Neural Networks 1

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This Lecture

- Non-linear decision boundaries
- Neural networks
- Backpropagation

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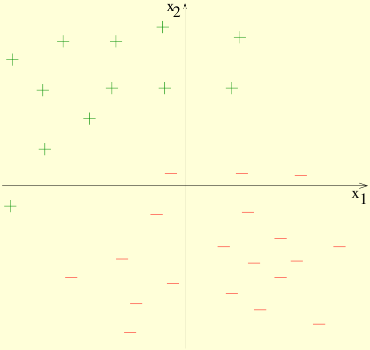
Supervised Learning



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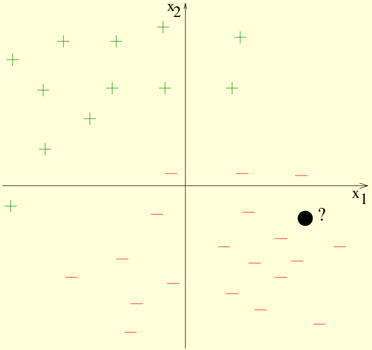

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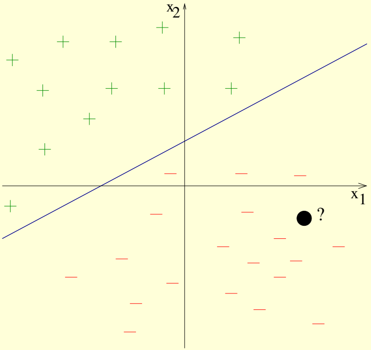

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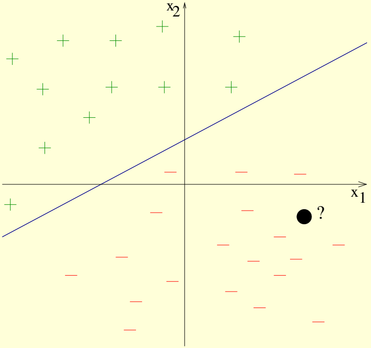

Given labeled training data $\{(x^1, y^1), (x^2, y^2), \ldots, (x^n, y^n)\}$,

Learn a model M such that

for an unseen test point x,

$M(x)$ will be a good prediction of x's (unknown) label y.

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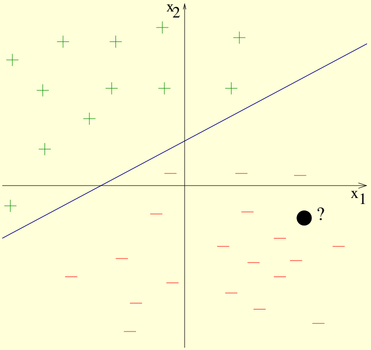

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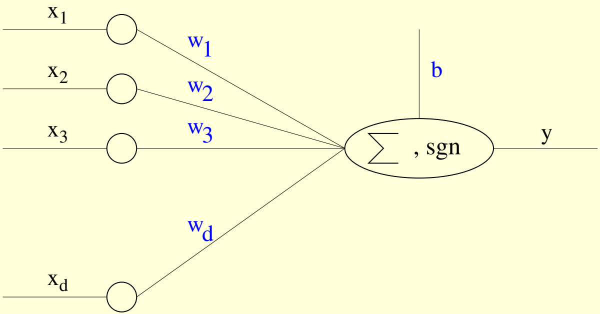
We have seen that if the classes are linearly separable, a perceptron can learn a separating hyperplane.

Perceptron Revisited



$$sgn(\alpha) \stackrel{\text{def}}{=} \begin{cases} 0 & \alpha < 0, \\ 1 & \alpha \geq 0. \end{cases}$$

$$y = \text{sgn}(w_1 x_1 + w_2 x_2 + \cdots + w_d x_d + b).$$

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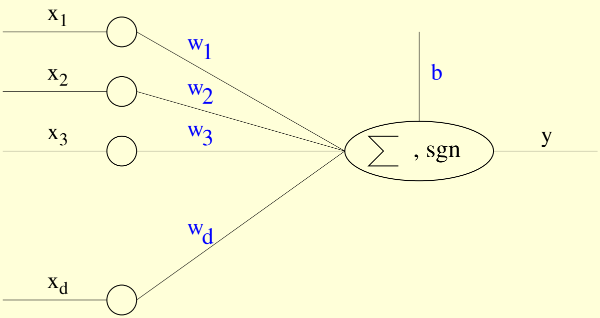

