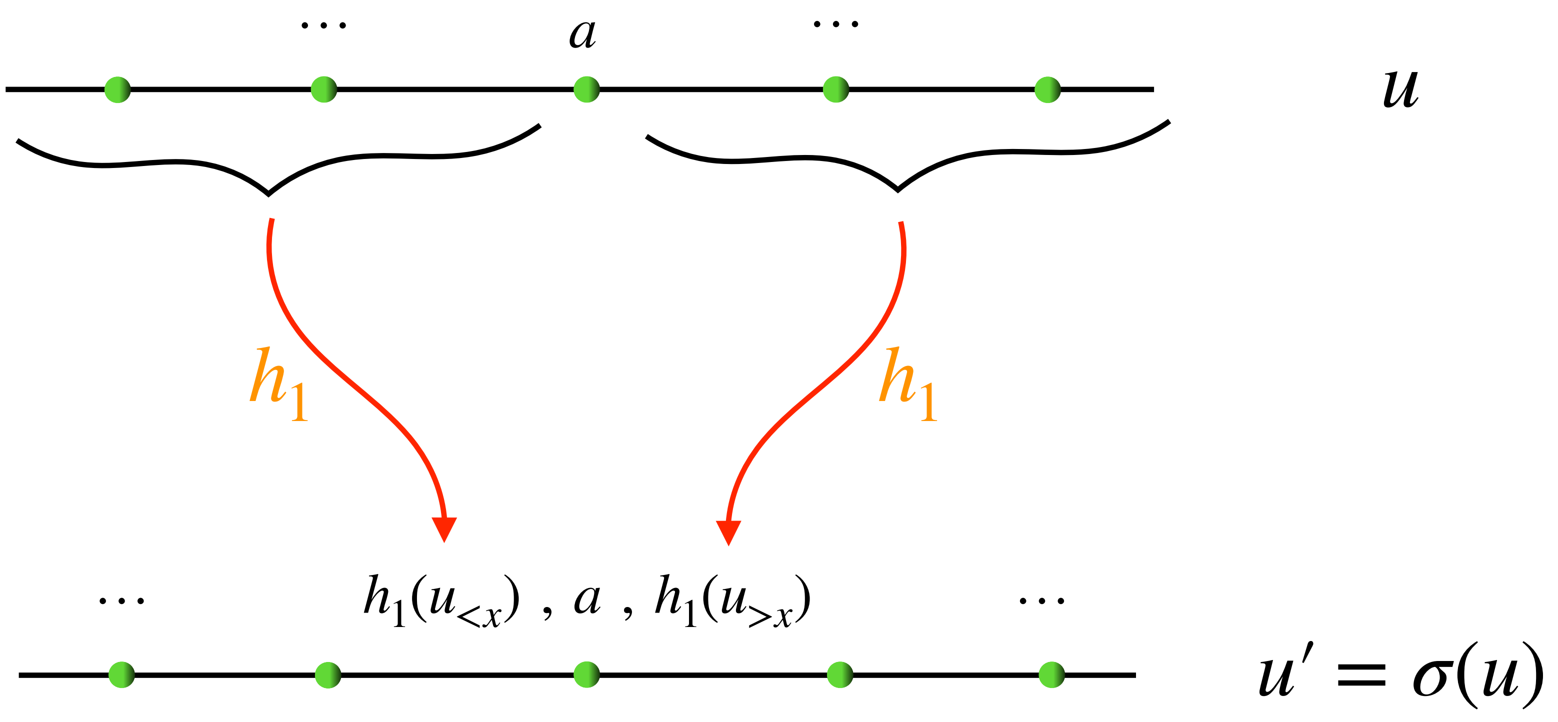
$\phi = \phi_1 \ \mathrm{S} \ \phi_2$

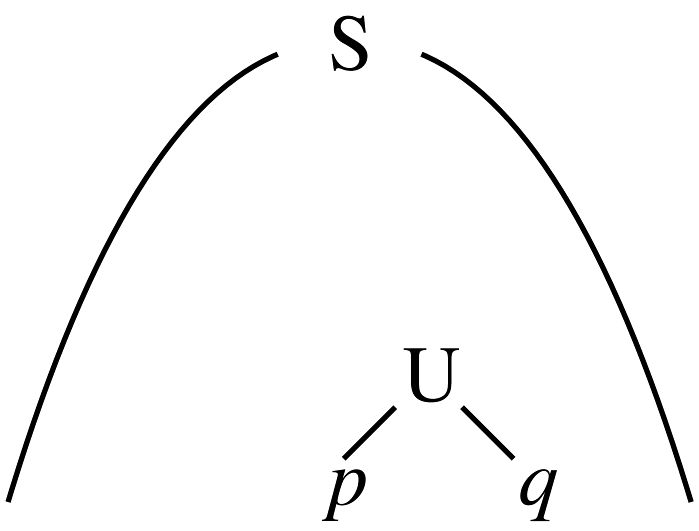


# LTL[S, U] (countable words)

$h_1 : \Sigma \to M^l$

$\phi = \phi_1 \text{ S } \phi_2$





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