CS 747, Autumn 2022: Lecture 9

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Autumn 2022
Markov Decision Problems

1. Review of linear programming

2. MDP planning through linear programming
Markov Decision Problems

1. Review of linear programming

2. MDP planning through linear programming
Linear Programming

To solve for real-valued variables $x_1, x_2, \ldots, x_m$ such that

- a given linear function of the variables is maximised, while
- given linear constraints on the variables are satisfied.

Maximise $x_1 + 2x_2$

subject to:

1. $x_1 + x_2 \leq 9$, (C1)
2. $4x_1 - 13x_2 \leq -75$, (C2)
3. $x_1 \leq 5$. (C3)

Well-studied problem with wide-ranging applications in mathematics, engineering. Today's solvers (commercial, as well as open source) can handle LPs with millions of variables.
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Maximise $x_1 + 2x_2$  //Objective function

subject to:  //Constraints

\begin{align*}
x_1 + x_2 & \leq 9, \quad \text{(C1)} \\
4x_1 - 13x_2 & \leq -75, \quad \text{(C2)} \\
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$$x_1 \leq 5.$$  (C1, C2, C3)

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Conceptual Steps towards Solving a Linear Program

- **Step 1**: Identify the **feasible set**, which contains all the points satisfying the constraints. Might be empty, but otherwise will be convex.

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Conceptual Steps towards Solving a Linear Program

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Conceptual Steps towards Solving a Linear Program

- **Step 1**: Identify the feasible set, which contains all the points satisfying the constraints. Might be empty, but otherwise will be convex.
- **Step 2**: Identify points within the feasible set that maximise the objective. Usually a single point.

Maximise $x_1 + 2x_2$
subject to:

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Actually Solving a Linear Program

- Common approaches: Simplex, interior-point methods.
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- LP with $d$ variables, $m$ constraints, $B$-bit representation of floats.
  - Can be solved in $\text{poly}(d, m, B)$ operations.
  - Can be solved in $\text{poly}(d, m) \cdot e^{O(\sqrt{d \log(m)})}$ expected "real RAM" operations.
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- Engineer’s focus is on formulating, rather than solving, LP.
Markov Decision Problems

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Bellman Optimality Equations as an LP

Bellman optimality equations: for \( s \in S \),

\[
V^*(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s') \{ R(s, a, s') + \gamma V^*(s') \}.
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Bellman Optimality Equations as an LP

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$$V^*(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s') \{ R(s, a, s') + \gamma V^*(s') \}.$$ 

Let us create $n$ variables $V(s_1), V(s_2), \ldots, V(s_n)$, and attempt to create an LP whose unique solution is $V^*$. 

Although the Bellman optimality equations are non-linear, we can easily create linear constraints. For $s \in S, a \in A$:

$$V(s) \geq \sum_{s' \in S} T(s, a, s') \{ R(s, a, s') + \gamma V(s') \}.$$ 

These are $nk$ linear constraints. Observe that $V^*$ is in the feasible set. Can we construct an objective function for which $V^*$ is the sole optimiser?
Bellman Optimality Equations as an LP

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Observe that \( V^* \) is in the feasible set.

Can we construct an objective function for which \( V^* \) is the sole optimiser?
Vector Comparison

For $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$ (equivalently $X, Y \in \mathbb{R}^n$), we define

\[ X \succeq Y \iff \forall s \in S : X(s) \geq Y(s), \]
\[ X \succ Y \iff X \succeq Y \text{ and } \exists s \in S : X(s) > Y(s). \]
Vector Comparison

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- For policies \( \pi_1, \pi_2 \in \Pi \), we define
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  \pi_1 \succeq \pi_2 \iff V^{\pi_1} \succeq V^{\pi_2},
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Note that we can have incomparable policies $\pi_1, \pi_2 \in \Pi$: that is, neither $\pi_1 \succeq \pi_2$ nor $\pi_2 \succeq \pi_1$. 
Vector Comparison

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Note that we can have incomparable policies $\pi_1, \pi_2 \in \Pi$: that is, neither $\pi_1 \succeq \pi_2$ nor $\pi_2 \succeq \pi_1$.

Also note that if $\pi_1 \succeq \pi_2$ and $\pi_2 \succeq \pi_1$, then $V^{\pi_1} = V^{\pi_2}$. 
B* Preserves \( \succeq \)

- **Fact.** For \( X : S \to \mathbb{R} \) and \( Y : S \to \mathbb{R} \),
  \[ X \succeq Y \implies B^*(X) \succeq B^*(Y). \]
\(B^*\) Preserves \(\succeq\)

**Fact.** For \(X : S \to \mathbb{R}\) and \(Y : S \to \mathbb{R}\),

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X \succeq Y \implies B^*(X) \succeq B^*(Y).
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As proof it suffices to show that if \(X \succeq Y\), then for \(s \in S\),

\[
(B^*(X))(s) - (B^*(Y))(s) \geq 0.
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**B* Preserves ≥**

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We use: $\max_a f(a) - \max_a g(a) \geq \min_a (f(a) - g(a))$. 
$B^*$ Preserves $\succeq$

**Fact.** For $X : S \rightarrow \mathbb{R}$ and $Y : S \rightarrow \mathbb{R}$,

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We use: $\max_a f(a) - \max_a g(a) \geq \min_a (f(a) - g(a))$.

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(B^*(X))(s) - (B^*(Y))(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s') \{ R(s, a, s') + \gamma X(s') \} - \\
\max_{a \in A} \sum_{s' \in S} T(s, a, s') \{ R(s, a, s') + \gamma Y(s') \} \\
\geq \gamma \min_{a \in A} \sum_{s' \in S} T(s, a, s') \{ X(s') - Y(s') \} \geq 0.
$$
Examining the Feasible Set of our LP

- Each $V : S \rightarrow \mathbb{R}$ in our feasible set satisfies $V \succeq B^*(V)$. 

By implication and by Banach's Fixed-point Theorem, $V \succeq \lim_{l \to \infty} (B^{\star l}(V)) = V^{\star}$. 

We "linearise" this result: for $V : S \rightarrow \mathbb{R}$ in the feasible set.

$P_s \in S$ 

$V(s) \geq P_s \in S (V^{\star}(s))$. 

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Examining the Feasible Set of our LP

- Each $V: S \rightarrow \mathbb{R}$ in our feasible set satisfies $V \succeq B^*(V)$.
- Since $B^*$ preserves $\succeq$, we get

$$V \succeq B^*(V)$$

$$\implies B^*(V) \succeq (B^*)^2(V)$$

$$\implies (B^*)^2(V) \succeq (B^*)^3(V)$$

$$\vdots$$
Examining the Feasible Set of our LP

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- Since \( B^* \) preserves \( \succeq \), we get

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- By implication and by Banach’s Fixed-point Theorem,

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V \succeq \lim_{l \to \infty} (B^*)^l(V) = V^*.
\]

- We “linearise” this result: for $V : S \to R$ in the feasible set.

\[
\sum_{s \in S} V(s) \geq \sum_{s \in S} V^*(s).
\]
Linear Programming Formulation

Maximise \(- \sum_{s \in S} V(s)\)

subject to

\[ V(s) \geq \sum_{s' \in S} T(s, a, s') \{ R(s, a, s') + \gamma V(s') \}, \forall s \in S, a \in A. \]

- This LP has \(n\) variables, \(nk\) constraints.
Linear Programming Formulation

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- This LP has \(n\) variables, \(nk\) constraints.
- There is also a dual LP formulation with \(nk\) variables and \(n\) constraints. See Littman et al. (1995) if interested.
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Next class: policy iteration.