Fitting Coupled Geometric Objects for Metric Vision

Paul O'Leary
Institute for Automation
University of Leoben
Leoben, Austria
automation@unileoben.ac.at

Matthew Harker
Institute for Automation
University of Leoben
Leoben, Austria

Paul Zsombor-Murray
Centre for Intelligent Machines
McGill University
Montreal, Canada
paul@cim.mcgill.ca

Abstract

A new approach to fitting coupled geometric objects, such as concentric circles, is presented. The objects can be coupled via common Grassmannian coefficients or through a correlation constraint on their coefficients. The implicit partitioning and partial block diagonal structure of the design matrix enables an efficient orthogonal residualization based on a generalized Eckart-Young-Mirsky matrix approximation. The residualization prior to eigen- or singular-value decomposition improves the numerical efficiency and makes the result invariant to the residuals of the independent portions.

Analysis is performed for the generalized case of coupled implicit equations and examples of parallel lines, concentric circles and coupled conics are given. Furthermore, numerical tests and applications in image processing are presented.

1. Introduction

The primary aim of metric vision is to gain quantitive information on the position, orientation and dimensions of objects from images. In general points of interest are extracted from one or more images and segmented prior to fitting. The points obtained are perturbed by noise or uncertainty in the images. Given the noisy points of interest it is then necessary to fit geometric objects.

Automatic inspection commonly requires the measurement of multiple objects simultaneously, whereby there may be coupling between the geometric objects, e.g. parallel lines, concentric circles, common orientation, etc. The coupling may be both spatial and temporal. Temporal coupling occurs, for example, when a mechanical component is being machined. From one image to the next some features change but many remain unchanged, i.e. these features are common to multiple images and represent a coupling for the geometric fitting procedure.

Much attention has been paid to the task of fitting individual objects, such as conics [1,2], and quadrics. Very little literature, however, is available on fitting multiple geometric objects which are coupled in some explicit form. Concentric circles and their projective concentric ellipses have been used by Kim et. al. [3] to calibrate cameras, and have also been used in remote sensing to determine dimensions of archeological sites [4]. However, only specific solutions to the tasks at hand were presented, and in the past no generalization has been performed.

This paper presents a new generalized method of fitting coupled geometric objects. The types of objects being considered are defined by their dual-Grassmannian coordinates and the specific object by its Grassmannian coefficients. Two mechanisms are proposed to implement the coupling:

1. Common-dual-Grassmannian coordinates and coefficients: The design matrix and coefficient vector are partitioned so as to reflect the coefficients which are common to both objects. The resulting scatter matrix, with its implicit partitioning, is solved in a two step procedure using the *Schur complement*.

2. Quadratic constrained total least squares: The total least square reduction is performed with the addition of a quadratic constraint on the coefficient vector. This enables the implementation of quadratic coupling between the coefficients of two objects.

Both mechanisms are implemented simultaneously by the method presented. Furthermore, a method of orthogonal residualization based on generalized Eckart-Young-Mirsky matrix approximation is presented which improves the numerical performance of the fitting procedures. Numerical tests and results from selected image processing tasks are presented.

2. Geometric background

Geometric objects can be represented as the product of a design vector $\boldsymbol{d}$ and a coefficient vector $\boldsymbol{v}$, furthermore, there may be a constraint on the coefficients. For Example, a hyperbola can be represented as,

$$\begin{bmatrix} x^2 & xy & y^2 & x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T = 0 \quad \text{subject to} \quad b^2 - 4ac > 0. \tag{1}$$

The design vector defines the general family of curves being considered, in this case a conic. The quadratic constraint on the coefficients determines the specific type of curve within the family, e.g. in (1) the hyperbolæ are defined, which belong to the family of conics. The coefficients define the specific example of the type of curve.