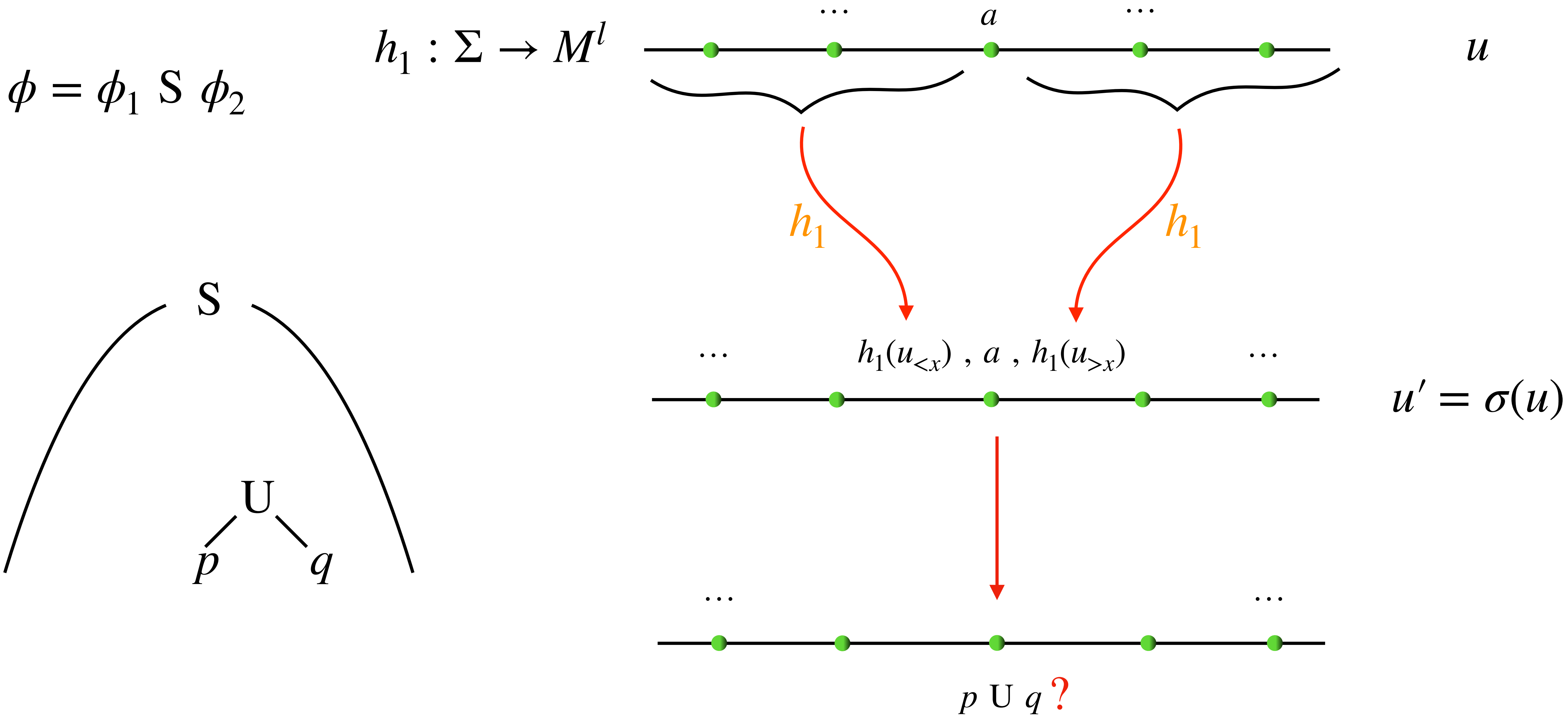

$\phi = \phi_1 \ \mathrm{S} \ \phi_2$

$h_1 : \Sigma \to M^l$

$\cdots$ $a$ $\cdots$

$u$

$h_1$ $h_1$

$\cdots$ $h_1(u_{<x})$ , $a$ , $h_1(u_{>x})$ $\cdots$

$u' = \sigma(u)$

$\cdots$ $\cdots$

$p \ \mathrm{U} \ q$ ?

