Theoretical Analysis of Policy Iteration

Shivaram Kalyanakrishnan

Department of Computer Science and Engineering
Indian Institute of Technology Bombay
shivaram@cse.iitb.ac.in

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Overview

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MDP Planning

- Markov Decision Problem: general abstraction of sequential decision making.

- An MDP comprises a tuple (S, A, R, T, γ), where
  S is a set of states (with $|S| = n$),
  A is a set of actions (with $|A| = k$),
  $R(s, a)$ is a bounded real number, $\forall s \in S, \forall a \in A$, and
  $T(s, a)$ is a probability distribution over S, $\forall s \in S, \forall a \in A$.

- A policy $\pi : S \to A$ specifies an action from each state, and yields trajectory

$$s^0, a^0 = \pi(s^0), r^0, s^1, a^1 = \pi(s^1), r^1, s^2, \ldots.$$

- The value of a policy π from state s is:

$$V^\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r^t \mid s^0 = s, a^t = \pi(s^t), t = 0, 1, 2, \ldots\right], \text{where}$$

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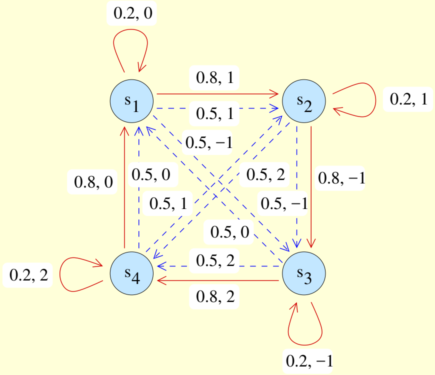
Illustration: MDPs as State Transition Diagrams



Notation: "transition probability, reward" marked on each arrow

States: s_1, s_2, s_3, and s_4.

Actions: Red (solid lines) and blue (dotted lines).

Transitions: Red action leads to same state with 20% chance, to next-clockwise state with 80% chance. Blue action leads to next-clockwise state or 2-removed-clockwise state with equal (50%) probability.

Rewards: $R(*, *, s_1) = 0$, $R(*, *, s_2) = 1$, $R(*, *, s_3) = -1$, $R(*, *, s_4) = 2$.

Discount factor: $\gamma = 0.9$.

Bellman's Equations

- Recall: $V^{\pi}(s) = \mathbb{E}[r^0 + \gamma r^1 + \gamma^2 r^2 + \ldots | s^0 = s, a^t = \pi(s^t)$ for t = 0, 1, $\ldots]$.

  Bellman's Equations: $\forall s \in S$,

  $$\boxed{V^{\pi}(s) = R(s, \pi(s)) + \gamma \sum_{s' \in S} T(s, \pi(s), s') V^{\pi}(s').}$$

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- Define: $\forall s \in S, \forall a \in A$,

$$\boxed{Q^\pi(s, a) = R(s, a) + \gamma \sum_{s' \in S} T(s, a, s') V^\pi(s').}$$

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> Given S, A, T, R, γ, and a fixed policy π, we can solve Bellman's Equations to obtain V^π and Q^π. This step is called Policy Evaluation.

Bellman's Optimality Equations

- The Optimal Value Function $V^\star \stackrel{\text{def}}{=} V^{\pi^\star}$ is unique solution of: $\forall s \in S$,

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- Given $Q^\star$, we may obtain $\pi^\star$ by setting, $\forall s \in S$:

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Given $\pi^{\star}$, how can we obtain $V^{\star}$ and $Q^{\star}$? By policy evaluation (previous slide).

Solution Strategies

- Value Iteration

$V_0 \leftarrow$ Arbitrary, element-wise bounded, n-length vector. $t \leftarrow 0$.
Repeat:
For $s \in S$**:**
$V_{t+1}(s) \leftarrow \max_{a \in A} \left(R(s,a) + \gamma \sum_{s' \in S} T(s,a,s') V_t(s')\right)$.
$t \leftarrow t + 1$.
Until $V_t \approx V_{t-1}$ (up to machine precision).

Convergence to $V^\star$ guaranteed using a max-norm contraction argument.

- Linear Programming

$$\text{Minimise} \quad \sum_{s \in S} V(s)$$

$$\text{subject to} \quad V(s) \geq \left(R(s,a) + \gamma \sum_{s'} T(s,a,s') V(s')\right), \forall s \in S, \forall a \in A.$$

n variables, nk constraints (or *dual* with nk variables, n constraints).

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- Bounds for Linear Programming-type approaches to MDP planning:
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  poly$(n, k) \cdot \exp(O(\sqrt{n \log(n)}))$ (Expected) [MSW96].
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- Appeal of Policy Iteration:
  Theoretical: naturally yields strong bounds (also enjoys good weak bounds [P94]).
  Practical: very fast on MDPs encountered in typical applications.