Given a set of n points the design matrix D and coefficient vector $\boldsymbol{v}$ can be defined,

$$\mathsf{D} \triangleq \begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^2 & x_ny_n & y_n^2 & x_n & y_n & 1 \end{bmatrix}, \tag{2}$$

$$\boldsymbol{v} \triangleq \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T. \tag{3}$$

The task of constrained fitting can now be formulated as,

$$\boldsymbol{v}^T \mathsf{D}^T \mathsf{D} \boldsymbol{v} = \min_{\|v\| \neq 0} \quad \text{subject to} \quad \boldsymbol{v}^T \mathsf{C} \boldsymbol{v} = \alpha \tag{4}$$

where C is the constraint matrix. Bookstein [5] showed in a seminal work that constrained problems of this sort can be solved using generalized eigenvectors. Differentiating (4) with respect to $\boldsymbol{v}$, gives the normal equations which need to be solved simultaneously,

$$\mathsf{D}^T \mathsf{D} \boldsymbol{v} = \mathbf{0} \tag{5}$$

$$\mathsf{C} \boldsymbol{v} = \mathbf{0} \tag{6}$$

$$\text{given} \quad \|v\| \neq 0.$$

Now using a Lagrange multiplier λ,

$$\mathsf{D}^T \mathsf{D} \boldsymbol{v} - \lambda \mathsf{C} \boldsymbol{v} = \mathbf{0} \quad \text{subject to} \quad \boldsymbol{v}^T \mathsf{C} \boldsymbol{v} = \alpha \tag{7}$$

defining the scatter matrix $\mathsf{S} \triangleq \mathsf{D}^T \mathsf{D}$ and rearranging (7),

$$\{\mathsf{S} - \lambda \mathsf{C}\} \boldsymbol{v} = \mathbf{0}. \tag{8}$$

The solutions are the eigenvectors of S with respect to C, which is a generalized eigenvector problem. Pre-multiplying by C^{-1},

$$\left\{\mathsf{C}^{-1} \mathsf{S} - \lambda \mathsf{I}\right\} \boldsymbol{v} = \mathbf{0} \tag{9}$$

and defining $\hat{\mathsf{S}} \triangleq \mathsf{C}^{-1} \mathsf{S}$ the equation becomes,

$$\left\{\hat{\mathsf{S}} - \lambda \mathsf{I}\right\} \boldsymbol{v} = \mathbf{0}. \tag{10}$$

This is a standard problem of finding the eigenvalues and eigenvectors of $\hat{\mathsf{S}}$. Such constrained minimization techniques have been successfully applied to the direct and specific fitting of hyperbolæ and ellipses [6].

3. Coupled geometric objects

This paper proposes two mechanisms to couple the geometric objects:

1. the objects have common Grassmannian coefficients, and/or
2. there is a quadratic constraint relating the coefficients of the two objects.

Both of these mechanisms are implemented simultaneously in the following formulation: consider two geometric objects O_1 and O_2 represented by implicit homogeneous equations,

$$O_1 \triangleq \begin{bmatrix} \boldsymbol{d}_c & \boldsymbol{d}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_c & \boldsymbol{v}_1 \end{bmatrix}^T = 0 \tag{11}$$

$$O_2 \triangleq \begin{bmatrix} \boldsymbol{d}_c & \boldsymbol{d}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_c & \boldsymbol{v}_2 \end{bmatrix}^T = 0. \tag{12}$$

The design and coefficient vectors have been partitioned such that $\boldsymbol{d}_c$ defines that portion of the dual-Grassmannian space common to both objects, and $\boldsymbol{v}_c$ contains the coefficients which the objects have in common. The common coefficients must not necessarily apply to a common design polynomial. The vectors $\boldsymbol{d}_1, \boldsymbol{v}_1$ and $\boldsymbol{d}_2, \boldsymbol{v}_2$ are the independent design and coefficient vectors associated with the objects O_1 and O_2 respectively.

The design and coefficient vectors can now be concatenated to define an extended Grassmannian space, i.e.

$$\boldsymbol{d} \triangleq \begin{bmatrix} \boldsymbol{d}_c & \boldsymbol{d}_1 & \boldsymbol{d}_2 \end{bmatrix} \tag{13}$$

$$\boldsymbol{v} \triangleq \begin{bmatrix} \boldsymbol{v}_c & \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}^T \tag{14}$$

Once again the constrained fitting task can be formulated as:

$$\boldsymbol{v}^T \mathsf{D}^T \mathsf{D} \boldsymbol{v} = \min_{\|\boldsymbol{v}\| \neq 0} \quad \text{subject to} \quad \boldsymbol{v}^T \mathsf{C} \boldsymbol{v} = \alpha \tag{15}$$

