Reinforcement Learning

1. Tile coding

2. Issues in control with function approximation

3. The case for policy search
Reinforcement Learning

1. Tile coding

2. Issues in control with function approximation

3. The case for policy search
How Good is Linear Function Approximation?

\[ V^\pi(x) = w_1 x + w_2. \]

Is \( V^\pi_3 \) the obvious choice? \( V^\pi_3 \) has the highest resolution, but does not generalise well. How to achieve high resolution along with generalisation?
How Good is Linear Function Approximation?

\[ \hat{V}_1(x) = w_1 x + w_2. \]
How Good is Linear Function Approximation?

\[ \hat{V}_2(x) = w_1 b_1 + w_2 b_2 + w_3 b_3. \]

\[ b_1 = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases} \]

\[ b_2 = \begin{cases} 1 & \text{if } 1 \leq x < 2, \\ 0 & \text{otherwise.} \end{cases} \]

\[ b_3 = \begin{cases} 1 & \text{if } 2 \leq x < 3, \\ 0 & \text{otherwise.} \end{cases} \]
How Good is Linear Function Approximation?

$\hat{V}_3(x)$: 18 piece-wise constants.
How Good is Linear Function Approximation?

\[ \hat{V}_3(x) : 18 \text{ piece-wise constants.} \]

Is \( \hat{V}_3 \) the obvious choice?

\( \hat{V}_3 \) has the highest resolution, but does not generalise well. How to achieve high resolution along with generalisation?
How Good is Linear Function Approximation?

Is $\hat{V}^3$ the obvious choice?

$\hat{V}^3$ has the highest resolution, but does not generalise well.
How Good is Linear Function Approximation?

Is $\hat{V}^3$ the obvious choice?
- $\hat{V}^3$ has the highest resolution, but does not generalise well.
- How to achieve high resolution along with generalisation?
A tiling partitions $x$ into equal-width regions called *tiles*.
A tiling partitions $x$ into equal-width regions called tiles.

Multiple tilings (say $m$) are created, each with an offset ($1/m$ tile width) from the previous.
Tile coding

- A tiling partitions $x$ into equal-width regions called tiles.
- Multiple tilings (say $m$) are created, each with an offset $(1/m$ tile width) from the previous.
- Each tile has an associated weight.
A tiling partitions $x$ into equal-width regions called tiles.

Multiple tilings (say $m$) are created, each with an offset ($1/m$ tile width) from the previous.

Each tile has an associated weight.

The function value of a point is the sum of the weights of the tiles intersecting it (one per tiling).
A tiling partitions $x$ into equal-width regions called tiles.

Multiple tilings (say $m$) are created, each with an offset ($1/m$ tile width) from the previous.

Each tile has an associated weight.

The function value of a point is the sum of the weights of the tiles intersecting it (one per tiling).
Tile coding

- Each tile is a binary feature.
- Tile width and the number of tilings determine generalisation, resolution.
- Observe that two points more than (tile width / number of tilings) apart can be given arbitrary function values.
Representing $\hat{Q}$

- Given a feature value $x$ as input, the corresponding set of tilings $F : \mathbb{R} \rightarrow \mathbb{R}$ returns the sum of the weights of the tiles activated by $x$. 

Usually, tile widths and the number of tilings are configured specifically for each feature. For example, in soccer, one could use 2 m as tile width for "distance" features, and 10 $^\circ$ as tile width for "angle" features.
Representing $\hat{Q}$

- Given a feature value $x$ as input, the corresponding set of tilings $F : \mathbb{R} \rightarrow \mathbb{R}$ returns the sum of the weights of the tiles activated by $x$.
- The usual practice is to have a separate set of tilings $F_{aj} : \mathbb{R} \rightarrow \mathbb{R}$ for each action $a$ and state feature $j \in \{1, 2, \ldots, d\}$. Hence

$$\hat{Q}(s, a) = \sum_{j=1}^{d} F_{aj}(x_j(s)).$$
Representing \( \hat{Q} \)

- Given a feature value \( x \) as input, the corresponding set of tilings \( F : \mathbb{R} \rightarrow \mathbb{R} \) returns the sum of the weights of the tiles activated by \( x \).
- The usual practice is to have a separate set of tilings \( F_{aj} : \mathbb{R} \rightarrow \mathbb{R} \) for each action \( a \) and state feature \( j \in \{1, 2, \ldots, d\} \). Hence

\[
\hat{Q}(s, a) = \sum_{j=1}^{d} F_{aj}(x_j(s)).
\]

- Usually, tile widths and the number of tilings are configured specifically for each feature. For example, in soccer, could use 2m as tile width for “distance” features, and 10° as tile width for “angle” features.
2-d Tile coding

- For representing more complex functions, can also have tilings on conjunctions of features (see below for 2 features).

