CS 747, Autumn 2022: Lecture 8

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Autumn 2022

Markov Decision Problems

1. Banach's fixed-point theorem
2. Bellman optimality operator
3. Value iteration

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Complete, Normed Vector Spaces

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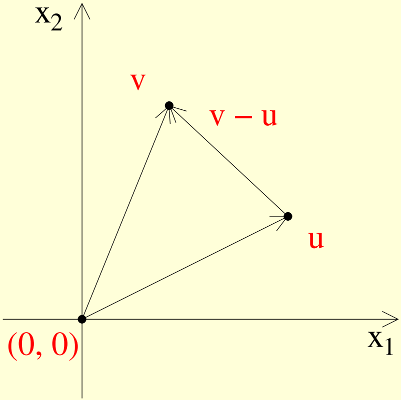

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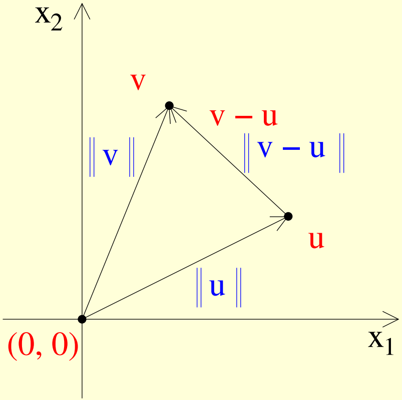

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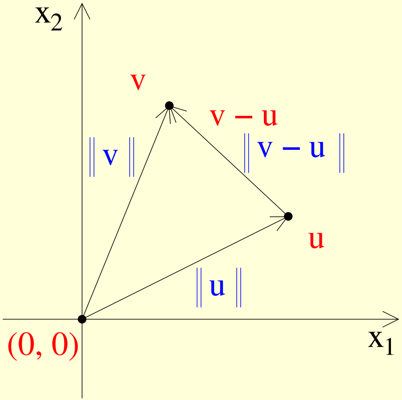

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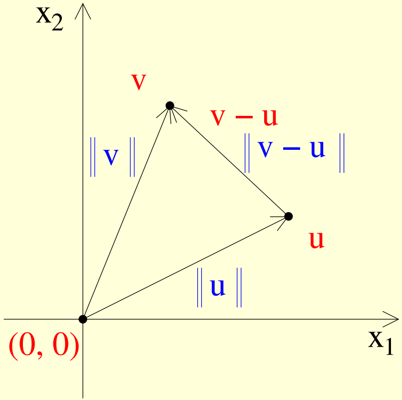

- A complete, normed vector space is called a Banach space.

Two Definitions

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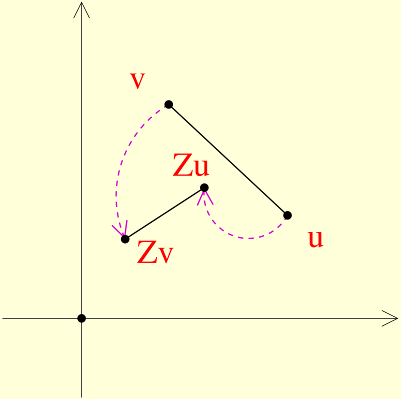

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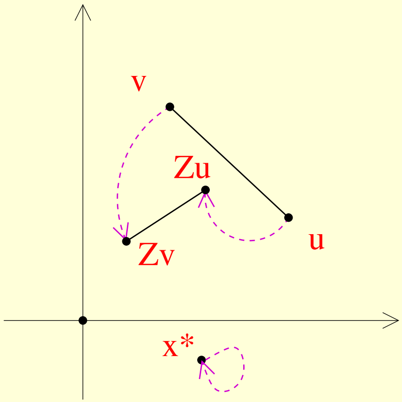

Banach's Fixed-point Theorem

(Adapted from Szepesvári, 2009 (see Appendix A.1).)

Let $(X, \|\cdot\|)$ be a Banach space, and let $Z : X \to X$ be a contraction mapping with contraction factor $\ell \in [0, 1)$. Then:

1. Z has a unique fixed point $x^\star \in X$.
2. For $x \in X, m \geq 0$: $\|Z^m x - x^\star\| \leq \ell^m \|x - x^\star\|$.

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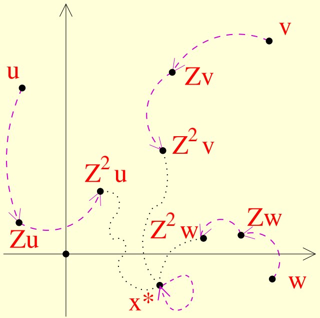

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$$(B^\star(F))(s) \stackrel{\text{def}}{=} \max_{a \in A} \sum_{s' \in S} T(s, a, s')\{R(s, a, s') + \gamma F(s')\}.$$

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> **Fact.** $B^\star$ is a contraction mapping in the $(\mathbb{R}^n, \|\cdot\|_\infty)$ Banach space with contraction factor γ.

Proof that $B^\star$ is a Contraction Mapping

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- **Fact.** $V^\star$ is the value function of every policy $\pi^\star : S \to A$ such that for $s \in S$:

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- We shall prove next week that every such policy $\pi^\star$ is an optimal policy. Hence $V^\star$ is the optimal value function.

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$V_0 \leftarrow$ Arbitrary, element-wise bounded, n-length vector.
$t \leftarrow 0$.
Repeat:
  For $s \in S$**:**
    $V_{t+1}(s) \leftarrow \max_{a \in A} \sum_{s' \in S} T(s, a, s') \left(R(s, a, s') + \gamma V_t(s')\right)$.
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Until $V_t \approx V_{t-1}$ (up to machine precision).

Value Iteration

- Iterative approach to compute $V^\star$.
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- Popular; easy to implement; quick to converge in practice.

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Next class: MDP planning through linear programming.