Whereby, there is an implicit partitioning of the design matrix,

$$\mathsf{D} \triangleq \begin{bmatrix} \mathsf{D}_{1,c} & \mathsf{D}_1 & 0 \\ \mathsf{D}_{2,c} & 0 & \mathsf{D}_2 \end{bmatrix} \tag{16}$$

the scatter matrix $\mathsf{S} \triangleq \mathsf{D}^T \mathsf{D}$,

$$\mathsf{S} = \left[\begin{array}{c|cc} \mathsf{D}_{1,c}^T \mathsf{D}_{1,c} + \mathsf{D}_{2,c}^T \mathsf{D}_{2,c} & \mathsf{D}_{1,c}^T \mathsf{D}_1 & \mathsf{D}_{2,c}^T \mathsf{D}_2 \\ \hline \mathsf{D}_1^T \mathsf{D}_{1,c} & \mathsf{D}_1^T \mathsf{D}_1 & 0 \\ \mathsf{D}_2^T \mathsf{D}_{2,c} & 0 & \mathsf{D}_2^T \mathsf{D}_2 \end{array}\right] \tag{17}$$

and also of the constraint matrix,

$$\mathsf{C} \triangleq \begin{bmatrix} \mathsf{C}_{c,c} & \mathsf{C}_{c,1} & \mathsf{C}_{c,2} \\ \mathsf{C}_{c,1} & \mathsf{C}_{1,1} & \mathsf{C}_{1,2} \\ \mathsf{C}_{c,2} & \mathsf{C}_{1,2} & \mathsf{C}_{2,2} \end{bmatrix}. \tag{18}$$

This formulation enables the simultaneous implementation of common coefficients and of the constraint:

$$\begin{aligned} & \boldsymbol{v}_c^T\, \mathsf{C}_{c,c}\, \boldsymbol{v}_c \\ & + \boldsymbol{v}_1^T\, \mathsf{C}_{1,1}\, \boldsymbol{v}_1 \\ & + \boldsymbol{v}_2^T\, \mathsf{C}_{2,2}\, \boldsymbol{v}_2 \\ & + 2\, \boldsymbol{v}_c^T\, \mathsf{C}_{c,1}\, \boldsymbol{v}_1 \\ & + 2\, \boldsymbol{v}_c^T\, \mathsf{C}_{c,2}\, \boldsymbol{v}_2 \\ & + 2\, \boldsymbol{v}_1^T\, \mathsf{C}_{1,2}\, \boldsymbol{v}_2 = \pm\alpha. \end{aligned} \tag{19}$$

Consequently, a constraint can be placed on all convolutions of the coefficient vectors $\boldsymbol{v}_c$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

4. Orthogonal residualization

Considering the design matrix from (16) once again.

$$\mathsf{D} \triangleq \left[\begin{array}{c|cc} \mathsf{D}_{1,c} & \mathsf{D}_1 & 0 \\ \mathsf{D}_{2,c} & 0 & \mathsf{D}_2 \end{array} \right] = \left[\begin{array}{c|c} \mathsf{D}_c & \mathsf{D}_{1,2} \end{array} \right], \tag{20}$$

a brute force method to find a solution for $\boldsymbol{v}$ is to apply singular value decomposition to D and select the right singular vector with the smallest singular value. However, this is both numerically inefficient and is not invariant to Euclidean transformations of the data.

The implicit partitioning of the matrix D and its partial block diagonal structure can be used to advantage in an orthogonal residualization. The magnitudes of the orthogonal projection of D_c onto $\mathsf{D}_{1,2}$ is given by

$$\mathsf{M} = \mathsf{D}_{1,2}^{+}\, \mathsf{D}_c, \tag{21}$$

where $\mathsf{D}_{1,2}^{+} \triangleq \left\{ \mathsf{D}_{1,2}^T\, \mathsf{D}_{1,2} \right\}^{-1} \mathsf{D}_{1,2}^T$ is the pseudo-inverse of $\mathsf{D}_{1,2}$. The orthogonal residual $\mathsf{D}_c^{\perp}$, i.e., the portion of D_c not predicted by $\mathsf{D}_{1,2}$, can be calculated as:

$$\mathsf{D}_c^{\perp} = \mathsf{D}_c - \mathsf{D}_{1,2}\, \mathsf{M}. \tag{22}$$

This residualization corresponds to a generalized Eckart-Young-Mirsky matrix approximation [7].

