Geometric Structure Computation from Conics

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Abstract

This paper presents several results on images of various configurations of conics. We extract information about the plane from single and multiple views of known and unknown conics, based on planar homography and conic correspondences. We show that a single conic section cannot provide sufficient information. Metric rectification of the plane can be performed from a single view if two conics can be identified to be images of circles without knowing their centers or radii. The homography between two views of a planar scene can be computed if two arbitrary conics are identified in them without knowing anything specific about them. The scene can be reconstructed from a single view if images of a pair of circles can be identified in two planes. Our results are simpler and require less information from the images than previously known results. The results presented here involve univariate polynomial equations of degree 4 or 8 and always have solutions. Applications to metric rectification, homography calculation, 3D reconstruction, and projective OCR are presented to demonstrate the usefulness of our scheme.

1. Introduction

The geometry of multiple views has been a subject of extensive research in the past few years. Relationships exist between corresponding entities of images of a scene in multiple viewpoints. These relationships are well understood for points, lines, and simple primitives [3, 4]. The images $\mathbf{x}$ and $\mathbf{x}'$ in two views of any point X lying on a plane are related as $\mathbf{x}' = H\mathbf{x}$ in homogeneous coordinates. The non-singular 3×3 projective transformation matrix H is called a homography. Homography can be recovered from sufficient number of point and line correspondences [4]. Non-planar points are related by an algebraic constraint $\mathbf{x}'^T F \mathbf{x} = 0$, where F is a 3×3 matrix of rank 2 called the fundamental matrix. Corresponding points are similarly related by a trifocal tensor when three views of a scene are given.

Metric rectification of planes involves finding its fronto-parallel view from a projectively transformed view. The properties of parallel and perpendicular lines have been used for metric rectification of planes [7]. Multiple views from uncalibrated cameras have been used for projective reconstruction of the scene using dense correspondence of points; correspondences between the scene and the image can lead to metric reconstruction of the scene [4]. Vanishing points, i.e., the images of points at infinity, have been used for single view affine reconstruction [2].

Point and line correspondences are not available in many situations or could be noisy. Higher order parametric curves can, however, be recovered robustly from images due to the large number of points defining them. Algorithms for projective and metric reconstruction of images of conics in the world from two or three views are presented in [8, 11]. The problem of reconstruction of quadrics from multiple views is addressed in [9]. We discuss the case of single view reconstruction in this paper. A method to recover the planar homography from conics and other curves given the epipolar geometry can be found in [12]. A discussion on the recovery of homography and fundamental matrix using correspondences of higher order parametric curves can be found in [5]. Their results can work for conics, but do not exploit any special property of conics. Their solution involves a set of complex multivariate non-linear equations which cannot be easily solved. For instance, homography calculation from conics using their formulation will result in 5 non-linear equations in 8 unknowns with infinite solutions. We show in this paper that the problem can be reduced to a univariate, fourth-degree polynomial equation for two conics. An earlier work [13] that has dealt with homography calculation from conics used a minimum of 7 conic correspondences. They used a 6×6 transformation relating conics and derived the homography in a linear fashion using it. We show that only two conic correspondences are enough for calculating the homography and present an efficient algorithm for doing so. A very recent paper describes a method to recover camera calibration given two parallel (i.e., not necessarily coplanar) circles [14].

We present results for metric rectification, homography estimation, and 3D reconstruction for various configurations of conic correspondences in this paper. First, we show that the minimum number of conics required for recovering the homography matrix, without any point correspondence, is two. Conic correspondences can be classified into four cases: (a) single view of coplanar conics, (b) multiple views of coplanar conics, (c) single view of non-coplanar conics, and (d) multiple views of non-coplanar conics. We describe an approach to perform metric rectification from a single view of coplanar conics using the information that the conics are circles. We also present a method for homography estimation from multiple views of general coplanar conics. Two conic correspondences are all that is required for it. We then present a reconstruction algorithm from conics. Unlike previous approaches, our method works for a single view of non-coplanar conics.

The next section establishes the minimum number of conic correspondences required for homography calculation as two. Section 3 presents metric rectification from a single view of two circles. Section 4 extends this method to finding the homography from two general conic correspondences. Section 5 presents 3D reconstruction from a single view of two planes with two circles each. Section 6 discusses an application of using conic correspondences in the form of projective OCR. Section 7 provides some concluding remarks.