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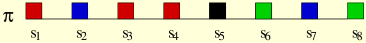

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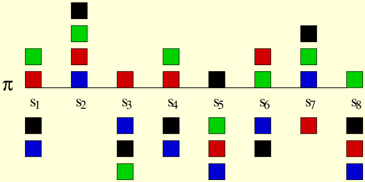

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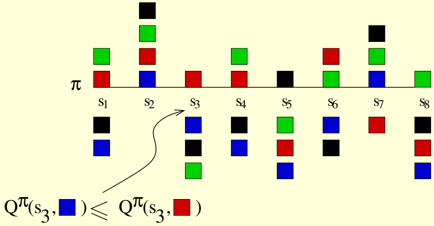

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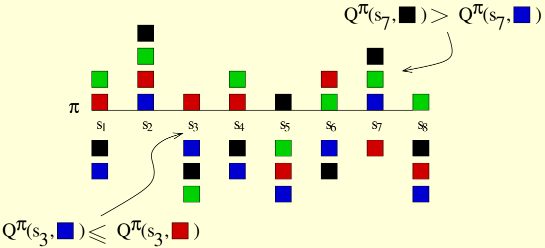

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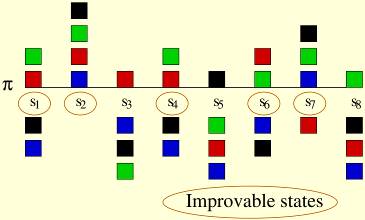

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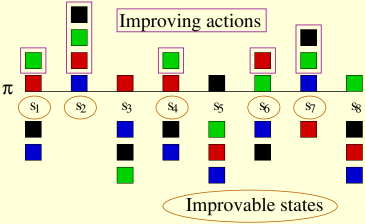

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  Pick one or more improvable states, and in them,
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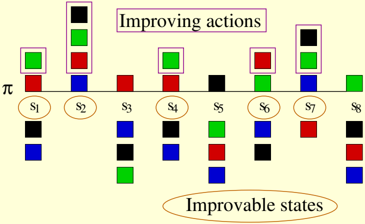
Let the resulting policy be π'.



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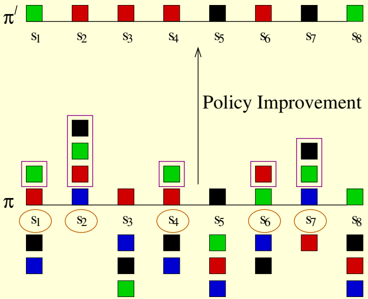

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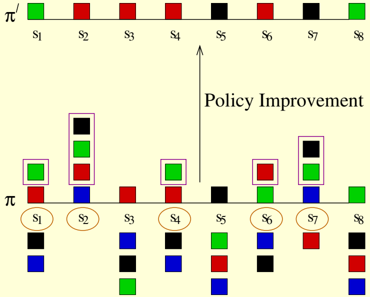

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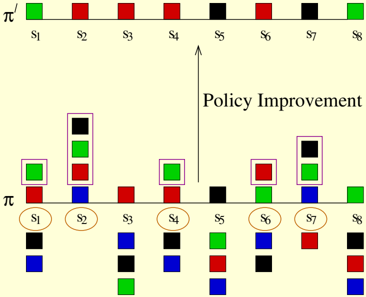

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$$\text{if } X \succeq Y, \text{ then } B^{\pi}(X) \succeq B^{\pi}(Y).$$

- **Fact 2**. For $\pi \in \Pi$ and $X : S \to \mathbb{R}$:

$$\lim_{l \to \infty} (B^{\pi})^{l}(X) = V^{\pi}.$$

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$\pi \leftarrow$ Arbitrary policy.
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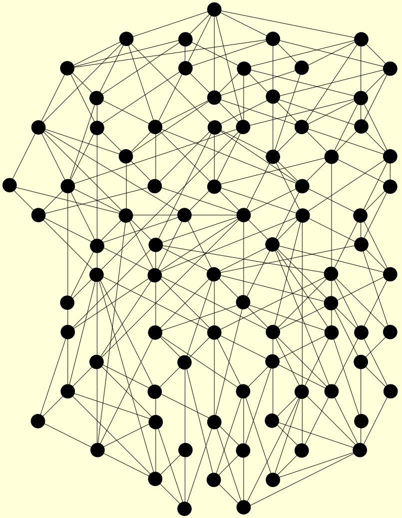