The right singular-vector of $\mathsf{D}_c^{\perp}$ with the smallest singular-value is the solution for the vector $\boldsymbol{v}_c$. The remaining coefficients are determined by back substitution,

$$\boldsymbol{v}_{1,2} = -\left\{ \mathsf{D}_{1,2}^{+}\, \mathsf{D}_c \right\} \boldsymbol{v}_c = -\mathsf{M}\, \boldsymbol{v}_c. \tag{23}$$

Taking advantage of the block diagonal nature of $\mathsf{D}_{1,2}$, the projection decomposes into two independent projections, i.e.,

$$\mathsf{M}_1 = \mathsf{D}_1^{+}\, \mathsf{D}_{1,c}, \tag{24}$$

$$\mathsf{M}_2 = \mathsf{D}_2^{+}\, \mathsf{D}_{2,c}, \tag{25}$$

$$\mathsf{D}_c^{\perp} = \begin{bmatrix} \mathsf{D}_{1,c}^{\perp} \\ \mathsf{D}_{2,c}^{\perp} \end{bmatrix} = \begin{bmatrix} \mathsf{D}_{1,c} - \mathsf{D}_1\, \mathsf{M}_1 \\ \mathsf{D}_{2,c} - \mathsf{D}_2\, \mathsf{M}_2 \end{bmatrix}, \tag{26}$$

and the back-substitution becomes,

$$\boldsymbol{v}_1 = -\mathsf{M}_1\, \boldsymbol{v}_c \quad \text{and} \quad \boldsymbol{v}_2 = -\mathsf{M}_2\, \boldsymbol{v}_c. \tag{27}$$

The resulting fitting algorithm can be summarized as follows:

1. Perform independent orthogonal residualization of each data set onto the common portion of the design matrix.

2. Concatenate the orthogonalized common portions to form $\mathsf{D}_c^{\perp}$

3. Solve the fitting on $\mathsf{D}_c^{\perp}$ to determine the common coefficients[1] $\boldsymbol{v}_c$ e.g. by applying singular value decomposition.

4. Perform independent back-substitution for each object to determine the remaining coefficients.

Assuming a total of n points are available, the vector $\boldsymbol{v}_c$ has the length m_c; the vectors $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$, have the lengths m_1 and m_2 respectively. The matrices D and $\mathsf{D}_c^{\perp}$ then have the dimensionality,

$$\mathsf{D} \Rightarrow n \times (m_c + m_1 + m_2), \tag{28}$$

$$\mathsf{D}_c^{\perp} \Rightarrow n \times m_c \tag{29}$$

respectively. Consequently, residualization has reduced the effort required to calculate the singular value decomposition, while improving the robustness of the result. Furthermore, determining the common coefficients $\boldsymbol{v}_c$ becomes invariant with respect to the residuals of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The method has been derived for two objects, however, it is generally applicable to any number of coupled objects.

5. Quadratic constrained fit

A solution to the problem of total least squares with a quadratic constraint has been presented by Gander [8]. However, he assumed that the constraint is of the form $\boldsymbol{v}^T\, \mathsf{K}^T\, \mathsf{K}\, \boldsymbol{v}$, which can only implement a magnitude constraint on the solution vectors. An alternative solution based on generalized eigenvectors is required if an orientational constraint is required, i.e., the matrix square root of C has complex entries.

In general the constraint does not apply to all coefficients and C is correspondingly sparse. Furthermore, C may not have an inverse. Nevertheless, the problem can be solved by an appropriate partitioning of the matrices. This paper proposes to first solve the system of equations for the

[1] It may be desirable to apply a quadratic constraint to the solution of $\mathsf{D}_c^{\perp}$, this for example would be the case when fitting hyperbolæ with common coefficients.

constrained coefficients, and then to determine the remaining coefficients via back-substitution. In many fitting problems, common coefficients can implement the constraint required and an additional constraint matrix need not be implemented, e.g. when fitting concentric circles.