- Introduces more parameters—which could help or hurt.
Tile Coding: Summary

- Linear function approximation does not restrict us to a representation that is linear in the given/raw features.

- Tile coding a standard approach to discretise input features and tune both resolution and generalisation.

- Many empirical successes, especially in conjunction with Linear Sarsa($\lambda$).

- Common to store weights in a hash table (collisions don’t seem to hurt much), whose size is set based on practical constraints.

- 1-d tilings most common; rarely see conjunction of 3 or more features.
Reinforcement Learning

1. Tile coding

2. Issues in control with function approximation

3. The case for policy search
Prediction problem (policy $\pi$).
Episodic, start state is $s_1$.
Observe that $V^\pi(s_1) = V^\pi(s_2) = 0$.
Linear function approximation with single parameter $w$: $x(s_1) = 1, x(s_2) = 2$; hence $\hat{V}(s_1) = w, \hat{V}(s_2) = 2w$. 

What's the optimal setting of $w$? $w = 0$ gives the exact answer!

We design an iteration $w_0 \to w_1 \to w_2 \to \ldots$, and see if it converges to 0.
A Counterexample (Tsitsiklis and Van Roy, 1996)

Prediction problem (policy $\pi$).
Episodic, start state is $s_1$.
Observe that $V^\pi(s_1) = V^\pi(s_2) = 0$.
Linear function approximation with single parameter $w$: $x(s_1) = 1, x(s_2) = 2$; hence $\hat{V}(s_1) = w, \hat{V}(s_2) = 2w$.
What’s the optimal setting of $w$?
A Counterexample (Tsitsiklis and Van Roy, 1996)

- **Prediction problem (policy \( \pi \)).**
- Episodic, start state is \( s_1 \).
- Observe that \( V^\pi(s_1) = V^\pi(s_2) = 0 \).
- Linear function approximation with single parameter \( w \):
  \[ x(s_1) = 1, x(s_2) = 2; \text{ hence } \hat{V}(s_1) = w, \hat{V}(s_2) = 2w. \]
- What's the optimal setting of \( w \)?
- \( w = 0 \) gives the exact answer!
Prediction problem (policy $\pi$).
- Episodic, start state is $s_1$.
- Observe that $V^\pi(s_1) = V^\pi(s_2) = 0$.
- Linear function approximation with single parameter $w$: $x(s_1) = 1, x(s_2) = 2$; hence $\hat{V}(s_1) = w, \hat{V}(s_2) = 2w$.
- What’s the optimal setting of $w$?
- $w = 0$ gives the exact answer!
- We design an iteration $w_0 \rightarrow w_1 \rightarrow w_2 \rightarrow \ldots$, and see if it converges to 0.
A Counterexample (Tsitsiklis and Van Roy, 1996)

From state $s$, let $s'$, $r$ be the (random) next state, reward.
From state $s$, let $s'$, $r$ be the (random) next state, reward.

If our current estimate of $V^\pi$ is $\hat{V}$, the bootstrapping idea suggests $\mathbb{E}_\pi[r + \gamma \hat{V}(s')]$ as a “better estimate” of $V^\pi(s)$.
A Counterexample (Tsitsiklis and Van Roy, 1996)

From state $s$, let $s'$, $r$ be the (random) next state, reward.

If our current estimate of $V^\pi$ is $\hat{V}$, the bootstrapping idea suggests $\mathbb{E}_\pi[r + \gamma \hat{V}(s')]$ as a “better estimate” of $V^\pi(s)$.

Starting with $w = w_0$, we update $w$ so it best-fits the bootstrapped estimate in terms of squared error on the states.
A Counterexample (Tsitsiklis and Van Roy, 1996)

From state $s$, let $s'$, $r$ be the (random) next state, reward.
If our current estimate of $V^\pi$ is $\hat{V}$, the bootstrapping idea suggests $E_\pi[r + \gamma \hat{V}(s')]$ as a “better estimate” of $V^\pi(s)$.
Starting with $w = w_0$, we update $w$ so it best-fits the bootstrapped estimate in terms of squared error on the states. For $k \geq 0$:

$$w_{k+1} \leftarrow \arg\min_{w \in \mathbb{R}} \sum_s \left( E_\pi[r + \gamma \hat{V}(w_k, x(s'))] - \hat{V}(w, x(s)) \right)^2.$$
A Counterexample (Tsitsiklis and Van Roy, 1996)