2. Number of Coplanar Conics

Conic sections are planar curves defined by a second degree parametric equation. Conics are abundantly available as many man-made structures follow them. A general conic section can be expressed as $x^T C x = 0$, where x is a point expressed in homogeneous coordinates and

$$C = \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix}$$

Any projective transformation H can be split into a similarity transform H_s, a pure affine transform H_a and a pure projective transform H_p such that $H = H_p H_a H_s$ [7]. In general H_s, H_a and H_p can respectively be written as

$$\begin{pmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_1 & v_2 & 1 \end{pmatrix}.$$

A conic C gets transformed by a homography H as $C' = H^{-T} C H^{-1}$. A circle C_1 first undergoes a similarity transform H_s to become another circle C_2, then an affine transform H_a to become an ellipse C_3, and a projective transform H_p to become conic section C_4. Let a, b, v_1 and v_2 be some parameters which define the matrices H_a^{-1} and H_p^{-1}. The equation of the circle C_2, the ellipse C_3, and the conic C_4 are given as

$$C_2 = \begin{pmatrix} A & 0 & G \\ 0 & A & F \\ G & F & C \end{pmatrix}$$

$$C_3 = H_a^{-T} C_2 H_a^{-1} = \begin{pmatrix} A' & H' & G' \\ H' & B' & F' \\ G' & F' & C' \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} a^2 A & abA & aG \\ abA & b^2 A + A/a^2 & bG + F/a \\ aG & bG + F/a & C \end{pmatrix}$$

$$C_4 = H_p^{-T} C_3 H_p^{-1} = \begin{pmatrix} A'' & H'' & G'' \\ H'' & B'' & F'' \\ G'' & F'' & C'' \end{pmatrix} = \quad (2)$$

$$\begin{pmatrix} A' + 2G'v_1 + C'v_1^2 & H' + G'v_2 + F'v_1 + C'v_1 v_2 & G' + C'v_1 \\ H' + G'v_2 + F'v_1 + C'v_1 v_2 & B' + 2F'v_2 + C'v_2^2 & F' + C'v_2 \\ G' + C'v_1 & F' + C'v_2 & C' \end{pmatrix}$$

There are infinite number of ellipses which undergo different pure projective transformations to result in any given conic. This is because the number of equations that an ellipse-conic correspondence offers is 5 (Eq. 2) and the number of unknowns is 7 (5 for the ellipse and 2 for the projective transform). Each of these ellipses can be generated by a unit circle by a similarity transformation followed by an affine transformation. Therefore, an infinite number of possible homographies differing by a general projective transform can exist between two given conics. In the case of imaging an unknown circle, we get only 5 equations in 7 unknowns (3 for C_2, 2 for affine, 2 for projective or 5 for C_3, 2 for projective) which again have infinite possible solutions. Thus, given only one conic correspondence no concrete information can be gathered about the projective, affine or similarity part of the homography.

We now consider the case of two conic correspondences. From Eq. 2, we get the following for one of the conics.

$$\begin{aligned} A'' &= A' + 2G'v_1 + Cv_1^2 \\ H'' &= H' + G'v_2 + F'v_1 + Cv_1 v_2 \\ G'' &= G' + Cv_1 \\ B'' &= B' + 2F'v_2 + Cv_2^2 \\ F'' &= F' + Cv_2 \\ C'' &= C \end{aligned}$$

Twelve equations in fourteen unknowns are obtained using two sets of the above equations. If a circle getting transformed as an ellipse using a pure affine transform, we can get from Eq. 1,

$$A'/B' = a^2 A/(b^2 + 1/a^2)A = a^2/(b^2 + 1/a^2) = \text{const}$$
$$A'/H' = a^2 A/abA = a/b = \text{constant}$$

Therefore, $A'_1/B'_1 = A'_2/B'_2$ and $A'_1/H'_1 = A'_2/H'_2$, which gives two more equations. We have 14 equations in 14 unknowns. It can be shown that these equations reduce to a univariate polynomial equation of degree 8 which has at most 8 solutions for H_p. This process can be extended to calculate the entire homography between two views using two conic correspondences. Having more than two conic correspondences would result in an over-determined set of equations which could provide more robust solutions. It can also be shown that one conic correspondence with a point correspondence on the conic is sufficient for homography calculation. However, it is difficult to obtain a point correspondence on a general conic and therefore, this method is of purely theoretical interest.

In summary, no information can be obtained from one or multiple views of one general conic. However, identification of two circles is sufficient to obtain the homography up to similarity.