5.1. Partitioned generalized eigenvectors

The implicit partitioning of partially constrained least square fitting was recognized by Halir et. al. [9]. Here their work is extended to include coupled geometric objects. It is important to note that C is in general sparse, since the constraint is only applied to some portion of the coefficient vector.

A numerically efficient and more robust solution to this problem is to repartition the coefficient vector $\boldsymbol{v}$ into its constrained $\boldsymbol{v}_a$ and unconstrained portions $\boldsymbol{v}_b$ resulting from Equation (19). The corresponding partitioning of the scatter and constraint matrices must also be performed, such that,

$$\left\{ \begin{bmatrix} \mathsf{S}_a & \mathsf{S}_c \\ \mathsf{S}_c^T & \mathsf{S}_b \end{bmatrix} - \lambda \begin{bmatrix} \mathsf{C}_a & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} \boldsymbol{v}_a \\ \boldsymbol{v}_b \end{bmatrix} = \mathbf{0}. \tag{30}$$

This can now be expanded and separated into a constrained and an unconstrained equation,

$$\mathsf{S}_a \, \boldsymbol{v}_a + \mathsf{S}_c \, \boldsymbol{v}_b - \lambda \, \mathsf{C}_a = \mathbf{0} \tag{31}$$
$$\mathsf{S}_c^T \, \boldsymbol{v}_a + \mathsf{S}_b \, \boldsymbol{v}_b = \mathbf{0}. \tag{32}$$

Solving (32) for $\boldsymbol{v}_b$ and back-substituting into (31) yields,

$$\boldsymbol{v}_b = -\mathsf{S}_b^{-1} \, \mathsf{S}_c^T \, \boldsymbol{v}_a \tag{33}$$
$$\left\{ \mathsf{S}_a - \mathsf{S}_c \mathsf{S}_b^{-1} \, \mathsf{S}_c^T \right\} \boldsymbol{v}_a - \lambda \, \mathsf{C}_a \, \boldsymbol{v}_a = \mathbf{0}. \tag{34}$$

Defining $\tilde{\mathsf{S}}_a \triangleq \mathsf{S}_a - \mathsf{S}_c \mathsf{S}_b^{-1} \, \mathsf{S}_c^T$, the Schur complement of S_b in S we have,

$$\left\{ \tilde{\mathsf{S}}_a - \lambda \, \mathsf{C}_a \right\} \boldsymbol{v}_a = \mathbf{0}. \tag{35}$$

In this case C_a is of full rank and the generalized eigenvectors and eigenvalues of (35) are found, the eigenvector with the smallest magnitude eigenvalue is selected as the solution for $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ is determined by back-substitution into (33).

6. Numerical tests and applications

6.1. Fitting parallel lines

Consider the task of fitting two parallel lines, $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, to two sets of data points ${}_1\mathsf{P}_i = \begin{bmatrix} {}_1x_i & {}_1y_i & 1 \end{bmatrix}^T$ and ${}_2\mathsf{P}_i = \begin{bmatrix} {}_2x_i & {}_2y_i & 1 \end{bmatrix}^T$ each having m and n points respectively. The two line equations can be defined as,

$$d_1 \, {}_1x + d_2 \, {}_1y + n_1 = 0 \tag{36}$$
$$d_1 \, {}_2x + d_2 \, {}_2y + n_2 = 0 \tag{37}$$

whereby, the orientation of the lines are common (d_1 and d_2) and normal distances to the origin (n_1 and n_2) are independent. The corresponding complete design matrix is,

$$\mathsf{D} \, \boldsymbol{v} = \left[\begin{array}{cc|cc} {}_1x_1 & {}_1y_1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ {}_1x_m & {}_1y_m & 1 & 0 \\ \hline {}_2x_1 & {}_2y_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ {}_2x_n & {}_2y_n & 0 & 1 \end{array} \right] \left[\begin{array}{c} d_1 \\ d_2 \\ \hline n_1 \\ n_2 \end{array} \right]. \tag{38}$$

