Image Reconstruction using Pollaczek Moments

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Abstract— **This paper proposes a new set of orthogonal moments based on continuous Pollaczek polynomials. In order to ensure numerical stability, Pollaczek polynomials are weighted using a weight function and a squared norm term associated with these polynomials. Weighted polynomials thus obtained are used to define continuous Pollaczek moments. In order to apply these moments for digital image reconstruction problem, a discrete representation of the continuous moments is considered for both binary as well as multilevel images. Further, it is observed that Legendre moments are a special case of Pollaczek moments. Hence the results are compared with Legendre moments only. It is also noted from the simulation results (peak signal to noise ratio, reconstruction error and universal image quality index) that Pollaczek moments are better than Legendre moments as par as image reconstruction problem is considered.**

Index Terms — ***Pollaczek polynomials, weighted Pollaczek polynomials, Pollaczek moments.***

I. INTRODUCTION

During the last few years, many moments such as Geometric[1], Legendre[2], Zernike[2], Tchebichef[3], Krawtchouk[4] have been used as shape descriptors of an image in a number of applications such as pattern recognition [1], image reconstruction [5], [6], shape identification[7], object classification[8], template matching[9], edge detection[10], pose estimation[11], robot vision[12], image watermarking[15],[16],[17] etc. Moments such as Geometric, Legendre, and Zernike fall under the class of continuous where as Tchebichef and Krawtchouk moments fall under discrete category. Hu [1] introduced Geometric moments, which are defined based on monomials. These moments are not orthogonal. Hence it is difficult to reconstruct an image from these moments. In order to solve the above problem, Teague [2] reported image reconstruction results using Legendre and Zernike moments which are based on continuous Legendre and Zernike polynomials. Mukundan et al. [3] proposed Tchebichef moments based on discrete Tchebichef polynomials and applied them for image reconstruction problem. Recently, Pew-Thian Yap et al. [4] proposed another moments known as Krawtchouk moments which are based on discrete Krawtchouk polynomials.

In this paper, a new set of orthogonal moments based on continuous weighted Pollaczek polynomials is proposed. The proposed moments are applied for image reconstruction problem. The obtained results are compared with Legendre moments since Legendre moments also belong to continuous moments class.

This paper is organized into four sections. Section II presents the details about Pollaczek polynomials, weighted Pollaczek polynomials and the proposed Pollaczek moments. Simulation results are reported in section III. Finally, the last section presents the conclusions about the work.

II. POLLACZEK MOMENTS

Pollaczek moments are defined using weighted Pollaczek polynomials, hence a detailed overview on weighted Pollaczek polynomials and Pollaczek moments is presented in this section.

Pollaczek polynomials [13] of order 'n' and parameters a, b are defined as

$$P_n(x;a,b) = e^{in\theta}\,{}_2F_1(-n,1/2+ih(\theta);1;1-e^{-2i\theta}) \quad (1)$$

where ${}_2F_1(.)$ is the hypergeometric function given by

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k z^k}{(\gamma)_k\, k!} \quad (2)$$

$$x = \cos\theta\ ,\ h(\theta) = \frac{a\cos\theta + b}{2\sin\theta}$$

and $(\alpha)_k$ is the Pochhammer symbol given by

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)....(\alpha+k-1) \quad (3)$$

These polynomials are orthogonal over the interval [-1,1] with respect to the weight function

$$w(\cos\theta;a,b) = e^{(2\theta-\pi)h(\theta)}\{\cosh[\pi h(\theta)]\}^{-1} \quad (4)$$

The orthogonality condition is given by

$$\int_{-1}^{1} P_p(x;a,b)P_q(x;a,b)w(x;a,b)dx = \rho(n,a)\,\delta_{pq} \quad (5)$$

where $\rho(n,a)$ is the squared norm that is given by

$$\rho(n,a) = [n + \frac{(a+1)}{2}]^{-1} \quad (6)$$

and δ_{pq} is the Kronecker function defined by

$$\delta_{pq} = \begin{cases} 1 & for\ p = q \\ 0 & for\ p \neq q \end{cases} \quad (7)$$

