Calculus of Variations and Computer Vision

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January 8, 2002

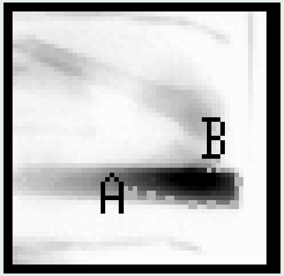
The Top Ten Algorithms of the Century

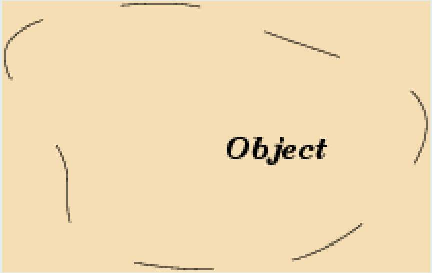
- Metropolis Algorithm for Monte Carlo
- Simplex Method for Linear Programming
- Krylov Subspace Iteration Methods
- The Decompositional Approach to Matrix Computations
- QR Algorithm for Computing Eigenvalues
- The Fortran Optimizing Compiler
- Quicksort Algorithm for Sorting
- Fast Fourier Transform
- Integer Relation Detection
- Fast Multipole Method

Overview

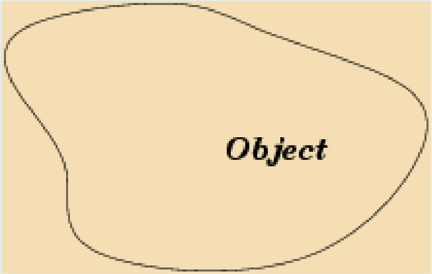
1. Sample problems from computer vision (fade out computer vision gurus)
   - The problem of shape
   - The problem of obtaining 3D information
   - The problem of motion
2. The beauty of mathematics (fade out math gurus)
   - The method of the calculus of variations
   - Allied mathematical methods: Singular Value Decomposition, and the Method of Lagrange Multipliers
3. Sample solution to problems posed (or how computer scientists teach/cheat)
4. Concluding remarks

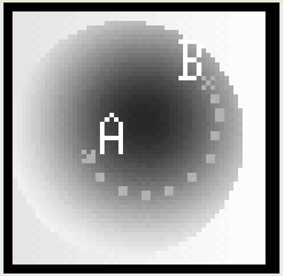
Shape





Idealized image

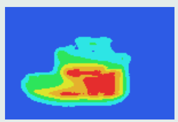


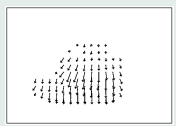


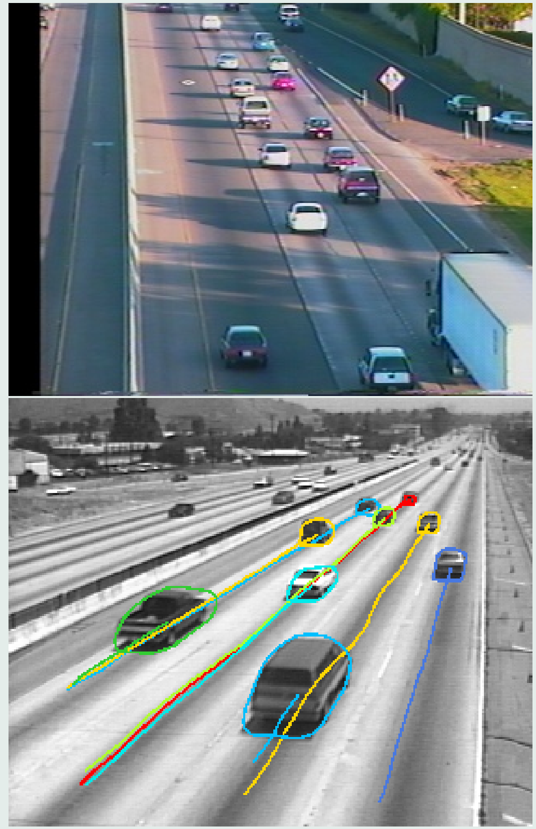
- When placed out of context, images can be quite involved
- What you get is not what you see (WYGINWYS)
- What you want to compute: A contour (useful for obtaining properties such as size)
- Key mathematical concept: parametric curve $x(s), y(s)$

Motion

- Optical flow is an intensity-based approximation to image motion from sequential time-ordered images
- Key mathematical concept: Two functions: $u(x, y)$ and $v(x, y)$
- What does it look like?
- How do we compute optical flow?

