Reinforcement Learning

1. Tile coding

2. Issues in control with function approximation

3. The case for policy search
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How Good is Linear Function Approximation?

$\hat{V}_3(x) = w_1 x + w_2$.

Is $\hat{V}_3$ the obvious choice?

$\hat{V}_3$ has the highest resolution, but does not generalise well.

How to achieve high resolution along with generalisation?
How Good is Linear Function Approximation?

\[ \hat{V}_1(x) = w_1 x + w_2. \]
How Good is Linear Function Approximation?

\[ \hat{V}_2(x) = w_1 b_1 + w_2 b_2 + w_3 b_3. \]

\[ b_1 = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{otherwise}. \end{cases} \]

\[ b_2 = \begin{cases} 1 & \text{if } 1 \leq x < 2, \\ 0 & \text{otherwise}. \end{cases} \]

\[ b_3 = \begin{cases} 1 & \text{if } 2 \leq x < 3, \\ 0 & \text{otherwise}. \end{cases} \]
How Good is Linear Function Approximation?

\[ \hat{V}_3(x) : 18 \text{ piece-wise constants.} \]
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Tile coding

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The function value of a point is the sum of the weights of the tiles intersecting it (one per tiling).
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Tile coding

- Each tile is a binary feature.
- **Tile width** and the **number of tilings** determine generalisation, resolution.
- Observe that two points more than (tile width / number of tilings) apart can be given arbitrary function values.
Representing $\hat{Q}$

- Given a feature value $x$ as input, the corresponding set of tilings $F : \mathbb{R} \rightarrow \mathbb{R}$ returns the sum of the weights of the tiles activated by $x$. 

Usually, tile widths and the number of tilings are configured specifically for each feature. For example, in soccer, could use 2 m as tile width for "distance" features, and 10 $^\circ$ as tile width for "angle" features.
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- The usual practice is to have a separate set of tilings $F_{aj} : \mathbb{R} \rightarrow \mathbb{R}$ for each action $a$ and state feature $j \in \{1, 2, \ldots, d\}$. Hence

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**2-d Tile coding**

- For representing more complex functions, can also have tilings on conjunctions of features (see below for 2 features).

![Diagram of 2-d tile coding](image)

- Introduces more parameters—which could help or hurt.
Tile Coding: Summary

- Linear function approximation does not restrict us to a representation that is linear in the given/raw features.

- Tile coding a standard approach to discretise input features and tune both resolution and generalisation.

- Many empirical successes, especially in conjunction with Linear Sarsa($\lambda$).

- Common to store weights in a hash table (collisions don’t seem to hurt much), whose size is set based on practical constraints.

- 1-d tilings most common; rarely see conjunction of 3 or more features.
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A Counterexample (Tsitsiklis and Van Roy, 1996)

Prediction problem (policy $\pi$).
Episodic, start state is $s_1$.
Observe that $V^\pi(s_1) = V^\pi(s_2) = 0$.
Linear function approximation with single parameter $w$: $x(s_1) = 1, x(s_2) = 2$; hence $\hat{V}(s_1) = w, \hat{V}(s_2) = 2w$. 

What's the optimal setting of $w$? $w = 0$ gives the exact answer!

We design an iteration $w_0 \rightarrow w_1 \rightarrow w_2 \rightarrow \ldots$, and see if it converges to 0.
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From state $s$, let $s'$, $r$ be the (random) next state, reward.
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From state \( s \), let \( s' \), \( r \) be the (random) next state, reward.

If our current estimate of \( V^\pi \) is \( \hat{V} \), the bootstrapping idea suggests
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E_\pi[r + \gamma \hat{V}(s')] 
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Starting with $w = w_0$, we update $w$ so it best-fits the bootstrapped estimate in terms of squared error on the states. For $k \geq 0$:

$$w_{k+1} \leftarrow \arg\min_{w \in \mathbb{R}} \sum_s \left( E_\pi[r + \gamma \hat{V}(w_k, x(s'))] - \hat{V}(w, x(s)) \right)^2.$$
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- For \( w_0 = 1, \epsilon = 0.1, \gamma = 0.99, \lim_{k \to \infty} w_k = \infty; \) divergence!
- The failure owes to the combination of three factors: off-policy updating, generalisation, bootstrapping.
- But these are almost always used together in practice!
## Summary of Theoretical Results

<table>
<thead>
<tr>
<th>Method</th>
<th>Tabular</th>
<th>Linear FA</th>
<th>Non-linear FA</th>
</tr>
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<tbody>
<tr>
<td>TD(0)</td>
<td>C, O</td>
<td>C</td>
<td>NK</td>
</tr>
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<td>TD($\lambda$), $\lambda \in (0, 1)$</td>
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<td>C, O</td>
<td>C, “Best”</td>
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<td>Sarsa(0)</td>
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<td>Q-learning(0)</td>
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</table>

(C: Convergent; O: Optimal; NK: Not known.)

*: to the best of your instructor’s knowledge.
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(\(m^{RED}, c^{RED}, m^{BLUE}, c^{BLUE}\)) a “good” approximation of \(Q^*\).

But induces non-optimal actions for \(x \in (A, B)\).

Perhaps we found \((\bar{m}^{RED}, \bar{c}^{RED}, \bar{m}^{BLUE}, \bar{c}^{BLUE})\) by Q-learning.

How to find \((\bar{m}^{RED}, \bar{c}^{RED}, \bar{m}^{BLUE}, \bar{c}^{BLUE})\)?

Next class: policy search.
So Near, Yet So Far

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