Recursive formula for obtaining Pollaczek polynomials is given by

$$nP_n(x;a,b) = [(2n-1+2a)x+2b]P_{n-1}(x;a,b) - (n-1)P_{n-2}(x;a,b) \quad (8)$$

for $n = 2,3,......$ with

$P_0(x;a,b) = 1\ \ and\ \ P_1(x;a,b) = (2a+1)x+2b$ as initial values. The set of polynomials presented above are not suitable for defining moments since their values grow as the order of the polynomial increases. Due to this, numerical fluctuations occur when these polynomials are used for defining moments. Hence, these polynomials are weighted in order to avoid the above numerical instability problem. The polynomials after weighting are called as weighted polynomials, which are defined as

$$\bar{P}_n(x;a,b) = P_n(x;a,b)\sqrt{\frac{w(x;a,b)}{\rho(n,a)}} \quad (9)$$

A plot of these weighted Pollaczek polynomials up to order 4 for parameters $a = -0.015\ and\ b = 0$ are shown in Fig.1. It is observed from the simulation results that better reconstruction of the image is obtained for these parameter values, Hence, the graph is plotted for these parameter values only.

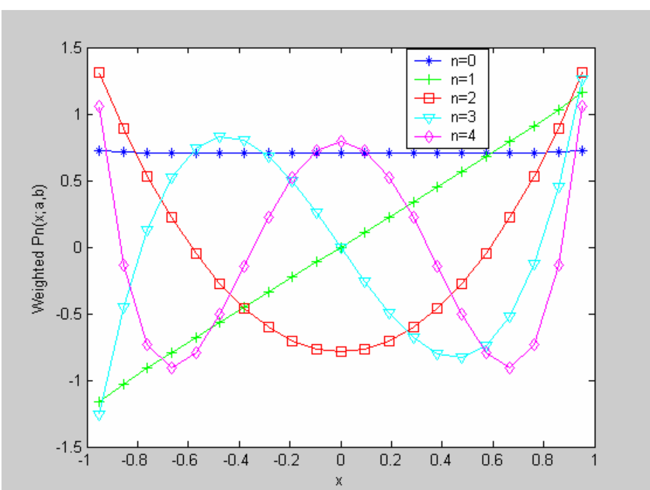

Fig.1. Plot of weighted Pollaczek polynomials for a = -0.015 and b = 0 up to order 4

The weighted Pollaczek polynomials satisfies the orthogonality condition given by

$$\int_{-1}^{1} \bar{P}_p(x;a,b)\bar{P}_q(x;a,b)dx = \delta_{pq} \quad (10)$$

The weighted Pollaczek polynomials presented above are used to define Pollaczek moments. Pollaczek moments of order (p,q) for an image $f(x,y)$ are defined as

$$S_{pq} = \int_{-1}^{+1}\int_{-1}^{+1} \bar{P}_p(x;a,b)\bar{P}_q(y;a,b) f(x,y)\,dx\,dy \quad (11)$$

where $p,q = 0,1,2,.....\infty$

We assume that the image function $f(x,y)$ is defined over the square [-1,1] x [-1,1]

In order to use the above continuous Pollaczek moments for digital images, a discrete form representation [6] given below is adapted.

$$S_{pq} = \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} \bar{P}_p(x_i;a,b)\bar{P}_q(y_j;a,b) f(x_i,y_j)\Delta x\,\Delta y \quad (12)$$

where $\Delta x = x_i - x_{i-1}$, $\Delta y = y_j - y_{j-1}$ are sampling intervals in the 'x' and 'y' directions respectively, (x_i, y_j) is the center of (i,j) pixel and N x M corresponds to the size of the image. Using (4), (6), (8) and (9), we can easily show that Legendre polynomials(moments) are special case of Pollaczek polynomials(moments) at the parameter values $a = b = 0$.

