Algorithms for MDP Planning

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Overview

1. Value Iteration
2. Linear Programming
3. Policy Iteration
    - Policy Improvement Theorem

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Value Iteration

$V_0 \leftarrow$ Arbitrary, element-wise bounded, n-length vector. $t \leftarrow 0$.
Repeat:
    For $s \in S$**:**
        $V_{t+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} T(s, a, s') \left(R(s, a, s') + \gamma V_t(s') \right)$.
        $t \leftarrow t + 1$.
Until $V_t \approx V_{t-1}$ (up to machine precision).

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Convergence to $V^\star$ guaranteed using a max-norm contraction argument.

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Linear Programming

$$\text{Minimise} \quad \sum_{s \in S} V(s)$$

$$\text{subject to} \quad V(s) \geq \sum_{s' \in S} T(s, a, s')\left(R(s, a, s') + \gamma V(s')\right), \forall s \in S, \forall a \in A.$$

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Can also be posed as *dual* with nk variables and n constraints.

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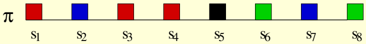

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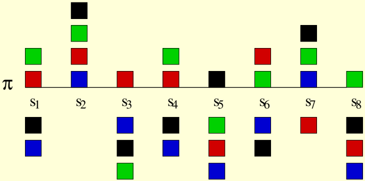

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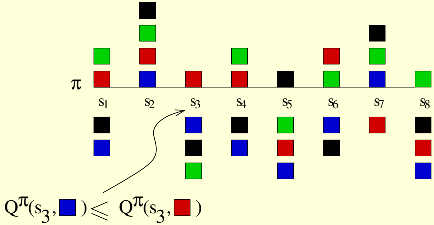

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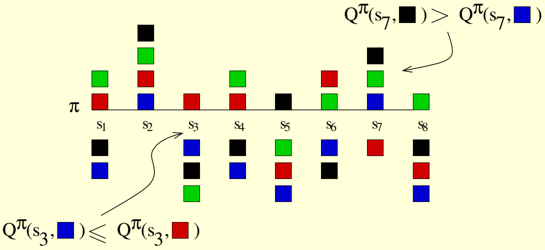

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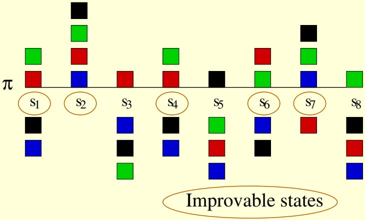

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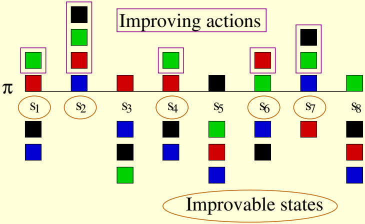

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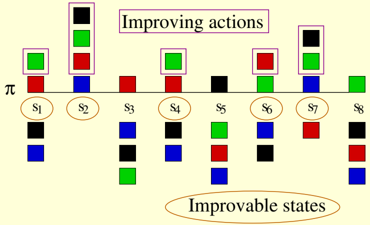
Given π,

Pick one or more improvable states, and in them,

Switch to an arbitrary improving action.

Let the resulting policy be π'.



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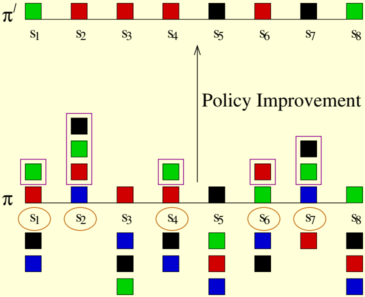

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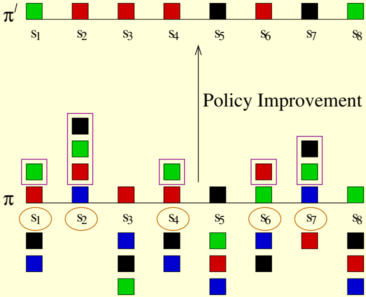

Policy Improvement Theorem:
(1) If π has no improvable states, then it is optimal, else
(2) if π' is obtained as above, then

$$\forall s \in S : V^{\pi'}(s) \geq V^{\pi}(s) \text{ and } \exists s \in S : V^{\pi'}(s) > V^{\pi}(s).$$

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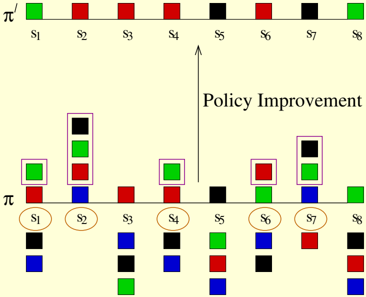

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Definitions and Basic Facts

- For $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$, we define $X \succeq Y$ if $\forall s \in S : X(s) \geq Y(s)$, and we define $X \succ Y$ if $X \succeq Y$ and $\exists s \in S : X(s) > Y(s)$.