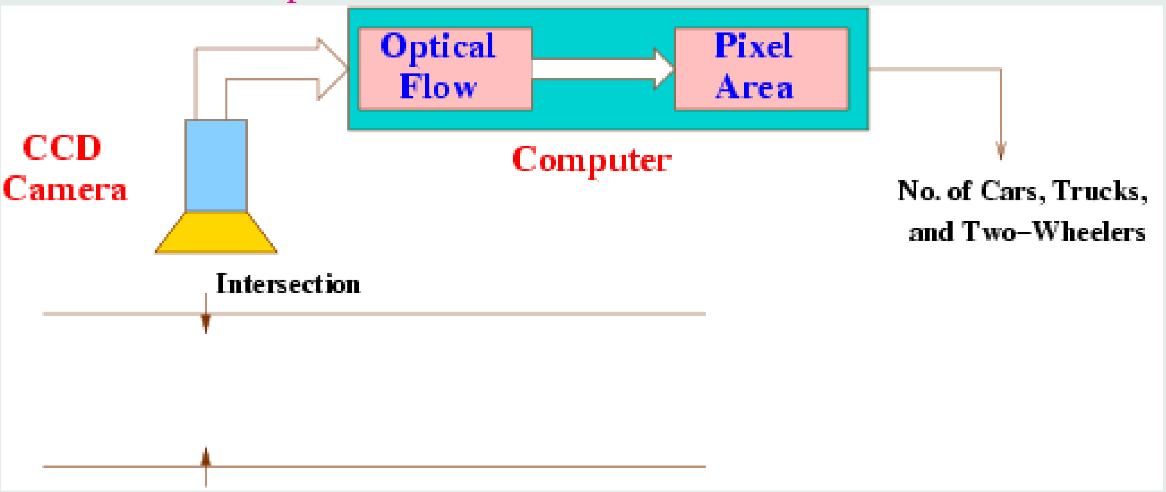




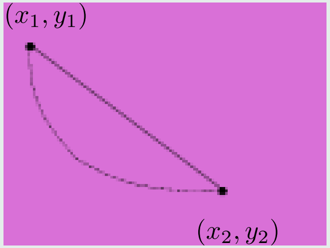


Application of Optical Flow

A short video segment
Specifications to help.



Two Problems



- Determine the equation of the curve joining two points (x_1, y_1) and (x_2, y_2) on the plane such that the length of the curve joining them is minimum
- Goal: Prove that your answer is correct
- Similar problem: Determine the equation of the support so that a ball placed at (x_1, y_1) reaches (due to gravity) (x_2, y_2) in the least possible time

Functions and Functionals

- Differential segment length is $(dx^2 + dy^2)^{1/2}$ where dx and dy are infinitesimal lengths in the x and y directions along the curve.
- Therefore the length of the curve joining the two end points (x_1, y_1) and (x_2, y_2) is

$$\begin{aligned} J &= \int_{x_1}^{x_2} \sqrt{(1 + y')}\, dx \\ &= \int_{x_1}^{x_2} F(x, y, y')\, dx \end{aligned}$$

- For the first problem, what should the function y be so that J is minimized?
- Compare: Find x_0 such that $y = f(x)$ is minimum.
- J is a *functional*, a quantity that depends on functions rather than dependent variables.

Solution to the Shortest Length Problem

- The Euler-Lagrange equation is a necessary condition for minimising J

$$F_y - \frac{d}{dx} F_{y'} = 0 \tag{1}$$

- In our problem, $F = \sqrt{1 + y'}$. Applying Equation 1 we get

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'}}\right) = 0$$

- Solution: y' is a constant.
- The shortest distance curve is a straight line.