Using the orthogonality condition of $\bar{P}_n(x;a,b)$ given in eq.(10), the image function $f(x,y)$ can be expressed as

$$f(x_i,y_j) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} S_{pq}\bar{P}_p(x_i;a,b)\bar{P}_q(y_j;a,b) \quad (13)$$

If the Pollaczek moments are computed for a finite order (i.e) $(N_{\max}, N_{\max})$, then the image function $f(x,y)$ given in eq.(13) can be approximated [6] as

$$f_R(x,y) = \sum_{p=0}^{N_{\max}}\sum_{q=0}^{p} S_{p-q,q}\bar{P}_{p-q}(x;a,b)\bar{P}_q(y;a,b) \quad (14)$$

The above expression corresponds to image reconstruction when the order $(N_{\max}, N_{\max})$ is given. The sequence of steps required for image reconstruction problem using the above equations are given below.

1. Input:
   Image
   $f(x,y),\ \ 0 \le x \le N-1\ \&\ 0 \le y \le M-1$
   $a = -0.015\ and\ b = 0$
2. Computation of weighted Pollaczek polynomials:
   $for\ p = 0 : N_{\max}$
   $for\ \ x = 0 : N-1$
   $compute\ \bar{P}_p(x;a,b)\ \ \ using\ \ (9)$
   end
   end

$$
\begin{aligned}
& for\, q = 0 : N_{\max} \\
& for\ y = 0 : M-1 \\
& compute\, \bar{P}_q(y; a, b) \quad using\ (9) \\
& end \\
& end
\end{aligned}
$$

2. Compute pollaczek moments:

$$
\begin{aligned}
& for\, p = 0 : N_{\max} \\
& for\, q = 0 : N_{\max} \\
& compute\, S_{pq} \quad using\ (12) \\
& end \\
& end
\end{aligned}
$$

3. Compute image reconstruction:

$$
\begin{aligned}
& for\, x = 0 : N-1 \\
& for\ y = 0 : M-1 \\
& compute\ f_R(x, y) \quad using \quad (14) \\
& end \\
& end
\end{aligned}
$$

4. Output:

Reconstructed image $f_R(x, y)$.

III. SIMULATION RESULTS

In order to verify the applicability of the proposed moments for image reconstruction problem, we have selected two images namely binary image Letter E (40 x 40 pixels) and multi level Lena image of size 256 x 256. The reconstructed images using various orders of the proposed moments with parameters $a = -0.015$ and $b = 0$ are shown in Figs.2 and 3. The obtained results are compared with reconstructed images obtained with the same orders of computed Legendre moments. These reconstructed images are also shown in Figs.2 and 3. For better appearance of the reconstructed multilevel image, histogram equalization is applied on the reconstructed image. In case of binary image, after reconstructing the image using (14) , it is passed through a threshold operator given by $f_R(x,y) = \begin{cases} 1 & if \quad f_R(x,y) \geq 0.5 \\ 0 & if \quad f_R(x,y) \prec 0.5 \end{cases}$ to convert the image pixel values to either zero or one. Further, in order to assess the quality of reconstructed image, we computed the Reconstruction Error (RE), Peak Signal to Noise Ratio (PSNR) as well as Universal Image Quality Index (UIQI) [18] between the original image and reconstructed image and the results are given in Tables I and II. The following formula [14] is used for computing the reconstruction error.