The orthogonal residualization delivers the projection magnitudes,

$${}_1\boldsymbol{p}_0 = \begin{bmatrix} {}_1\overline{x} \\ {}_1\overline{y} \end{bmatrix} \quad \text{and} \quad {}_2\boldsymbol{p}_0 = \begin{bmatrix} {}_2\overline{x} \\ {}_2\overline{y} \end{bmatrix}, \tag{39}$$

which are the coordinates of the centroids of the clouds of points. The two lines must pass through their respective centroids. Subtracting the coordinates of the respective centroid for each cloud of points centers the data at the origin (i.e. mean free data). Defining the mean free data as:

$${}_j\tilde{x}_i \triangleq {}_jx_i - \overline{{}_jx}, \tag{40}$$
$${}_j\tilde{y}_i \triangleq {}_jy_i - \overline{{}_jy}, \tag{41}$$
$$\text{for} \quad i = 1 \ldots n \; (\text{or} \, m),$$

the design matrix is now redefined on the mean free data,

$$\tilde{\mathsf{D}} \, \boldsymbol{v}_c = \left[\begin{array}{cc} {}_1\tilde{x}_1 & {}_1\tilde{y}_1 \\ \vdots & \vdots \\ {}_1\tilde{x}_m & {}_1\tilde{y}_m \\ \hline {}_2\tilde{x}_1 & {}_2\tilde{y}_1 \\ \vdots & \vdots \\ {}_2\tilde{x}_n & {}_2\tilde{y}_n \end{array} \right] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}. \tag{42}$$

Applying singular value decomposition[2] (or eigenvector analysis) and selecting the right singular-vector with the smallest singular value yields the vector of common coefficients $\boldsymbol{v}_c = [d_1 \; d_2]$. Back substitution can now be performed to determine the remaining coefficients

$$n_1 = -\boldsymbol{v}_c \, {}_1p_0 = -{}_1\overline{x} \, d_1 - {}_1\overline{y} \, d_2 \tag{43}$$
$$n_2 = -\boldsymbol{v}_c \, {}_2p_0 = -{}_2\overline{x} \, d_1 - {}_2\overline{y} \, d_2. \tag{44}$$

The two lines have the coordinates

$$\boldsymbol{l}_1 = \begin{bmatrix} \boldsymbol{v}_c & n_1 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & n_1 \end{bmatrix} \tag{45}$$
$$\boldsymbol{l}_2 = \begin{bmatrix} \boldsymbol{v}_c & n_2 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 & n_2 \end{bmatrix}. \tag{46}$$

[2] It should be noted that the matrix $\tilde{\mathsf{D}}$ is of dimension $(m+n) \times 2$ whereas D is of dimension $(m+n) \times 4$, reducing the numerical effort required to perform the singular value decomposition.

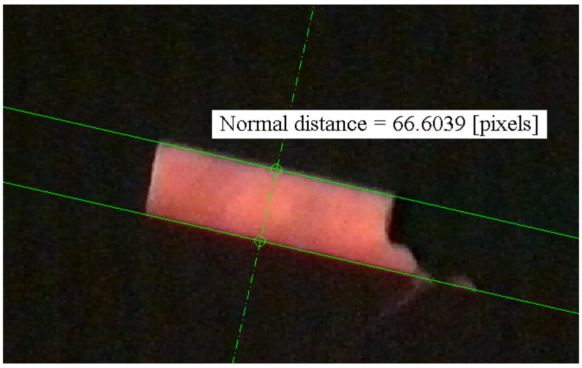

Figure 1: Example of fitting two parallel lines in image processing.

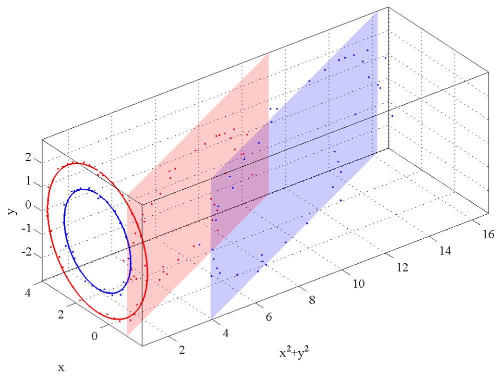

Figure 2: Concentric circles correspond to parallel hyperplanes in their dual-Grassmannian space.