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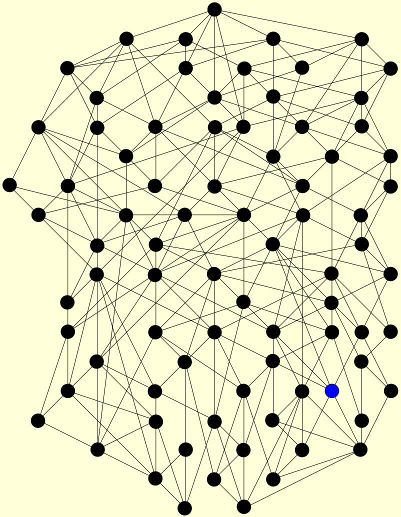

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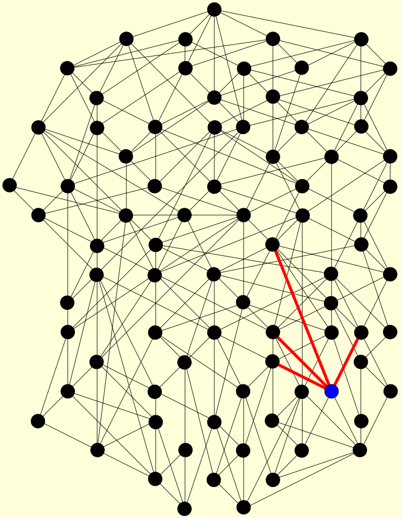

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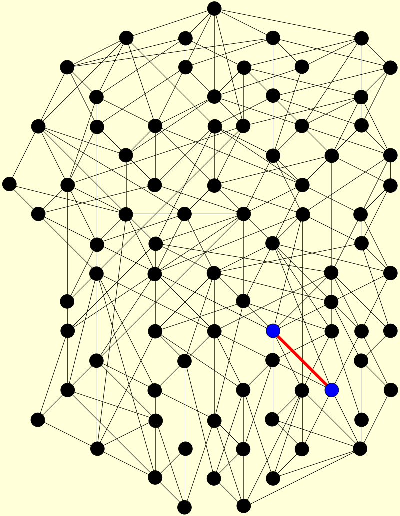

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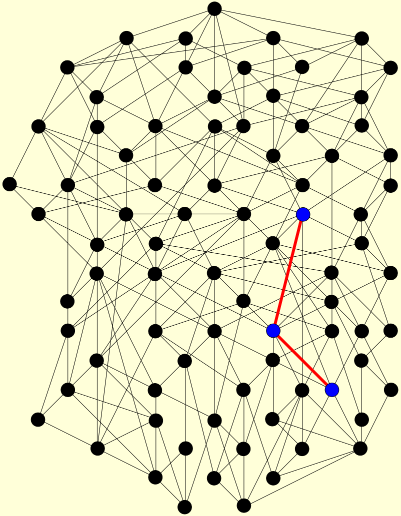

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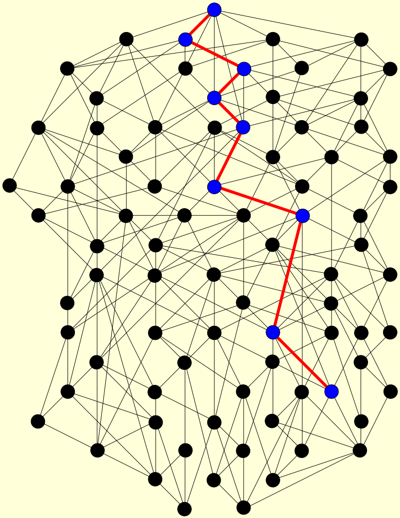

Switching Strategies and Bounds

Upper bounds on number of iterations

PI Variant	**Type**	$k = 2$	**General** k
Howard's PI [H60, MS99]	Deterministic	$O\left(\frac{2^n}{n}\right)$	$O\left(\frac{k^n}{n}\right)$
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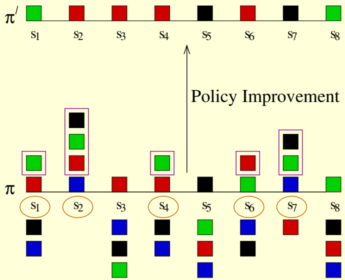
Overview

Basic Tool: Policy Improvement and Policy Deprovement

$$\boxed{\pi' \succ \pi.}$$



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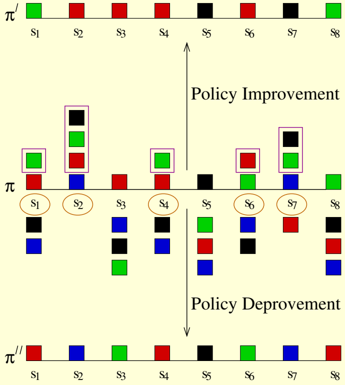

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Basic Tool: Property of Improvement sets in 2-action MDPs

Consider $\pi, \pi' \in \Pi$. If $V^{\pi} \neq V^{\pi'}$, then π and π' cannot have the same set of improvable states.