Detour: Why the Euler equation

- $J = \int_{x_1}^{x_2} F(x, f, f')\, dx$ with boundary conditions $f(x_1) = f_1$ and $f(x_2) = f_2$.
- Let $\eta(x)$ be a test function. Deform f by $\epsilon\eta(x)$. Then, $dJ/\,d\epsilon = 0$ at the "right" place.
- This is true for all test functions $\eta(x)$. The boundary conditions assert $\eta(x_1) = \eta(x_2) = 0$
- If $f(x)$ is replaced by $f(x) + \epsilon\eta(x)$ then $f'(x)$ will be replaced by $f'(x) + \epsilon\eta'(x)$.
- The integral then becomes

$$J = \int_{x_1}^{x_2} F(x, f + \epsilon\eta, f' + \epsilon\eta')\, dx$$

Detour: Why the Euler equation

- If F is differentiable, expand the integrand in a Taylor series
- Differentiate with respect to ϵ and set to zero
- Apply integration by parts to get

$$\int_{x_1}^{x_2} \eta'(x) F_{f'} dx = [\eta(x) F'_f]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d(F_f)}{dx} dx,$$

- The first term is zero due to the boundary conditions
- Hence

$$\int_{x_1}^{x_2} \eta(x)(F_f - \frac{dF'_f}{dx}) dx = 0$$

Bottom Line Euler Equations

- Various generalizations are possible
  - Higher order derivatives: $J = \int_{x_1}^{x_2} F(x, f, f', f'', \ldots)dx$
  - Integrand may depend on several functions instead of only one
  - Finding functions that have two independent variables

$$J = \int\int F(x, y, f, f_x, f_y)dxdy$$

- The bottom line

Function to optimize	The Euler-Lagrange equations
$\int F(x, u_x)dx$	$F_u - \frac{d}{dx}F_{u_x} = 0$
$\int F(x, u_x, u_{xx})dx$	$F_u - \frac{d}{dx}F_{u_x} - \frac{d^2}{dx^2}F_{u_{xx}} = 0$
$\int F(x, u_x, v_x)dx$	$F_u - \frac{d}{dx}F_{u_x} = 0$ $F_v - \frac{d}{dx}F_{v_x} = 0$
$\int\int F(x, y, u_x, u_y)dxdy$	$F_u - \frac{d}{dx}F_{u_x} - \frac{d}{dy}F_{u_y} = 0$

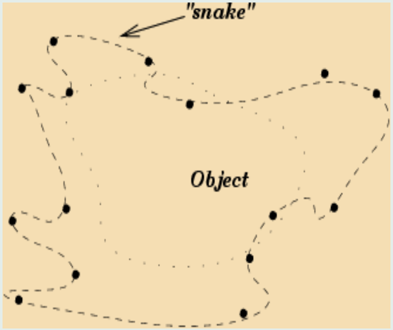
The Snake Formulation

- Start with any old curve (a "snake"). Mathematically deform it to be the desired shape (based on image data).
- Curve is $\mathbf{v}(s) = (x(s), y(s))$. Define energy of the curve to be

$$E(\mathbf{v}(s)) = \int_0^1 E_{snake}(\mathbf{v}(s))\, ds \qquad (2)$$

where $E_{snake} = E_{int} + E_{ext}$

- Find the curve that minimizes Equation 2



Remember, actual image is not so clean!

More on Formulation

- The energy functional is given by $E = \int_0^1 (E_{int}(\mathbf{v}(s)) + E_{img}(\mathbf{v}(s)) + E_{con}(\mathbf{v}(s)))\, ds$
- $E_{int} = (\alpha(s)|\mathbf{v}_s(s)|^2 + \beta(s)|\mathbf{v}_{ss}(s)|^2)/2$
- $E_{con} = -k(\mathbf{v}(s) - \bar{x})^2$
- $E_{img} = w_{line}E_{line}(\mathbf{v}(s)) + w_{edge}E_{edge}(\mathbf{v}(s)) + w_{term}E_{term}(\mathbf{v}(s))$
  - Line energy is $E_{line}(\mathbf{v}(s)) = \mathcal{I}(x, y)$
  - Edge energy is $E_{edge}(\mathbf{v}(s)) = -|\nabla \mathcal{I}(x, y)|^2$