For binary images

$$
\text{RE} = \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} \left| f(x,y) - f_R(x,y) \right| \qquad (15)
$$

and for gray level images

$$
\text{RE} = \sqrt{\sum_{x=0}^{N-1} \sum_{y=0}^{M-1} [f(x,y) - f_R(x,y)]^2} \qquad (16)
$$

We use the following formula for computing PSNR.

$$
PSNR = 10 \log_{10} (f(x,y)^2_{\max} / (\frac{1}{MN} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} (f(x,y) - f_R(x,y))^2)) \qquad (17)
$$

UIQI is computed using the MATLAB code available online at http://anchovy.ece.utexas.edu/~zwang/research/quality_index/demo.html. It is noted from the reconstruction error results that this error is less using the proposed moments compared with Legendre moments. The computed PSNR as well as UIQI with the proposed Pollaczek moments are better than the Legendre moments. Hence, we conclude that proposed moments are performing better in terms of reconstructed image quality as compared with Legendre moments.

IV. CONCLUSIONS

A new set of continuous orthogonal moments based on weighted Pollaczek polynomials is proposed in this paper. A discrete form representation of the proposed moments is used for image reconstruction problem. The results are reported and observed that the reconstructed image is better than the Legendre moments. Even though the standard procedure is adopted for calculation of Pollaczek moments, it is also possible to use the symmetry property of weighted Pollaczek polynomials and matrix form representation of (12) and (14) for reducing the number of computations. Discretization error is visible on the reconstructed Lena image only as the order of the moments is increased. This problem can be solved using either the selection of small size image (Letter 'E' in this paper) or the selection of discrete orthogonal moments like Tchebichef and Krawtchouk. In order to test the quality of reconstructed image, we have computed the reconstruction error, peak signal to noise ratio and universal image quality index and found that they are better for reconstructed images using Pollaczek moments.

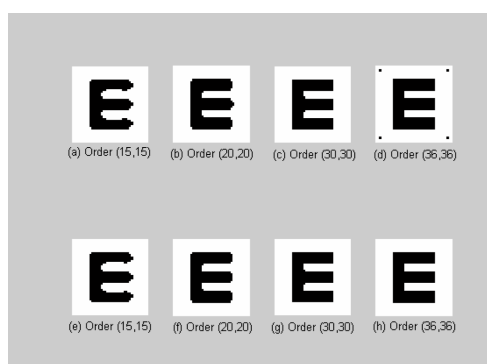

Fig.2. Reconstructed images of Letter 'E' with different orders (a)-(d) Legendre moments, (e)-(h) Pollaczek moments.

Fig.3. Reconstructed images of Lena with different orders (a)-(d) Legendre moments, (e)-(h) Pollaczek moments.

Table I. Computed RE, PSNR and UIQI for Letter 'E' using up to various orders of Legendre and Pollaczek moments.

Order	Legendre moments			Pollaczek moments		
	RE	PSNR	UIQI	RE	PSNR	UIQI
(5,5)	218	8.66	0.38	238	8.28	0.34
(10,10)	142	10.52	0.60	148	10.34	0.59
(15,15)	36	16.48	0.88	38	16.24	0.87
(20,20)	12	21.25	0.95	10	22.04	0.96
(25,25)	4	26.02	0.99	4	26.02	0.99
(30,30)	2	29.03	0.99	2	29.03	0.99
(36,36)	4	26.02	0.99	0	Inf	1

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Table II. Computed RE, PSNR and UIQI for Lena image using up to various orders of Legendre and Pollaczek moments.

Order	Legendre moments			Pollaczek moments		
	RE	PSNR	UIQI	RE	PSNR	UIQI
(20,20)	6.9188 e+003	18.89	0.17	7.1443 e+003	18.61	0.17
(40,40)	5.2932 +003	21.22	0.34	5.3223 e+003	21.17	0.34
(60,60)	5.1486 e+003	21.46	0.45	4.4529 e+003	22.72	0.46
(80,80)	6.0415 +003	20.07	0.47	5.1987 e+003	21.38	0.49
(100, 100)	8.4332 e+003	17.18	0.51	6.1508 e+003	19.92	0.52
(120, 120)	14.294 e+003	12.59	0.58	10.085 e+003	15.62	0.63