Algorithms for MDP Planning

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Overview

1. Value Iteration

2. Linear Programming

3. Policy Iteration
   Policy Improvement Theorem
Overview

1. Value Iteration

2. Linear Programming

3. Policy Iteration
   - Policy Improvement Theorem
Value Iteration

\[ V_0 \leftarrow \text{Arbitrary, element-wise bounded, } n\text{-length vector. } t \leftarrow 0. \]

Repeat:

For \( s \in S \):

\[ V_{t+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} T(s, a, s') (R(s, a, s') + \gamma V_t(s')). \]

\( t \leftarrow t + 1. \)

Until \( V_t \approx V_{t-1} \) (up to machine precision).
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Until \( V_t \approx V_{t-1} \) (up to machine precision).

Convergence to \( V^* \) guaranteed using a max-norm contraction argument.
Overview

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   Policy Improvement Theorem
Linear Programming

Minimise \[ \sum_{s \in S} V(s) \]

subject to \[ V(s) \geq \sum_{s' \in S} T(s, a, s') (R(s, a, s') + \gamma V(s')) , \forall s \in S, \forall a \in A. \]
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Minimise \[ \sum_{s \in S} V(s) \]
subject to \[ V(s) \geq \sum_{s' \in S} T(s, a, s') \left( R(s, a, s') + \gamma V(s') \right), \forall s \in S, \forall a \in A. \]

Let \(|S| = n\) and \(|A| = k\).
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\( n \) variables, \( nk \) constraints.
Linear Programming

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Let \( |S| = n \) and \( |A| = k \).

\( n \) variables, \( nk \) constraints.

Can also be posed as dual with \( nk \) variables and \( n \) constraints.
Overview

1. Value Iteration

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3. Policy Iteration
   Policy Improvement Theorem
Policy Improvement

Given $\pi$, pick one or more improvable states, and in them, switch to an arbitrary improving action. Let the resulting policy be $\pi'$.

**Policy Improvement Theorem:**
1. If $\pi$ has no improvable states, then it is optimal, else
2. if $\pi'$ is obtained as above, then $\forall s \in S: V_{\pi'}(s) \geq V_\pi(s)$ and $\exists s \in S: V_{\pi'}(s) > V_\pi(s)$.
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\begin{itemize}
  \item PI: if $\pi$ has no improvable states, then it is optimal,
  \itemPolicy Improvement Theorem: if $\pi'$ is obtained as above, then
  \begin{align*}
  \forall s \in S: V_{\pi'}(s) &\geq V_{\pi}(s) \\
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$$\forall s \in S : V^{\pi'}(s) \geq V^\pi(s) \text{ and } \exists s \in S : V^{\pi'}(s) > V^\pi(s).$$
Definitions and Basic Facts

For $X : S \rightarrow \mathbb{R}$ and $Y : S \rightarrow \mathbb{R}$, we define $X \succeq Y$ if $\forall s \in S : X(s) \geq Y(s)$, and we define $X \succ Y$ if $X \succeq Y$ and $\exists s \in S : X(s) > Y(s)$.

Bellman Operator.

For $\pi \in \Pi$, we define $B_\pi : (S \rightarrow \mathbb{R}) \rightarrow (S \rightarrow \mathbb{R})$ as follows:

for $X : S \rightarrow \mathbb{R}$ and $\forall s \in S$,

$$(B_\pi(X))(s) = \sum_{s' \in S} T(s, \pi(s), s') (R(s, \pi(s), s') + \gamma X(s'))$$

Fact 1. For $\pi \in \Pi$, $X : S \rightarrow \mathbb{R}$, and $Y : S \rightarrow \mathbb{R}$, if $X \succeq Y$, then $B_\pi(X) \succeq B_\pi(Y)$.

Fact 2. For $\pi \in \Pi$ and $X : S \rightarrow \mathbb{R}$,

$$\lim_{l \to \infty} (B_\pi)^l(X) = V_\pi$$

(from Banach's FP Theorem)
Definitions and Basic Facts

For $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$, we define $X \succeq Y$ if $\forall s \in S : X(s) \geq Y(s)$, and we define $X \succ Y$ if $X \succeq Y$ and $\exists s \in S : X(s) > Y(s)$.