# $\mathrm{LTL}[S, U]$ (countable words)

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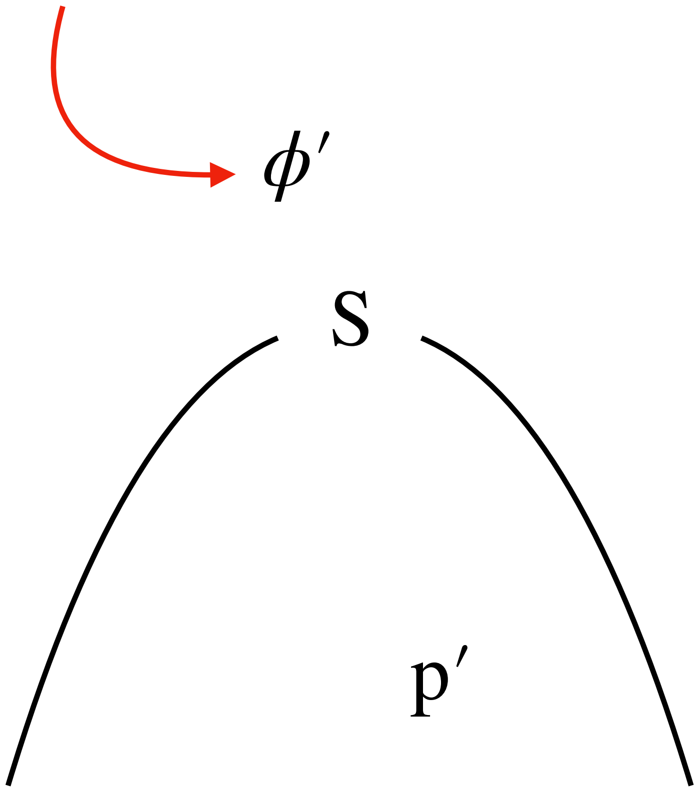
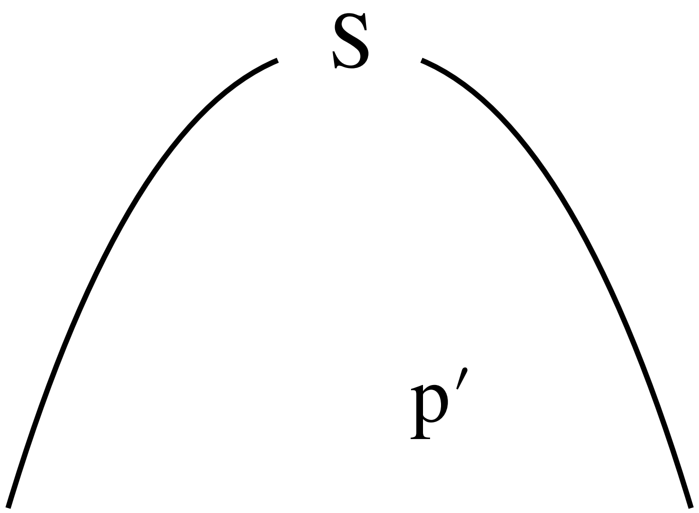
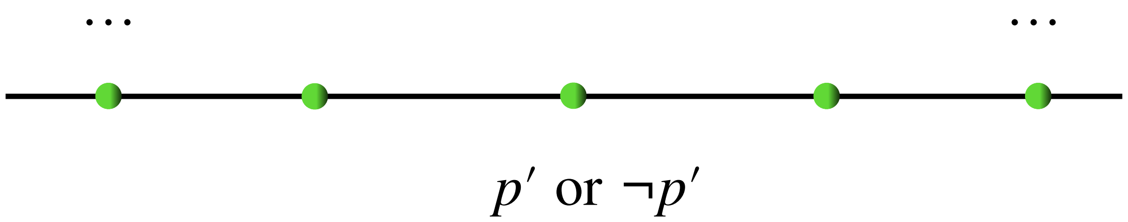