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- **Bellman Operator.** For $\pi \in \Pi$, we define $B^{\pi} : (S \to \mathbb{R}) \to (S \to \mathbb{R})$ as follows: for $X : S \to \mathbb{R}$ and $\forall s \in S$,

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- **Fact 2**. For $\pi \in \Pi$ and $X : S \to \mathbb{R}$:

$$\lim_{l \to \infty} (B^\pi)^l(X) = V^\pi. \text{ (from Banach's FP Theorem)}$$

Proof of Policy Improvement Theorem

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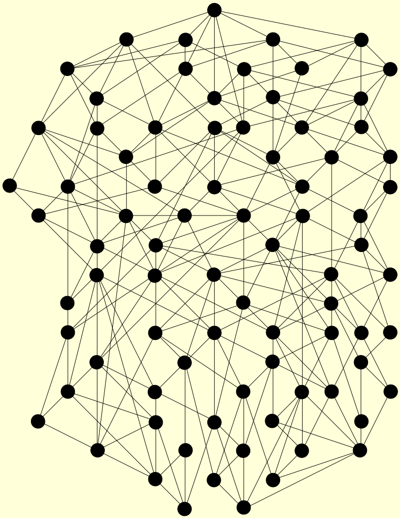

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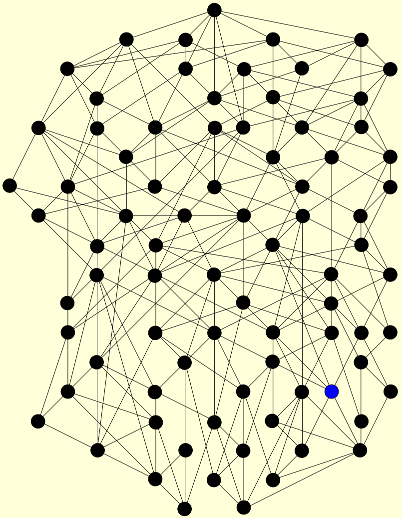

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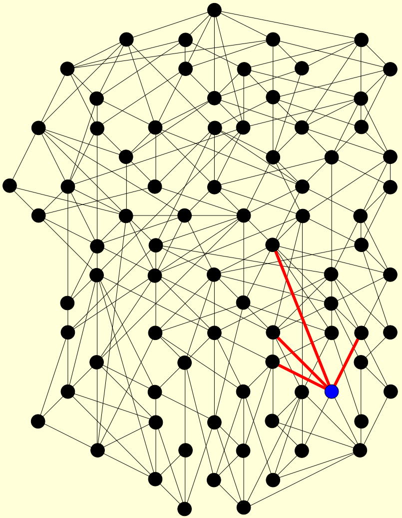

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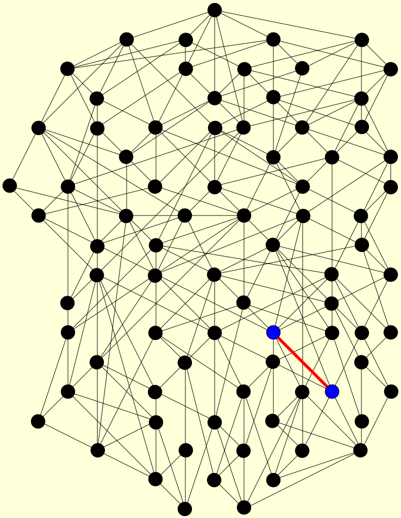

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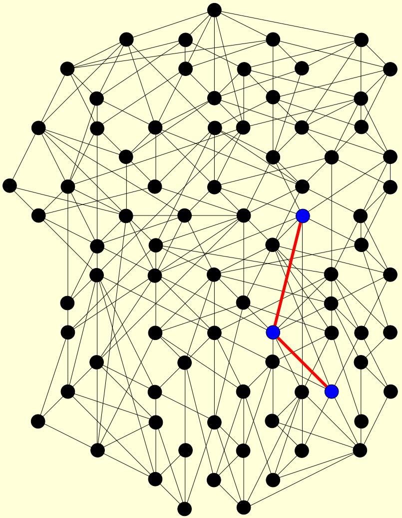

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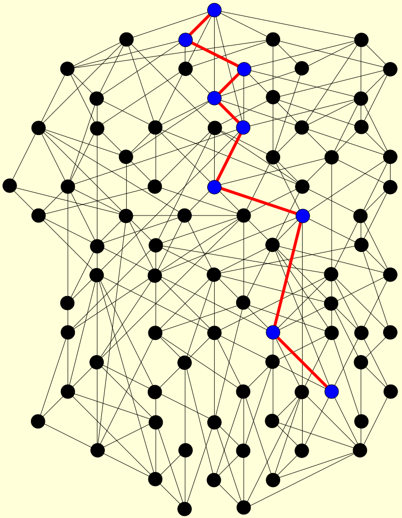

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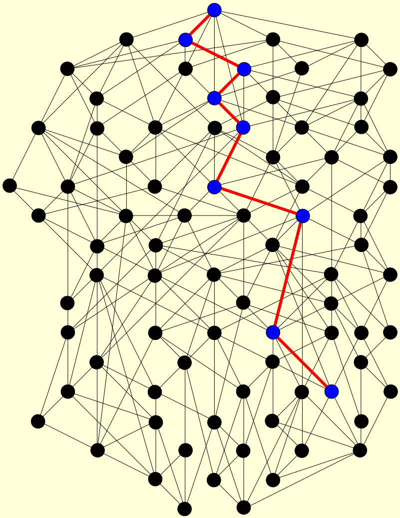

Number of iterations depends on switching strategy. Current bounds quite loose.