Applying Euler-Lagrange to Snakes

We minimize

$$J_1 = \int (\alpha(s)x_s^2 + \beta(s)x_{ss}^2)/2 + E_x(s)ds \;\; = \int X(s,x,x',x'')ds$$

and

$$J_2 = \int (\alpha(s)y_s^2 + \beta(s)y_{ss}^2)/2 + E_y(s)ds \;\; = \int Y(s,y,y',y'')ds$$

For $X(s,x,x',x'')$ the Euler equation is

$$X_x - \frac{d}{ds}X_{x'} + \frac{d^2}{ds^2}X_{x''} = 0$$

More on Application

- $X_x = \frac{\partial E}{\partial s}$, $X_x{}' = \alpha(s)x'(s)$, and $X_{x''} = \beta(s)x''(s)$
  - $\frac{d}{ds}X_{x'} = \alpha' x' + \alpha x''$ and
  - $\frac{d}{ds}(X''(s)) = \beta' x'' + \beta x'''$ and
  - $\frac{d^2}{ds^2}(X''(s)) = \beta'' x'' + 2\beta' x''' + \beta x''''$
- For purpose of illustration use only two terms: $\frac{\partial E}{\partial s} - \alpha' x' - \alpha x'' = 0$
- Numerically approximating we have
  - $\frac{\partial E}{\partial x} \simeq E_{i+1} - E_i$ at location i
  - $\alpha'(s) \simeq \alpha_{i+1} - \alpha_i$
  - $x'(s) \simeq x_{i+1} - x_i$
  - $\alpha' x' = (\alpha_{i+1} - \alpha_i)(x_{i+1} - x_i) = (\alpha_{i+1} - \alpha_i)x_{i+1} - (\alpha_{i+1} - \alpha_i)x_i$
  - $x''(s) \simeq x_{i+1} - 2x_i + x_{i-1}$

Reducing Snakes To A Matrix Equation

$$
\begin{pmatrix} -1 & 1 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{pmatrix} -
$$

$$
\begin{pmatrix} -(\alpha_2 - \alpha_1) & (\alpha_2 - \alpha_1) & 0 & \dots & 0 \\ 0 & -(\alpha_3 - \alpha_2) & (\alpha_3 - \alpha_2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -(\alpha_{n+1} - \alpha_n) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
$$

$$
- \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}^T \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0
$$

This can be written as $Ax = B$, and solved using SVD.

The Method of Lagrange Multipliers

Minimize a function $f(x, y)$ subject to $g(x, y) = 0$.

- For example, find a point on the line $x \sin\theta - y\cos\theta + p = 0$ closest to origin
- That is, minimize $x^2 + y^2$ subject to the given constraint.

In MOLM, we generate a new function $E = f + \lambda g$ and set the partial derivatives with respect to x, y and λ to zero.

In the example

- $2x + \lambda\sin\theta = 0$
- $2y + \lambda\cos\theta = 0$
- $x\sin\theta - y\cos\theta + p = 0$

The solution is the obvious one

- $x = -p\sin\theta$
- $y = +p\cos\theta$

The method can be generalized to larger number of constraints.

Formulation for Motion

- Let $E(x, y, t)$ be the intensity value at point (x, y) in the image
- Due to motion, let this point correspond to a point $(x + \delta x, y + \delta y)$ at time instance $t + \delta t$
- Key assumption: $E(x, y, t) = E(x + \delta x, y + \delta y, t + \delta t)$
- Two parts to determine $u(x, y)$ and $v(x, y)$
  - Use Taylor's theorem to get $E_x u + E_y v + E_t = 0$
  - Introduce a smoothness constraint: $f(u, v, u_x, v_x, u_y, v_y) = {u_x}^2 + {u_y}^2 + {v_x}^2 + {v_y}^2$
- Using MOLM we have a functional that needs to be minimized

$$\int\!\!\int f(u, v, u_x, v_x, u_y, v_y) + \lambda g(u, v, t)^2 \, dx dy$$

$$\int\!\!\int {u_x}^2 + {u_y}^2 + {v_x}^2 + {v_y}^2 + \lambda(E_x u + E_y v + E_t)^2 \, dx dy$$