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$\boxed{1}$ $\boxed{0}$ $\boxed{1}$ $\boxed{1}$ 0 1 1 0 $\quad$ $\boxed{1}$ $\boxed{1}$ $\boxed{0}$ $\boxed{0}$ 1 1 0 1

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1 1 0 0 0 1 1 0

$\succ$

$\boxed{\mathbf{1}}$ $\boxed{\mathbf{0}}$ $\boxed{\mathbf{1}}$ $\boxed{\mathbf{1}}$ **0 1 1 0** $\boxed{\mathbf{1}}$ $\boxed{\mathbf{1}}$ $\boxed{\mathbf{0}}$ $\boxed{\mathbf{0}}$ **1 1 0 1**

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$$\succ$$

$$\boxed{\mathbf{1}} \quad \boxed{\mathbf{0}} \quad \boxed{\mathbf{1}} \quad \boxed{\mathbf{1}} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{0} \qquad \boxed{\mathbf{1}} \quad \boxed{\mathbf{1}} \quad \boxed{\mathbf{0}} \quad \boxed{\mathbf{0}} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{1}$$

$$\succeq$$

$$\mathbf{1} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{1}$$

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1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
 & & & \succ & & & & \\
\boxed{1} & \boxed{0} & \boxed{1} & \boxed{1} & 0 & 1 & 1 & 0 \\
 & & & \succeq & & & & \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1
\end{array}
\qquad
\begin{array}{cccccccc}
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
 & & & \succ & & & & \\
\boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & 1 & 1 & 0 & 1 \\
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\end{array}
\qquad
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1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
 & & & \succ & & & & \\
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1 & 1 & 0 & 0 & 0 & 1 & 1 & 0
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Contradiction!

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1 1 0 0 0 1 1 0   **1 0 1 1 1 1 0 1**

$\succ$ $\succ$

$\boxed{1}$ $\boxed{0}$ $\boxed{1}$ $\boxed{1}$ **0 1 1 0** $\boxed{1}$ $\boxed{1}$ $\boxed{0}$ $\boxed{0}$ **1 1 0 1**

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1 0 1 1 1 1 0 1   **1 1 0 0 0 1 1 0**

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\boxed{1}\ \boxed{0}\ \boxed{1}\ \boxed{1}\ 0\ 1\ 1\ 0 \qquad \boxed{1}\ \boxed{0}\ \boxed{1}\ \boxed{1}\ 1\ 1\ 0\ 1
$$

Equal value functions.

Howard's Policy Iteration (2-action MDPs)

Switch actions in every improvable state.

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π **0 0 0 0 0 0 0** $\boxed{\mathbf{0}}$ $\boxed{\mathbf{0}}$ $\boxed{\mathbf{0}}$ $\boxed{\mathbf{0}}$ $\boxed{\mathbf{0}}$

Howard's Policy Iteration (2-action MDPs)

Switch actions in every improvable state.

π'	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**1**
π	**0**	**0**	**0**	**0**	**0**	**0**	**0**	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$

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Possible?

π'	**0**	$\boxed{\mathbf{0}}$	**0**	**0**	$\boxed{\mathbf{0}}$	**0**	**0**	$\boxed{\mathbf{1}}$	$\boxed{\mathbf{1}}$	$\boxed{\mathbf{1}}$	$\boxed{\mathbf{1}}$	$\boxed{\mathbf{1}}$
π	**0**	**0**	**0**	**0**	**0**	**0**	**0**	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$

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π'	**0**	$\boxed{0}$	**0**	**0**	$\boxed{0}$	**0**	**0**	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	**1**
π	**0**	**0**	**0**	**0**	**0**	**0**	**0**	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$

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π'	0	$\boxed{0}$	0	0	$\boxed{0}$	0	0	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	1
π_1	0	0	0	0	0	0	0	1	1	1	1	0
π	0	0	0	0	0	0	0	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$

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π_1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**1**	**0**
π_2	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	**0**	**0**
π	**0**	**0**	**0**	**0**	**0**	**0**	**0**	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$

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π_1	0	0	0	0	0	0	0	1	1	1	1	0
π_2	0	0	0	0	0	0	0	1	1	1	0	0
π_3	0	0	0	0	0	0	0	1	1	0	0	0
π	0	0	0	0	0	0	0	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$

Howard's Policy Iteration (2-action MDPs)

Switch actions in every improvable state.