6.2. Fitting concentric circles

A circle can be defined on the dual-Grassmannian coordinates and coefficients:

$$\left[x^2+y^2 \quad x \quad y \quad 1\right] \left[c_1 \quad c_2 \quad c_3 \quad c_4\right]^T = 0. \tag{47}$$

whereby the relationships to the centre point and radius are:

$$x_0 = -\frac{c_2}{2c_1}, \quad y_0 = -\frac{c_3}{2c_1}, \quad \text{and} \quad r^2 = \frac{c_4}{2c_1} - x_0^2 - y_0^2, \tag{48}$$

assuming $c_1 \neq 0$. Consider two concentric circles $\mathsf{C}_1 = [x_0, y_0, r_1]$ and $\mathsf{C}_2 = [x_0, y_0, r_2]$, which can be coupled by defining c_1, c_2 and c_3 to be common and implementing independent c_4's. This corresponds to two parallel planes in the dual-Grassmannian space. The corresponding complete design matrix is,

$$\mathsf{D}\,\boldsymbol{v} = \left[\begin{array}{ccc|cc} {}_1x_1^2 + {}_1y_1^2 & {}_1x_1 & {}_1y_1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {}_1x_m^2 + {}_1y_m^2 & {}_1x_m & {}_1y_m & 1 & 0 \\ \hline {}_2x_1^2 + {}_2y_1^2 & {}_2x_1 & {}_2y_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {}_2x_n^2 + {}_2y_n^2 & {}_2x_n & {}_2y_n & 0 & 1 \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ \hline n_1 \\ n_2 \end{array}\right]. \tag{49}$$

The orthogonal residualization delivers,

$${}_1\boldsymbol{p}_0 = \begin{bmatrix} \overline{{}_1x^2 + {}_1y^2} \\ {}_1\overline{x} \\ {}_1\overline{y} \end{bmatrix} \quad \text{and} \quad {}_2\boldsymbol{p}_0 = \begin{bmatrix} \overline{{}_2x^2 + {}_2y^2} \\ {}_2\overline{x} \\ {}_2\overline{y} \end{bmatrix}. \tag{50}$$

These are the coordinates of the centroid of the clouds of points in the hyperspace. Defining,

$${}_j\widetilde{x^2y^2}_i \triangleq {}_jx_i^2 + {}_jy_i^2 - \overline{{}_jx^2 + {}_jy^2} \tag{51}$$

$${}_j\tilde{x}_i \triangleq {}_jx_i - \overline{{}_jx} \tag{52}$$

$${}_j\tilde{y}_i \triangleq {}_jy_i - \overline{{}_jy} \tag{53}$$

$$\text{for} \quad i = 1 \ldots n \ (\text{or}\, m),$$

The design matrix is now redefined on the mean free data,

$$\tilde{\mathsf{D}}\,\boldsymbol{v}_c = \left[\begin{array}{ccc} {}_1\widetilde{x^2y^2}_1 & {}_1\tilde{x}_1 & {}_1\tilde{y}_1 \\ \vdots & \vdots & \vdots \\ {}_1\widetilde{x^2y^2}_m & {}_1\tilde{x}_m & {}_1\tilde{y}_m \\ \hline {}_2\widetilde{x^2y^2}_1 & {}_2\tilde{x}_1 & {}_2\tilde{y}_1 \\ \vdots & \vdots & \vdots \\ {}_2\widetilde{x^2y^2}_n & {}_2\tilde{x}_n & {}_2\tilde{y}_n \end{array}\right] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{54}$$

Applying singular value decomposition and selecting the right singular-vector with the smallest singular value yields the vector of common coefficients $\boldsymbol{v}_c = [c_1, c_2, c_3]$. Back-substitution can now be performed to determine the remaining coefficients

$$n_1 = -\boldsymbol{v}_c\ {}_1\boldsymbol{p}_0 = -\overline{{}_1x^2 + {}_1y^2}\, c_1 - {}_1\overline{x}\, c_2 - {}_1\overline{y}\, c_3 \tag{55}$$

$$n_2 = -\boldsymbol{v}_c\ {}_2\boldsymbol{p}_0 = -\overline{{}_2x^2 + {}_2y^2}\, c_1 - {}_2\overline{x}\, c_2 - {}_2\overline{y}\, c_3. \tag{56}$$

The coordinates for the two circles are:

$$\boldsymbol{C}_1 = \left[\boldsymbol{v}_c \quad n_1\right] = \left[c_1 \quad c_2 \quad c_3 \quad n_1\right] \quad \text{and} \tag{57}$$

$$\boldsymbol{C}_2 = \left[\boldsymbol{v}_c \quad n_2\right] = \left[c_1 \quad c_2 \quad c_3 \quad n_2\right]. \tag{58}$$

6.3. Coupled conics

The last example presented here is the case of two coupled conics: which, are coupled in their quadratic portion, i.e.