Applying Euler-Lagrange for Optical Flow

- We have

  $F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0 \qquad F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y} = 0$

- Applying these we get

  $F_u = 2\lambda E_x (E_x u + E_y v + E_t)$

  $F_{u_x} = 2u_x, \frac{\partial}{\partial x} F_{u_x} = 2u_{xx} \qquad F_{u_y} = 2u_y, \frac{\partial}{\partial y} F_{u_y} = 2u_{yy}$

- And $F_v = 2\lambda E_x (E_x v + E_y v + E_t)$

  $F_{v_x} = 2v_x, \frac{\partial}{\partial x} F_{v_x} = 2v_{xx} \qquad F_{v_y} = 2v_y, \frac{\partial}{\partial y} F_{v_y} = 2v_{yy}$

- This implies a coupled system of equations

$$\begin{aligned} \Delta^2 u &= \lambda (E_x u + E_y v + E_t) E_x \\ \Delta^2 v &= \lambda (E_x u + E_y v + E_t) E_y \end{aligned}$$

Solution using Euler Lagrange

Discretizing we have

- $u_{k,l}$, $u_{k+1,l}$ $u_{k,l+1}$ $u_{k-1,l}$ $u_{k,l-1}$: u component of the velocity at points (k,l), $(k+1,l)$, $(k,l+1)$, $(k-1,l)$ and $(k,l-1)$ respectively.
- $v_{k,l}$, $v_{k+1,l}$ $v_{k,l+1}$ $v_{k-1,l}$ $v_{k,l-1}$: v component of the velocity at points (k,l), $(k+1,l)$, $(k,l+1)$, $(k-1,l)$ and $(k,l-1)$ respectively.

$$u_{k,l} = \frac{\overline{u_{k,l}} - \lambda E_x E_t}{1 + \tilde{\lambda} {E_x}^2} - \frac{v_{k,l} E_x E_t}{1 + \tilde{\lambda} {E_x}^2}$$

$$v_{k,l} = \frac{\overline{v_{k,l}} - \lambda E_x E_t}{1 + \tilde{\lambda} {E_x}^2} - \frac{u_{k,l} E_x E_t}{1 + \tilde{\lambda} {E_x}^2}$$

Uses of Euler Lagrange

Problem	*Regularization principle*
Contours	$\int E_{snake}(\mathbf{v}(s))ds$
Area based Optical flow	$\int[(u_x{}^2 + u_y{}^2 + v_x{}^2 + v_y{}^2) + \lambda(E_x u + E_y v + i_t)^2)]dxdy$
Edge detection	$\int[(Sf - i)^2 + \lambda\,(f_{xx})^2]dx$
Contour based Optical flow	$\int[(V \cdot N - V^N)^2 + \lambda(\frac{\delta V}{\delta x})^2]$
Surface reconstruction	$\int[(S \cdot f - d^2 + \lambda(f_{xx} + 2f_{xy}{}^2 + f_{yy}{}^2)]dxdy$
Spatiotemporal approximation	$\int[(S \cdot f - i)^2 + \lambda(\nabla f \cdot V + ft)^2]dxdydt$
Colour	$\|I^y - Ax\|^2 + \lambda\|Pz\|^2$
Shape from shading	$\int[(E - R(f,g))^2 + \lambda(f_x{}^2 + f_y{}^2 + g_x{}^2 + g_y{}^2)]dxdy$
Stereo	$\int\{[\nabla^2 G * (L(x,y) - R(x + d(x,y),y))]^2 + \lambda(\nabla d)^2\}dxdy$

Concluding Remarks

- Two interesting problems have been described
- Defined and derived the Euler Lagrange equations
- Used mathematics to solve the computer vision problem