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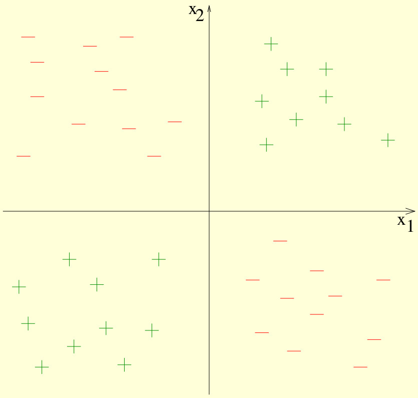
$w = (w_1, w_2, \ldots, w_d)$ and b are parameters learned from data.

The XOR Problem



Minsky and Papert, 1972.

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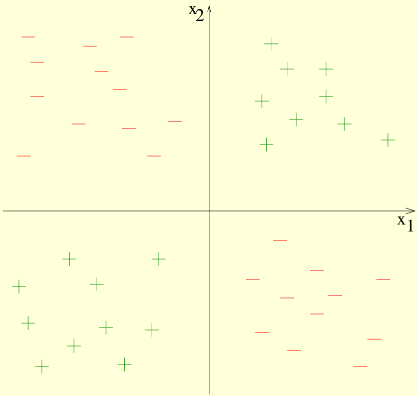

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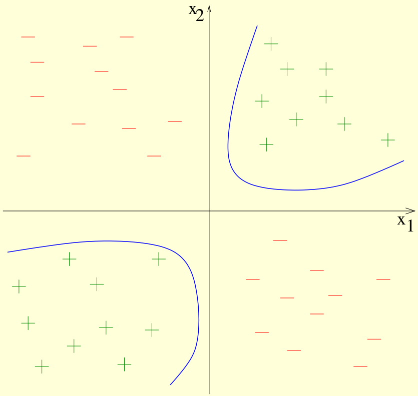

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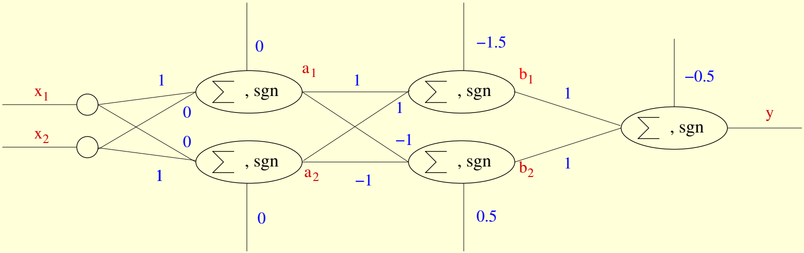

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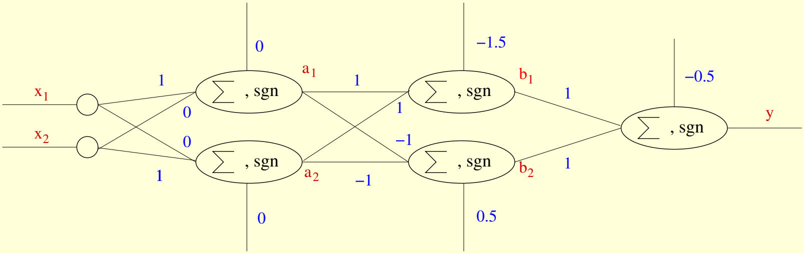

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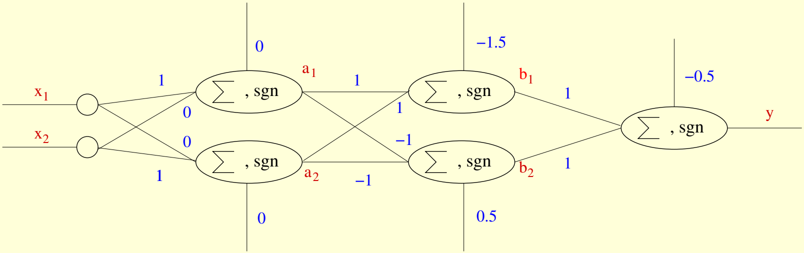