For policies $\pi_1, \pi_2 \in \Pi$, we define $\pi_1 \succeq \pi_2$ if $V^{\pi_1} \succeq V^{\pi_2}$, and we define $\pi_1 \succ \pi_2$ if $V^{\pi_1} \succ V^{\pi_2}$.
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Bellman Operator. For $\pi \in \Pi$, we define $B^\pi : (S \to \mathbb{R}) \to (S \to \mathbb{R})$ as follows: for $X : S \to \mathbb{R}$ and $\forall s \in S$,

$$(B^\pi(X))(s) \overset{\text{def}}{=} \sum_{s' \in S} T(s, \pi(s), s') (R(s, \pi(s), s') + \gamma X(s')) .$$
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- For \( X : S \rightarrow \mathbb{R} \) and \( Y : S \rightarrow \mathbb{R} \), we define \( X \succeq Y \) if \( \forall s \in S : X(s) \geq Y(s) \), and we define \( X \succ Y \) if \( X \succeq Y \) and \( \exists s \in S : X(s) > Y(s) \).

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- **Bellman Operator.** For \( \pi \in \Pi \), we define \( B^\pi : (S \rightarrow \mathbb{R}) \rightarrow (S \rightarrow \mathbb{R}) \) as follows: for \( X : S \rightarrow \mathbb{R} \) and \( \forall s \in S \),

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- **Fact 1.** For \( \pi \in \Pi \), \( X : S \rightarrow \mathbb{R} \), and \( Y : S \rightarrow \mathbb{R} \):

if \( X \succeq Y \), then \( B^\pi(X) \succeq B^\pi(Y) \).
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- **Fact 1.** For $\pi \in \Pi$, $X : S \rightarrow \mathbb{R}$, and $Y : S \rightarrow \mathbb{R}$:

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- **Fact 2.** For $\pi \in \Pi$ and $X : S \rightarrow \mathbb{R}$:

$$\lim_{l\to\infty} (B^\pi)^l(X) = V^\pi. \text{ (from Banach's FP Theorem)}$$
Proof of Policy Improvement Theorem

Observe that for $\pi, \pi' \in \Pi, \forall s \in S$: $B^{\pi'}(V^\pi)(s) = Q^\pi(s, \pi'(s))$. 

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Proof of Policy Improvement Theorem

Observe that for $\pi, \pi' \in \Pi, \forall s \in S$: $B^{\pi'}(V^\pi)(s) = Q^\pi(s, \pi'(s)).$

$\pi$ has no improvable states

$\implies \forall \pi' \in \Pi: V^\pi \succeq B^{\pi'}(V^\pi)$
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$\implies \forall \pi' \in \Pi : V^\pi \succeq V^{\pi'}$.

$\pi$ has improvable states and policy improvement yields $\pi'$
Proof of Policy Improvement Theorem

Observe that for \( \pi, \pi' \in \Pi, \forall s \in S: B^{\pi'}(V^\pi)(s) = Q^\pi(s, \pi'(s)). \)

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$\equiv \forall \pi' \in \Pi : V^\pi \succeq V^\pi'$.

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$$\implies (B^{\pi'})^2(V^\pi) \succeq B^{\pi'}(V^\pi) \succ V^\pi$$

$$\implies \lim_{l \to \infty} (B^{\pi'})^l(V^\pi) \succeq (B^{\pi'})^2(V^\pi) \succeq B^{\pi'}(V^\pi) \succ V^\pi$$

$$\implies V^\pi' \succ V^\pi.$$
Policy Iteration Algorithm

\[ \pi \leftarrow \text{Arbitrary policy.} \]

\textbf{While } \pi \text{ has improvable states:}

\[ \pi \leftarrow \text{PolicyImprovement}(\pi). \]
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Policy Iteration Algorithm

$\pi \leftarrow$ Arbitrary policy.

**While** $\pi$ has improvable states:

$\pi \leftarrow$ PolicyImprovement($\pi$).

Number of iterations depends on **switching strategy**. Current bounds quite loose.