- From state $s$, let $s'$, $r$ be the (random) next state, reward.
- If our current estimate of $V^\pi$ is $\hat{V}$, the bootstrapping idea suggests $\mathbb{E}_\pi[r + \gamma \hat{V}(s')]$ as a “better estimate” of $V^\pi(s)$.
- Starting with $w = w_0$, we update $w$ so it best-fits the bootstrapped estimate in terms of squared error on the states. For $k \geq 0$:

$$w_{k+1} \leftarrow \arg\min_{w \in \mathbb{R}} \sum_s \left( \mathbb{E}_\pi[r + \gamma \hat{V}(w_k, x(s'))] - \hat{V}(w, x(s)) \right)^2.$$ 

- Is $\lim_{k \to \infty} w_k = 0$?
A Counterexample (Tsitsiklis and Van Roy, 1996)

From state $s$, let $s'$, $r$ be the (random) next state, reward. If our current estimate of $V^\pi$ is $\hat{V}$, the bootstrapping idea suggests $\mathbb{E}_\pi[r + \gamma \hat{V}(s')]$ as a “better estimate” of $V^\pi(s)$.

Starting with $w = w_0$, we update $w$ so it best-fits the bootstrapped estimate in terms of squared error on the states. For $k \geq 0$:

$$w_{k+1} \leftarrow \arg\min_{w \in \mathbb{R}} \sum_s \left( \mathbb{E}_\pi[r + \gamma \hat{V}(w_k, x(s'))] - \hat{V}(w, x(s)) \right)^2.$$ 

Is $\lim_{k \to \infty} w_k = 0$? Let’s see.
A Counterexample (Tsitsiklis and Van Roy, 1996)

\[ w_{k+1} = \arg\min_{w \in \mathbb{R}} \sum_s \left( \mathbb{E}_\pi [r + \gamma \hat{V}(w_k, x(s'))] - \hat{V}(w, x(s)) \right)^2 \]

\[ = \arg\min_{w \in \mathbb{R}} \left( (2\gamma w_k - w)^2 + (2\gamma (1 - \varepsilon) w_k - 2w)^2 \right) = \gamma \frac{6 - 4\varepsilon}{5} w_k. \]
A Counterexample (Tsitsiklis and Van Roy, 1996)

\[ w_{k+1} = \arg\min_{w \in \mathbb{R}} \mathbb{E}_{\pi} \left[ r + \gamma \hat{V}(w_k, x(s')) \right] - \hat{V}(w, x(s)) \]

\[ = \arg\min_{w \in \mathbb{R}} \left( (2\gamma w_k - w)^2 + (2\gamma (1 - \epsilon) w_k - 2w)^2 \right) = \gamma \frac{6 - 4\epsilon}{5} w_k. \]

For \( w_0 = 1, \epsilon = 0.1, \gamma = 0.99, \lim_{k \to \infty} w_k = \infty; \text{ divergence!} \]
A Counterexample (Tsitsiklis and Van Roy, 1996)

\[ w_{k+1} = \arg\min_{w \in \mathbb{R}} \sum_s \left( \mathbb{E}_\pi [r + \gamma \hat{V}(w_k, x(s'))] - \hat{V}(w, x(s)) \right)^2 \]

\[ = \arg\min_{w \in \mathbb{R}} (2\gamma w_k - w)^2 + (2\gamma (1 - \epsilon) w_k - 2w)^2 = \gamma \frac{6 - 4\epsilon}{5} w_k. \]

- For \( w_0 = 1, \epsilon = 0.1, \gamma = 0.99, \lim_{k \to \infty} w_k = \infty; \) divergence!
- The failure owes to the combination of three factors: off-policy updating, generalisation, bootstrapping.
A Counterexample (Tsitsiklis and Van Roy, 1996)

\[ w_{k+1} = \arg\min_{w \in \mathbb{R}} \sum_s \left( \mathbb{E}_\pi [r + \gamma \hat{V}(w_k, x(s'))] - \hat{V}(w, x(s)) \right)^2 \]

\[ = \arg\min_{w \in \mathbb{R}} \left( (2\gamma w_k - w)^2 + (2\gamma (1 - \epsilon) w_k - 2w)^2 \right) = \gamma \frac{6 - 4\epsilon}{5} w_k. \]