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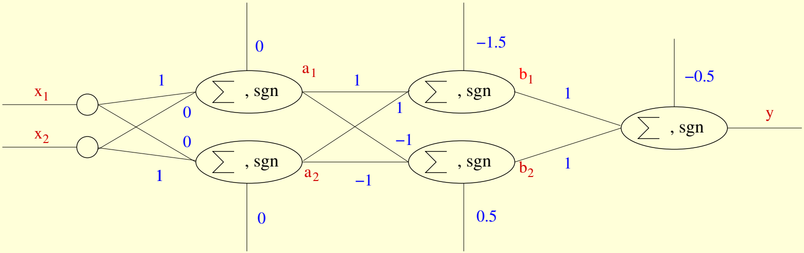

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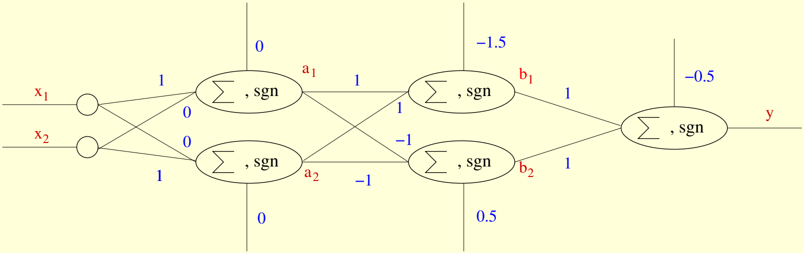

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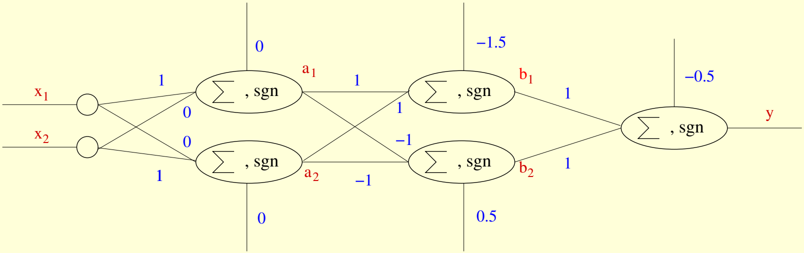

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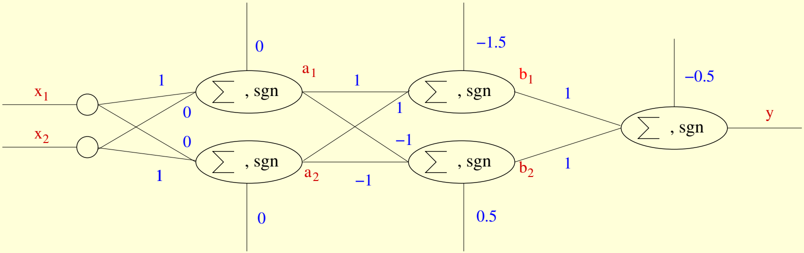

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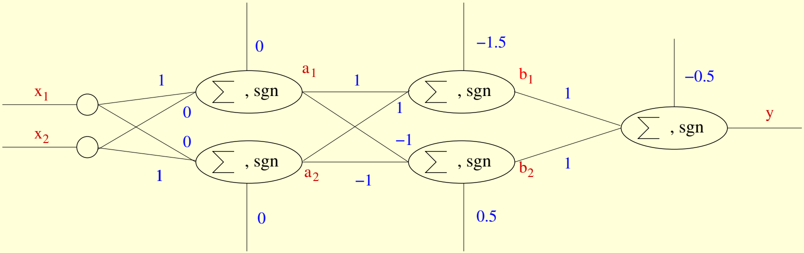

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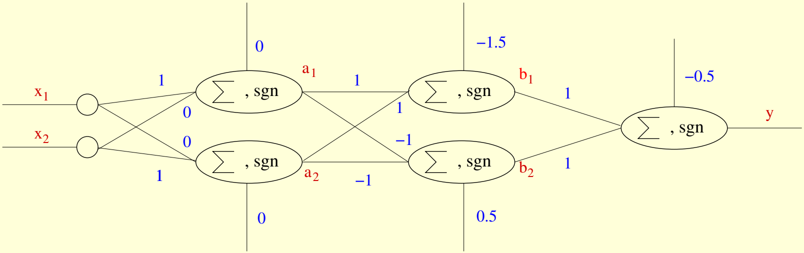

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But we'll make an effort . . . .