π'	0	$\boxed{0}$	0	0	$\boxed{0}$	0	0	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	1
π_1	0	0	0	0	0	0	0	1	1	1	1	0
π_2	0	0	0	0	0	0	0	1	1	1	0	0
π_3	0	0	0	0	0	0	0	1	1	0	0	0
π_4	0	0	0	0	0	0	0	1	0	0	0	0
π	0	0	0	0	0	0	0	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$

Howard's Policy Iteration (2-action MDPs)

Switch actions in every improvable state.

π'	0	$\boxed{0}$	0	0	$\boxed{0}$	0	0	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	1
π_1	0	0	0	0	0	0	0	1	1	1	1	0
π_2	0	0	0	0	0	0	0	1	1	1	0	0
π_3	0	0	0	0	0	0	0	1	1	0	0	0
π_4	0	0	0	0	0	0	0	1	0	0	0	0
π	0	0	0	0	0	0	0	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$

If π has m improvable states and $\pi \xrightarrow{\text{Howard's PI}} \pi'$, then there exist m policies π'' such that $\pi' \succeq \pi'' \succ \pi$.

Howard's Policy Iteration (2-action MDPs)

- Take $m^{\star} = \frac{n}{3}$.

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$$\leq \frac{2^n}{m^\star} = \frac{2^n}{n/3}.$$

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- Number of policies with fewer than $m^\star$ improvable states visited

$$\leq \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m^\star - 1}$$

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- Take $m^\star = \frac{n}{3}$.
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$$\leq \frac{2^n}{m^\star} = \frac{2^n}{n/3}.$$

- Number of policies with fewer than $m^\star$ improvable states visited

$$\leq \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m^\star - 1} \leq 3\frac{2^n}{n}.$$

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$$\leq \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{m^\star - 1} \leq 3\frac{2^n}{n}.$$

Number of iterations taken by Howard's PI: $O\left(\frac{2^n}{n}\right)$ [MS99, HGDJ14].

Randomised Policy Iteration (2-action MDPs)

From the set of improving states, pick a non-empty subset S_I uniformly at random.
Switch actions of all states in S_I .

Randomised Policy Iteration (2-action MDPs)

From the set of improving states, pick a non-empty subset S_I uniformly at random. Switch actions of all states in S_I.

π **0 0 0 0 0 0 0 0 0** $\boxed{\mathbf{0}}$ $\boxed{\mathbf{0}}$ $\boxed{\mathbf{0}}$

Randomised Policy Iteration (2-action MDPs)

From the set of improving states, pick a non-empty subset S_I uniformly at random.
Switch actions of all states in S_I .

π_7	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	$1/7$
π_6	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	$1/7$
π_5	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	$1/7$
π_4	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	$1/7$
π_3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	$1/7$
π_2	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	$1/7$
π_1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	$1/7$
π	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	Probability

Randomised Policy Iteration (2-action MDPs)

From the set of improving states, pick a non-empty subset S_I uniformly at random. Switch actions of all states in S_I .

π_7	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	$1/7$
π_6	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	$1/7$
π_5	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	$1/7$
π_4	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	$1/7$
π_3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	$1/7$
π_2	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	$1/7$
π_1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	$1/7$
π	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	Probability

If π has m improvable states and $\pi \xrightarrow{\text{Randomised PI}} \pi'$, then with probability 1/2, there exist 2^{m-1} policies π'' such that $\pi'' \succ \pi$ and $\neg(\pi'' \succ \pi')$.

Randomised Policy Iteration (2-action MDPs)

From the set of improving states, pick a non-empty subset S_I uniformly at random. Switch actions of all states in S_I .

π_7	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**1**	$1/7$
π_6	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	$1/7$
π_5	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	$1/7$
π_4	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	$1/7$
π_3	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	$1/7$
π_2	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	$1/7$
π_1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	$1/7$
π	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	Probability

If π has m improvable states and $\pi \xrightarrow{\text{Randomised PI}} \pi'$, then with probability 1/2, there exist 2^{m-1} policies π'' such that $\pi'' \succ \pi$ and $\neg(\pi'' \succ \pi')$.

Number of policies eliminated is exponential in m. As before, $m^\star$ can be tuned such that the expected number of iterations taken by Randomised PI = $O(1.7172^n)$ [MS99].