3. Single View of Coplanar Conics

Many documents and billboards contain circles which result in conics when imaged. The exact equation of the circles is not known. We show that a fronto-parallel view of the scene can be generated given 2 circles on a plane. Two conics, in general, intersect in four points, real or imaginary. Consider the conic sections

$$\begin{aligned} ax^2 + 2x(hy+g) + by^2 + 2fy + c &= 0, \\ a'x^2 + 2x(h'y+g') + b'y^2 + 2f'y + c' &= 0 \end{aligned} \quad (3)$$

Eliminating x from these equations results in an equation of the fourth degree in y, giving four values, real or imaginary, for y. Eliminating x^2 from these two equations, we see that there is only one value of x for each value of y. Thus, there are only four points of intersection. All circles pass through the circular points $I = (1, i, 0)^T$ and $J = (1, -i, 0)^T$ [4]. Therefore, two of the points of intersection of any two circles are the circular points. Consider the case where the circular points are subjected to a general projective transform $H = \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix}$.

Consider the points $HI = (h_1 + jh_2, h_4 + jh_5, h_7 + jh_8)$ and $HJ = (h_1 - jh_2, h_4 - jh_5, h_7 - jh_8)$. Multiplying HI by $(h_7 - jh_8)$ and HJ by $(h_7 + jh_8)$ we get $HI = (h_1h_7 + h_2h_8 + j(h_2h_7 - h_1h_8), h_4h_7 + h_5h_8 + j(h_5h_7 - h_4h_8), h_7^2 + h_8^2)$ and $HJ = (h_1h_7 + h_2h_8 - j(h_2h_7 - h_1h_8), h_4h_7 + h_5h_8 - j(h_5h_7 - h_4h_8), h_7^2 + h_8^2)$. These circular points cannot get mapped onto points with both real x and y coordinates because then the matrix H becomes rank deficient. This is because then $h_2h_7 - h_1h_8 = 0$ and $h_5h_7 - h_4h_8 = 0$ which implies $h_1/h_2 = h_4/h_5 = h_7/h_8$. (Or, $h_7 = h_8 = 0$, in which case H is affine, which is handled later.) That is, the first two columns are related by a scale, which implies that the matrix is rank deficient. Also, the x and y coordinates of the projected circular points are conjugate to each other. Thus, two or four of the points of intersection will have either the x or the y coordinate (or both) as imaginary. We get one or two pairs of imaginary points such that within each, the x and y coordinates are conjugates of each other.

Let the two conics present in the view be given by Eq. 3. Since they are obtained from imaging circles, solving them would give us the circular points. These can be used to find the dual conic $C^*_\infty = IJ^T + JI^T$ [4]. The matrix H_pH_a can be obtained using the dual conic [7]. This degenerate dual conic is invariant to similarity transform and is given as

$$\begin{aligned} C^*_\infty &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ (C^*_\infty)' &= (H_pH_aH_s)C^*_\infty(H_pH_aH_s)^T \\ &= (H_pH_a)C^*_\infty(H_pH_a)^T \\ &= \begin{pmatrix} KK^T & K^T\mathbf{v} \\ \mathbf{v}^TK & 0 \end{pmatrix}, \end{aligned}$$

where $\mathbf{K} = \begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix}$ defines the affine transformation and $\mathbf{v} = [v_1 v_2]$ defines the projective transformation. The projective and affine components are determined directly from the image of C^*_∞ but the similarity component is undetermined. The circular points get projected to points at infinity if the transformation is either similarity or affine. Solving the above equations will not yield the correct result. For this case, consider Eq. 3 of the conics using homogeneous coordinates. Solve for points of intersection at infinity by setting $w = 0$. This results in the following equations.

$$\begin{aligned} ax^2 + 2hxy + by^2 &= 0 \\ a'x^2 + 2h'xy + b'y^2 &= 0. \end{aligned}$$

This would give either one or two pairs of circular points. Thus, we obtain a maximum of four pairs of circular points, two for the projective case and two for handling the affine and similarity cases. The degenerate dual conic is obtained in each case to rectify the image. One of the four possible resultant images is the required rectified image. The exact equations or the relative positions of the circles in the scene need not be known. The ease of specifying a conic section makes it possible to obtain the result with minimal correspondences. This method does not result in complex multivariate non-linear equation and guarantees a result for every input. We cannot determine which of the four possible solutions is the correct one without additional information about the scene such as the ratio of the radii of the circles or the distance between their centers.

The algorithm for metric rectification can be summarized as follows.

1. Obtain the equations of the two conics from the image.
2. Solve them using non-homogeneous coordinates and homogeneous coordinates with $w = 0$.
3. Obtain a maximum of four pairs of imaginary points with conjugate x and y coordinates.
4. Treating each pair as the circular points, obtain four different matrices for $H_p H_a$.
5. One of these four matrices results in the actual rectification.

Figures 1 shows the single view of some of the images containing unknown coplanar circles and their rectified images using the above algorithm. In all our experiments, the equations of the conics in the image were computed from their boundary points by the method described in [6]. The method has the advantage of explicitly modelling noise and providing a reliability score.