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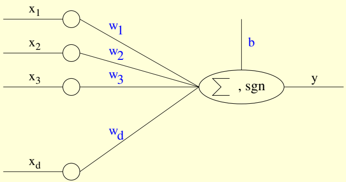
Artificial Neuron

Perceptron



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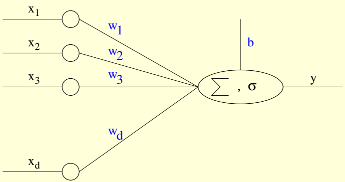
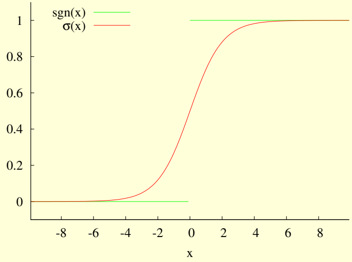
Artificial neuron



$$\sigma(x) \stackrel{\text{def}}{=} \frac{1}{1+\exp(-x)}.$$



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  Since $\sigma(x) = \frac{1}{1+e^{-x}}$, we get

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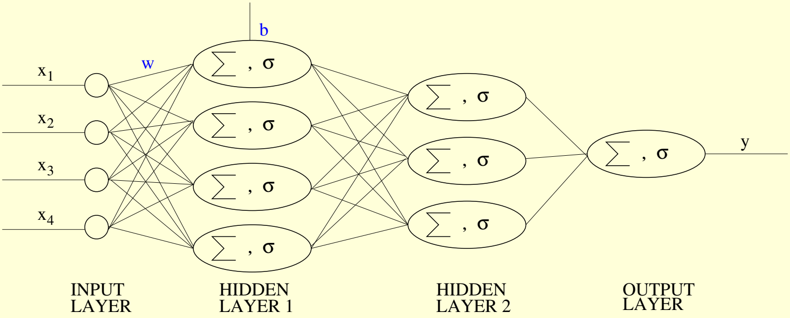
- There are many other choices of activation function.

Artificial Neural Networks



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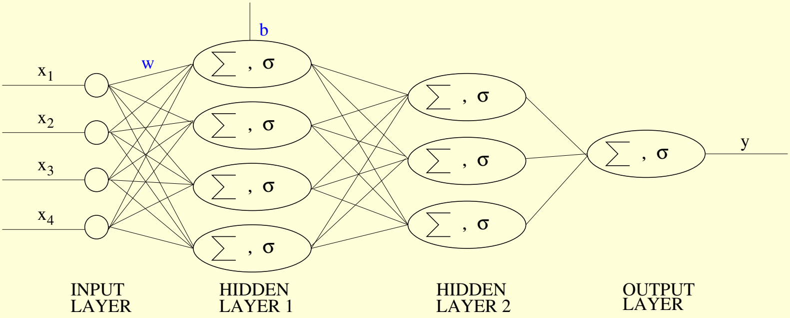

Artificial neural networks are Universal Approximators.

For any function on a finite data set, there exists a single-hidden-layer neural network that fits it exactly (Hornik *et al.*, 1989).

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Backpropagation Algorithm (Rumelhart *et al.*, 1986)

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For each data point r,

the true label is y^r,

the prediction is $p^r(\mathbf{w})$; and

thus, the error is $(y^r - p^r(\mathbf{w}))^2$.

The aggregate error $E(\mathbf{w})$ is $\sum_{r=1}^{n}(y^r - p^r(\mathbf{w}))^2$.
We move a step in the direction minimising error:

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and iterate until convergence.

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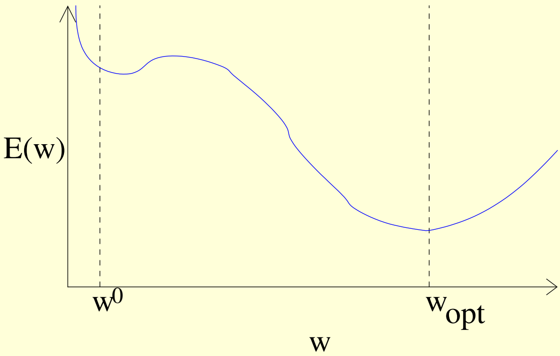
For a given neural network, $\nabla_{\mathbf{w}} E(\mathbf{w})$ can be easily computed (coming up).

Convergence of Backprop

Backprop will converge to a local minimum: it need not find $w_{\text{opt}} = \text{argmin}_w \, E(\mathbf{w})$.



References

- Chapter 10, **A Course in Machine Learning**, Hal Daumé III. Available on-line at `http://ciml.info/`.