Batch-Switching Policy Iteration (BSPI)

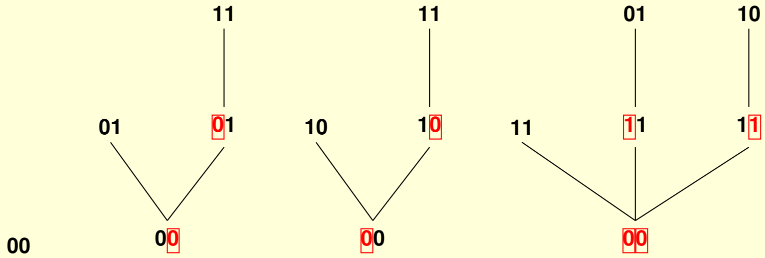
Howard's Policy Iteration takes at most ___ iterations on a 2-state MDP!

Batch-Switching Policy Iteration (BSPI)

Howard's Policy Iteration takes at most _3_ iterations on a 2-state MDP!

Batch-Switching Policy Iteration (BSPI)

Howard's Policy Iteration takes at most _3_ iterations on a 2-state MDP!



Batch-Switching Policy Iteration (BSPI)

Partition the states into 2-sized batches; arranged from left to right.

Given a policy, improve the rightmost set containing an improvable state.

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$$\pi_1 \quad \left\| \begin{array}{cc} \mathbf{0} & \boxed{\mathbf{1}} \\ s_1 & s_2 \end{array} \right\| \begin{array}{cc} \mathbf{1} & \mathbf{0} \\ s_3 & s_4 \end{array} \left\| \begin{array}{cc} \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \\ s_5 & s_6 \end{array} \right\| \begin{array}{cc} \mathbf{1} & \boxed{\mathbf{0}} \\ s_7 & s_8 \end{array} \left\| \begin{array}{cc} \boxed{\mathbf{0}} & \mathbf{0} \\ s_9 & s_{10} \end{array} \right\|$$

Batch-Switching Policy Iteration (BSPI)

Partition the states into 2-sized batches; arranged from left to right.

Given a policy, improve the rightmost set containing an improvable state.

π_2	**0**	$\boxed{\mathbf{1}}$	**1**	**0**	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	**1**	**0**	**1**	**0**
π_1	**0**	$\boxed{\mathbf{1}}$	**1**	**0**	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	**1**	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	**0**
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}

Batch-Switching Policy Iteration (BSPI)

Partition the states into 2-sized batches; arranged from left to right.

Given a policy, improve the rightmost set containing an improvable state.

π_3	$\boxed{0}$	$\boxed{1}$	1	0	1	$\boxed{1}$	1	$\boxed{0}$	1	0
π_2	0	$\boxed{1}$	1	0	$\boxed{0}$	$\boxed{0}$	1	0	1	0
π_1	0	$\boxed{1}$	1	0	$\boxed{0}$	$\boxed{0}$	1	$\boxed{0}$	$\boxed{0}$	0
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}

Batch-Switching Policy Iteration (BSPI)

Partition the states into 2-sized batches; arranged from left to right.

Given a policy, improve the rightmost set containing an improvable state.

π_4	$\boxed{0}$	1	1	0	1	1	1	1	1	0
π_3	$\boxed{0}$	$\boxed{1}$	1	0	1	$\boxed{1}$	1	$\boxed{0}$	1	0
π_2	0	$\boxed{1}$	1	0	$\boxed{0}$	$\boxed{0}$	1	0	1	0
π_1	0	$\boxed{1}$	1	0	$\boxed{0}$	$\boxed{0}$	1	$\boxed{0}$	$\boxed{0}$	0
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}

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Partition the states into 2-sized batches; arranged from left to right.

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	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
π_4	$\boxed{0}$	1	1	0	1	1	1	1	1	0
π_3	$\boxed{0}$	$\boxed{1}$	1	0	1	$\boxed{1}$	1	$\boxed{0}$	1	0
π_2	0	$\boxed{1}$	1	0	$\boxed{0}$	$\boxed{0}$	1	0	1	0
π_1	0	$\boxed{1}$	1	0	$\boxed{0}$	$\boxed{0}$	1	$\boxed{0}$	$\boxed{0}$	0

- Left-most batch can change only when all other columns are non-improvable.

Batch-Switching Policy Iteration (BSPI)

Partition the states into 2-sized batches; arranged from left to right.

Given a policy, improve the rightmost set containing an improvable state.

π_4	$\boxed{0}$	1	1	0	1	1	1	1	1	0
π_3	$\boxed{0}$	$\boxed{1}$	1	0	1	$\boxed{1}$	1	$\boxed{0}$	1	0
π_2	0	$\boxed{1}$	1	0	$\boxed{0}$	$\boxed{0}$	1	0	1	0
π_1	0	$\boxed{1}$	1	0	$\boxed{0}$	$\boxed{0}$	1	$\boxed{0}$	$\boxed{0}$	0
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}

- Left-most batch can change only when all other columns are non-improvable.
- Left-most batch can change at most 3 times (following previous result).