# LTL[S, U] (countable words)

$\phi = \phi_1 \text{ S } \phi_2$

$\phi'$





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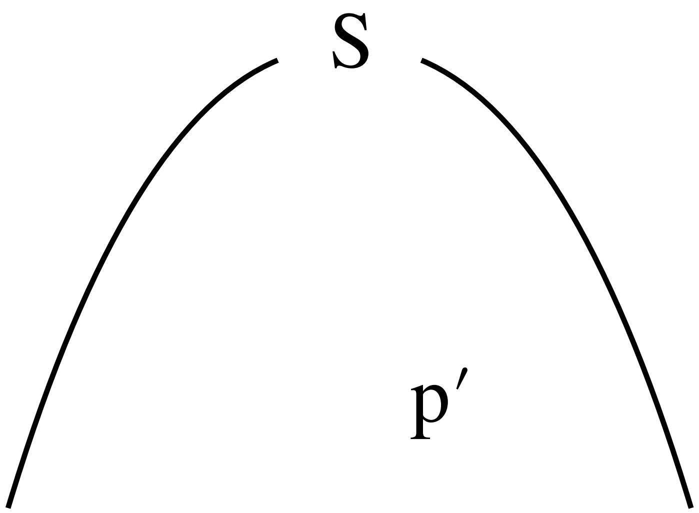

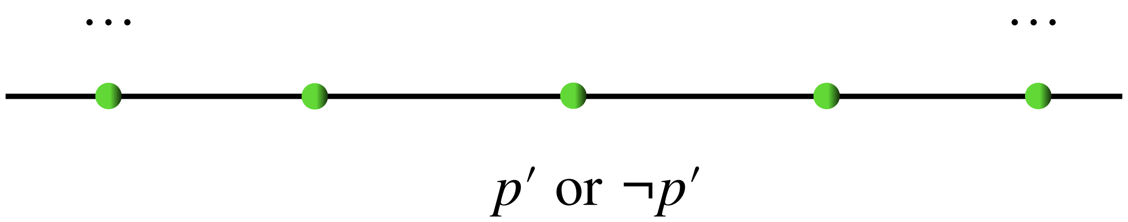

$M^l \square N$

# Summary

- Block product operation for $\circledast -$Algebra

- $\mathscr{L}(\Box^* U_1)$ $=$ FO definable countable languages

  $=$ countable languages of marked star free expressions

- $\mathscr{L}(\Box_w^* U_1) =$ FO$_2$ definable languages

- $\mathscr{L}(\Box_w^* \{M^r, M^l\})$ $=$ LTL[S, U] definable languages

# Future Work

1) Extend to more expressive logic e.g. FO[cut]

$\exists X \; \phi(X)$

2) Get finer characterization of LTL or FO

3) Get equational characterization of LTL[S, U]

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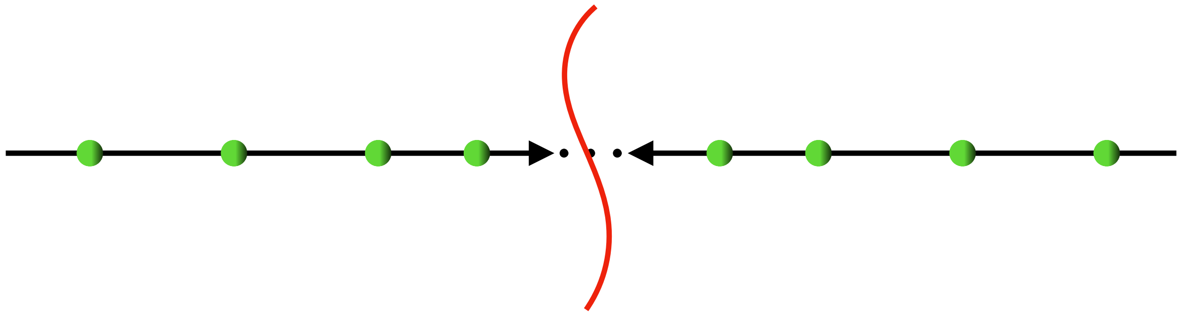

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2) Get finer characterization of LTL or FO

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