Figure 1: Circles in the left images were used to rectify them to get the right images. The circles representing the nodes of the graph in the book and the 'O's in the classroom board were used for rectification by projectively transforming them to result in circles. The rectification is quite good; the images on the right show the quadrilaterals to which the rectangular images on the left map.

4. Multiple Views of Coplanar Conics

Multiple views of a plane containing conics can help in recovering the homography between the views. Previous approaches to homography calculation from conics require a minimum of 7 correspondences [13]. Other approaches to homography calculation from higher order parametric curves fail in the case of conics as they result in insufficient number of equations [5].

Two given conics intersect at 4 points. If these points can be recovered and corresponded, the homography can be calculated directly. If not, both images can metric rectified by assuming that the given conics are images of circles. Note that this assumption is not valid when the conics intersect at 4 points because any two circles can intersect at a maximum of 2 real points. The method works even when the conics are not obtained by imaging circles. Even though the rectification would be incorrect, both the views are transformed incorrectly in a similar way and the final homography is calculated using this rectified views.

Given a view with conics C_1^1 and C_1^2, obtain the homography H_1 such that $C_1^1 = H_1^{-T} C_1^1 H_1^{-1}$ and $C_1^2 = H_1^{-T} C_1^2 H_1^{-1}$ where C_1^1 and C_1^2 are circles. Similarly, for the second view containing conics C_2^1 and C_2^2, we obtain the homography H_2 such that $C_2^1 = H_2^{-T} C_2^1 H_2^{-1}$ and $C_2^2 = H_2^{-T} C_2^2 H_2^{-1}$ where C_2^1 and C_2^2 are circles. If the views are of the same scene, there exists a similarity transform H_s such that

$$C_1^1 = H_s^{-T} C_2^1 H_s^{-1} \text{ and } C_1^2 = H_s^{-T} C_2^2 H_s^{-1}$$

We can solve for H_s using two point correspondences, such as the centers of the circles in the rectified view as similarity transform preserves the centers. The center of a circle $ax^2 + ay^2 + 2gx + 2fy + c = 0$ is given as $(-g/a, -f/a)$. The homography between the two views is the product $H_1 H_s H_2^{-1}$. We can obtain a unique H_s if the radii of the two circles are different. If the radii are the same, we cannot determine which center in one view corresponds to which center in the other. This results in the rotation and translation being unknown. If the circles are concentric, the scale and translation can be determined from the radii and the center, but the rotation ambiguity will remain. This approach involves simple univariate polynomial equations of degree 4 which always have a solution.

In summary, the algorithm for homography computation from 2 views is

1. Obtain the equations of the conics in both views.
2. Rectify both views assuming that the conics are images of circles. The results are correct even if this is not so. Let H_1 and H_2 be the rectifying homographies.
3. Calculate the similarity transform H_s between the two rectified views using two point correspondences obtained by finding the centers of the two circles.
4. The homography between the two views is obtained as $H_1 H_s H_2^{-1}$.

Figure 2 shows the results of homography calculation for a real image. The two 'O's in the sign were assumed to be circles. The image obtained by applying the homography to the left image is shown at the bottom.

5. Single View of Non-coplanar Conics

Lastly, we consider the case of imaging non-coplanar conics, specifically the case where there are two planes containing at least two conics each. We show that we can obtain the 3D reconstruction of the scene from them, provided

Figure 2: The homography between the top images was computed. The bottom image is the result of applying it to the top left one.

they are images of circles. The previous approaches to 3D reconstruction from conics [8, 11] are restricted to the case of multiple views. Our approach is similar to the one in [2], which finds 3D affine measurements from minimal geometric information such as the vanishing line l of a reference plane and a vanishing line for another plane not parallel to the previous plane v. To calculate the height of an object, we use a reference height Z_r specified by a base point b_r and a top point t_r, a parameter α such that

$$\alpha = \frac{||b_r \times t_r||}{Z_r(l \cdot b_r)||v \times t_r||}$$

The height Z_x of an object specified by the base point b_x and the top point t_x is given as

$$Z_x = \frac{||b_x \times t_x||}{\alpha(l \cdot b_x)||v \times t_x||}$$

The measurements are restricted to the ratio of the heights of objects, ratio of the lengths along parallel lines on the plane and the ratio of areas on the plane. The complete reconstruction of the 3D scene containing conics can be obtained from its multiple views using the method described in [11].

The vanishing line is the last row of the matrix used for rectification. Metric rectification can be performed as described in Section 3. The user has to specify a reference height used for the reconstruction. The algorithm is summarized as follows.