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Partition the states into 2-sized batches; arranged from left to right.

Given a policy, improve the rightmost set containing an improvable state.

π_4	0	1	1	0	1	1	1	1	1	0
π_3	0	1	1	0	1	1	1	0	1	0
π_2	0	1	1	0	0	0	1	0	1	0
π_1	0	1	1	0	0	0	1	0	0	0
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}

- Left-most batch can change only when all other columns are non-improvable.
- Left-most batch can change at most 3 times (following previous result).
- $T(n) \leq 3 \times T(n-2) \leq \sqrt{3}^n$.

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Howard's Policy Iteration takes at most 5 iterations on a 3-state MDP!

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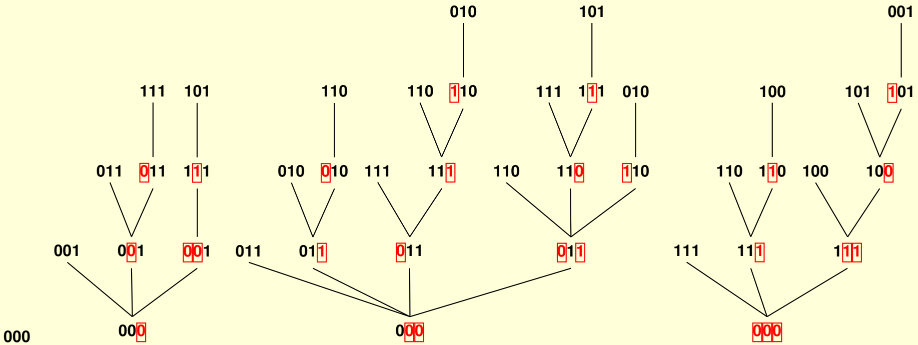

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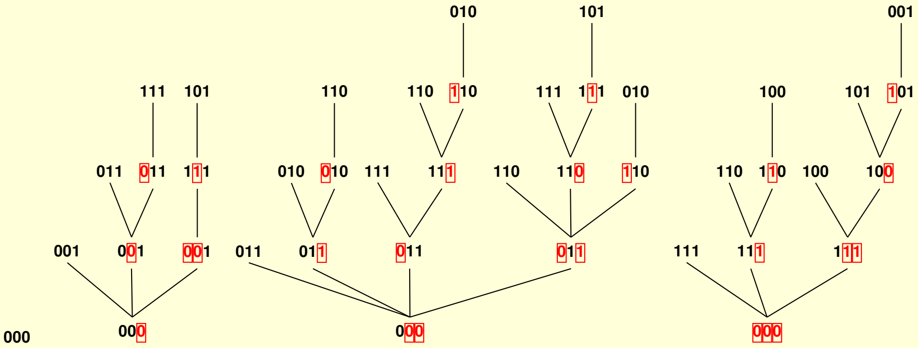

The structures drawn above are called Trajectory-bounding Trees (TBTs) [KMG16a] (and correspond to the Order Regularity Problem [H12, GHDJ15]).

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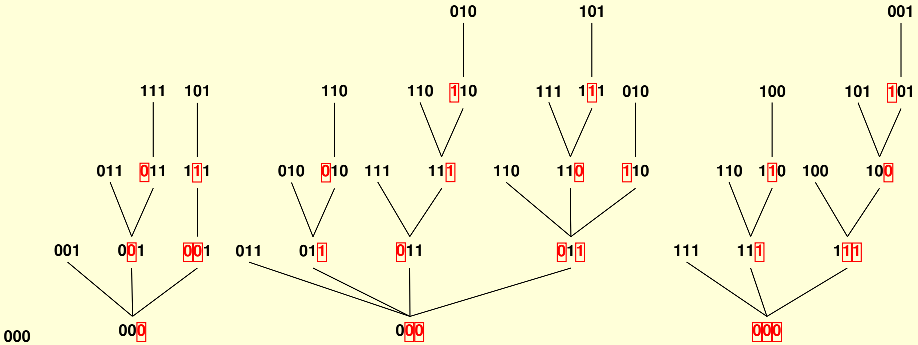

The structures drawn above are called Trajectory-bounding Trees (TBTs) [KMG16a] (and correspond to the Order Regularity Problem [H12, GHDJ15]).

BSPI with 3-sized batches gives $T(n) \leq 5 \times T(n-3) \leq 1.71^n$.