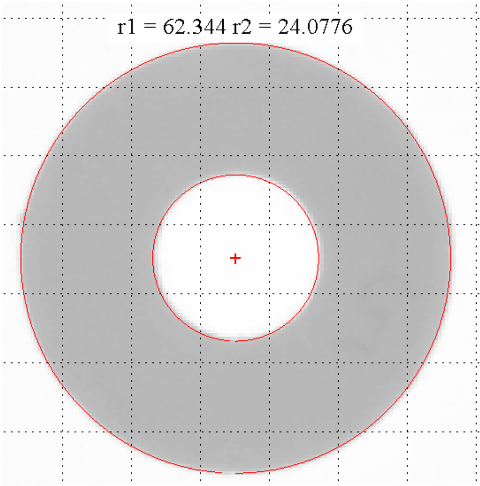

Figure 3: Fitting two concentric circles, in this application the volume of a ruffian is determined automatically via digital image processing.

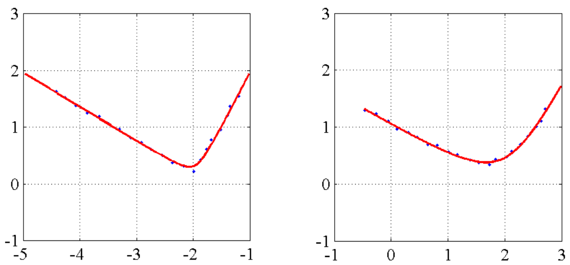

Figure 4: Two conics coupled so that they have the same quadratic components

they are forced to have the same orientation, however, their centre points and radius are independent. The common design portion is,

$$[x^2 \quad xy \quad y^2] \tag{59}$$

and the corresponding coefficients a, b and c are subject to the constraint $b^2 - 4ac = 1$ (this constraint forces the conic fit to be a hyperbola [6]), Figure (4) shows two hyperbolæ with asymptotes of identical direction.

7. Conclusions

A new method of fitting coupled geometric objects has been presented. The analysis shows that any implicit equations with common coefficients or with constraints on the correlations of their coefficients can be fitted using this technique. It is a generalization of quadratic constrained total least squares to coupled fitting.

The method has been explicitly demonstrated for parallel lines, concentric circles and coupled conics.

References

[1] Andrew Fitzgibbon, Maurizio Pilu, and Robert Fischer. Direct least square fitting of ellipses. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 21(5):476–480, 1999.

[2] Walter Gander, Gene Golub, and Rolf Strebel. Fitting of circles and ellipses least squares solution. Technical Report doc/tech-reports/1994/217.ps, ETH Zurich, Departement Informatik, June 1994 1994.

[3] Jun-Sik Kim, Ho-Won Kim, and In So Kweon. A camera calibration method using concentric circles of vision applications. In *5th Asian Conference on Computer Vision*, Melbourne, Australia, 2002.

[4] Siri Diyupathi Dampegama. Determination of original size of the abhayagiriya stupa. In *21st. Asian Conference on Remote Sensing*, Taipei, 2000.

[5] Fred Bookstein. Fitting conic sections to scattered data. *Computer Graphics and Image Processing*, 9:56–71, 1987.

[6] Paul O'Leary and Paul Zsombor-murray. Direct and specific least-square fitting of hyperbolae and ellipses. *Journal of Electronic Imaging*, In press, 2004.

[7] G.H. Golub, A. Hoffman, and G.W. Stewart. A generalization of the eckart-young-mirsky matrix approximation theorem. *Linear Algebra and its Applications*, 88:317–327, 1987.

[8] Walter Gander. Least squares with a quadratic constraint. *Numer. Math*, 36:291–307, 1981.

[9] Radim Halir and Jan Flusser. Numerically stable direct least squares fitting of ellipses. Technical Report citeseer.nj.nec.com/350661.html, Department of Software Engineering, Charles University, Czech Republic, 2000.