- For \( w_0 = 1, \epsilon = 0.1, \gamma = 0.99, \lim_{k \to \infty} w_k = \infty; \text{ divergence!} \)
- The failure owes to the combination of three factors: off-policy updating, generalisation, bootstrapping.
- But these are almost always used together in practice!
### Summary of Theoretical Results

<table>
<thead>
<tr>
<th>Method</th>
<th>Tabular</th>
<th>Linear FA</th>
<th>Non-linear FA</th>
</tr>
</thead>
<tbody>
<tr>
<td>TD(0)</td>
<td>C, O</td>
<td>C</td>
<td>NK</td>
</tr>
<tr>
<td>TD((\lambda), (\lambda \in (0, 1)))</td>
<td>C, O</td>
<td>C</td>
<td>NK</td>
</tr>
<tr>
<td>TD(1)</td>
<td>C, O</td>
<td>C, “Best”</td>
<td>C, Local optimum</td>
</tr>
<tr>
<td>Sarsa(0)</td>
<td>C, 0</td>
<td>Chattering</td>
<td>NK</td>
</tr>
<tr>
<td>Sarsa((\lambda), (\lambda \in (0, 1)))</td>
<td>NK</td>
<td>Chattering</td>
<td>NK</td>
</tr>
<tr>
<td>Sarsa(1)</td>
<td>NK</td>
<td>NK</td>
<td>NK</td>
</tr>
<tr>
<td>Q-learning(0)</td>
<td>C, 0</td>
<td>NK</td>
<td>NK</td>
</tr>
</tbody>
</table>

(C: Convergent; O: Optimal; NK: Not known.)

*: to the best of your instructor’s knowledge.
Reinforcement Learning

1. Tile coding

2. Issues in control with function approximation

3. The case for policy search
(\(m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}}\)) is a “good” approximation of \(Q^*\).
(\(m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}}\)) a “good” approximation of \(Q^*\). But induces non-optimal actions for \(x \in (A, B)\).
(\(m^{\text{RED}}\), \(c^{\text{RED}}\), \(m^{\text{BLUE}}\), \(c^{\text{BLUE}}\)) a “good” approximation of \(Q^*\). But induces non-optimal actions for \(x \in (A, B)\).

(\(\bar{m}^{\text{RED}}\), \(\bar{c}^{\text{RED}}\), \(\bar{m}^{\text{BLUE}}\), \(\bar{c}^{\text{BLUE}}\)) a “bad” approximation of \(Q^*\). But induces optimal actions for all \(x\)!
So Near, Yet So Far

- \((m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}})\) a “good” approximation of \(Q^*\). But induces non-optimal actions for \(x \in (A, B)\).
- \((\bar{m}^{\text{RED}}, \bar{c}^{\text{RED}}, \bar{m}^{\text{BLUE}}, \bar{c}^{\text{BLUE}})\) a “bad” approximation of \(Q^*\). But induces optimal actions for all \(x\)!
- Perhaps we found \((m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}})\) by Q-learning.
So Near, Yet So Far

- $(m_{\text{RED}}, c_{\text{RED}}, m_{\text{BLUE}}, c_{\text{BLUE}})$ a “good” approximation of $Q^*$. But induces non-optimal actions for $x \in (A, B)$.
- $\overline{(m_{\text{RED}}, c_{\text{RED}}), \overline{m_{\text{BLUE}}, c_{\text{BLUE}}}}$ a “bad” approximation of $Q^*$. But induces optimal actions for all $x$!
- Perhaps we found $(m_{\text{RED}}, c_{\text{RED}}, m_{\text{BLUE}}, c_{\text{BLUE}})$ by Q-learning.
- How to find $\overline{(m_{\text{RED}}, c_{\text{RED}}, m_{\text{BLUE}}, c_{\text{BLUE}})}$?
(\(m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}}\)) a “good” approximation of \(Q^*\). But induces non-optimal actions for \(x \in (A, B)\).

(\(\bar{m}^{\text{RED}}, \bar{c}^{\text{RED}}, \bar{m}^{\text{BLUE}}, \bar{c}^{\text{BLUE}}\)) a “bad” approximation of \(Q^*\). But induces optimal actions for all \(x\)!

Perhaps we found \((m^{\text{RED}}, c^{\text{RED}}, m^{\text{BLUE}}, c^{\text{BLUE}})\) by Q-learning.

How to find \((\bar{m}^{\text{RED}}, \bar{c}^{\text{RED}}, \bar{m}^{\text{BLUE}}, \bar{c}^{\text{BLUE}})\)? Next week: policy search.