Batch-Switching Policy Iteration (BSPI)

Principle of constructing TBTs:

$L^{+}_{\pi,IS} \stackrel{\text{def}}{=} \{\pi' \in \Pi : \exists s \in IS(\pi'(s) \neq \pi(s)) \wedge \forall s \in (S \setminus IS)(\pi'(s) = \pi(s))\}$;

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$L^{+}_{\pi,IS}$		**1**	**1**	**1**	**0**	**0**
		1	**1**	**0**	**0**	**0**
		1	**0**	**1**	**0**	**0**
		1	**0**	**0**	**0**	**0**
		0	**1**	**1**	**0**	**0**
		0	**1**	**0**	**0**	**0**
		0	**0**	**1**	**0**	**0**
$L^{-}_{\pi,IS}$	π, IS:	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	**0**	**0**
		0	**0**	**0**	**0**	**1**
		0	**0**	**0**	**1**	**0**
		0	**0**	**0**	**1**	**1**

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$L^{+}_{\pi,IS}$		**1**	**1**	**1**	**0**	**0**
		1	**1**	**0**	**0**	**0**
		1	**0**	**1**	**0**	**0**
		1	**0**	**0**	**0**	**0**
		0	**1**	**1**	**0**	**0**
		0	**1**	**0**	**0**	**0**
		0	**0**	**1**	**0**	**0**
$L^{-}_{\pi,IS}$	π, IS:	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	$\boxed{\mathbf{0}}$	**0**	**0**
		0	**0**	**0**	**0**	**1**
		0	**0**	**0**	**1**	**0**
		0	**0**	**0**	**1**	**1**

If $(\pi_1, IS_1), (\pi_2, IS_2), \ldots, (\pi_t, IS_t)$ is a trajectory encountered by PI, it must satisfy, for $1 \leq i < j \leq t$:

$$L^{-}_{\pi_i, IS_i} \cap L^{+}_{\pi_j, IS_j} = \emptyset.$$

BSPI: Bounds

Batch size	Depth of TBT	Bound on number of iterations
1	2	2^n
2	3	1.7321^n
3	5	1.7100^n
4	8	1.6818^n
5	13	1.6703^n
6	21	1.6611^n
7	33	1.6479^n

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Depth of TBT for batch size 7 due to Gerencsér *et al.* [GHDJ15].

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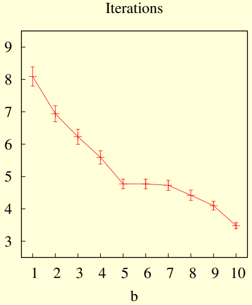
Depth of TBT for batch size 7 due to Gerencsér *et al.* [GHDJ15].

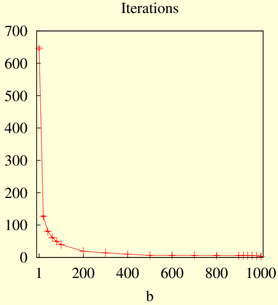
Will the bound continue to be non-increasing in the batch size?

If so, 1.6479^n would be a bound for Howard's Policy Iteration!

BSPI: Effect of Batch Size b



$n = 10$



$n = 1000$

Averaged over n-state, 2-action MDPs with randomly generated transition and reward functions. Each point is an average over 100 randomly-generated MDP instances and initial policies [KMG16a].

Overview

Policy Iteration on k-action MDPs

- What are the main differences between 2-action and k-action MDPs ($k > 2$)?

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- Mansour and Singh's analysis makes no assumption on which improving action is picked, only that one is picked at all, in the states selected to be switched.

  Bound for Howard's PI: $O\left(\frac{k^n}{n}\right)$ iterations [MS99, HGDJ14].

  Bound for Randomised PI: $O\left(\left(\left(1 + \frac{2}{\log(k)}\right)\frac{k}{2}\right)^n\right)$ expected iterations [MS99].

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  Bound for Howard's PI: $O\left(\frac{k^n}{n}\right)$ iterations [MS99, HGDJ14].

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Policy Iteration on k-action MDPs

- What are the main differences between 2-action and k-action MDPs ($k > 2$)?

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- **Randomised Simple PI** [KMG16b]: Switch only the "rightmost" improvable state; switch to an improving action picked uniformly at random.

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- **Recursive BSPI** [GK17]: Deterministic switching strategy based on a binary hierarchy of actions (that facilitates reusing the 2-action MDP analysis).

  Bound: $k^{0.7207n}$ iterations.

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Conclusion

- Policy Iteration is an elegant family of algorithms for MDP Planning.
- Under the infinite precision arithmetic computation model, it naturally yields strong running time bounds, which depend only on the number of states and actions.
- This tutorial is prompted by some recent progress that has resulted in exponential improvements in upper bounds.
- The main tool of analysis remains basic: the well-known Policy Improvement Theorem.
- Both theory and experiments suggest that Howard's Policy Iteration could be more efficient than it has formally been proven.
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Thank you!