1. Obtain the equations of the conics on each plane and rectify each plane.
2. Obtain the vanishing lines of the two planes from the rectification transformation.
3. Use the procedure in [2] to obtain the 3D reconstruction of the scene from these vanishing lines.

Figure 3: The 3D reconstruction of the top image was used to render the bottom views.

Figure 3 shows the results of reconstructing the image using the two circles on the wall. The lines on the floor were used to get its vanishing line. Two different views generated from the 3D model starting with the first view are shown. The results are similar to those reported in [2] which used parallel lines to obtain vanishing lines. Our approach gave the height of the leftmost wall as 16.1 units compared to 16 units from [2].

6. Application to Projective OCR

Optical Character Recognition (OCR) is the process of converting a document image to text. Normally, the input image is scanned and presented to the OCR system. The document might be skewed or scaled slightly. Projective OCR is one that identifies the text from the image of a document taken from a general view using a perspective camera. Conventional OCR engine can be used on the document image after metric rectification of the projectively distorted one. Earlier approaches, such as [10], find the transformation assuming that the imaging system has the vertical vanishing point at infinity. Others map the quadrilateral containing the text in the image to a rectangle [1]. We assume that there are two circles – either from character ‘O’ or otherwise – in the text of the document. These can be identified automatically by looking for conics in the image. Rectification is then performed on the image by assuming ‘O’s to be circles. The recognition is independent of the aspect ratio; thus this assumption produces good results. Figure 4 shows two sample pages along with their rectified image. The filled ‘O’s in the heading were used to rectify the image. We took 10 different views of 7 pages, rectified them and fed them to an existing OCR system with and without rectification. An overall accuracy of 94.75% was achieved with rectification when compared to an accuracy of 42.68% without rectification.

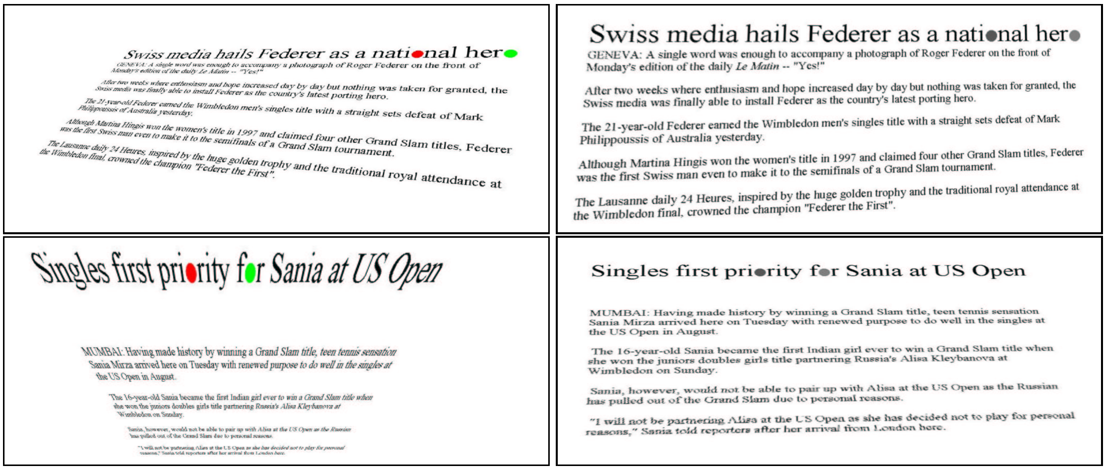

Figure 4: Right images give the rectified versions of the left ones.

Type	Configuration	#Views	Application	Reference
4 Conics	Non-coplanar	2	Fundamental Matrix Computation	[5]
2 Circles	Coplanar	1	Metric Rectification	In this paper
7 Conics	Coplanar	2	Homography Computation	[13]
2 Conics	Coplanar	2	Homography Computation	In this paper
Conics	Non-coplanar	2	3D Reconstruction	[8, 11]
Circles	Non-coplanar	1	3D Reconstruction	In this paper

Table 1: Summary of applications of various configurations of conics

7. Conclusions and Future Work

This paper discussed various configurations of conics in the scene. We presented several applications of geometric structure estimation from conics in the image, including metric rectification of a single view of unknown circles, homography calculation from multiple views of unknown conics or single view of known conics, and 3D reconstruction from single view of unknown non-coplanar conics. The algorithms are simple and do not require correspondences of a large number of conics or the solution of multivariate polynomial equations. A summary of the results is shown in Table 1. Extensions to these methods which handle over determined sets of equations, obtained due to the presence of more than two coplanar conics, need to be explored.

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