Computer-Aided Geometric Design

Problem Sheet

Euclidean Geometry

Let $v = (x, y, z)$ be a typical element of $\mathbb{R}^3$. We define $|v|$ as $\sqrt{x^2 + y^2 + z^2}$. Given two vector $v = (x, y, z)$ and $w = (x', y', z')$, the inner product is defined as $(v, w) = xx' + yy' + zz'$. Vector v is called **orthogonal** to w if $(v, w) = 0$. Vector v is called a **unit vector** if $|v| = 1$.

1. Show that $(v, w) \leq |v||w|$.

2. Recall that the cross product $v \times w$ is defined as the vector

$$det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ x' & y' & z' \end{bmatrix}$$

   Prove that $v \times w$ is orthogonal to both v and w.

3. A Line in $\mathbb{R}^3$ may be represented as a tuple (v, w) with both $v, w \in \mathbb{R}^3$. This denotes the line $v + tw$ where $t \in \mathbb{R}$. (i) Given $L_1 = (v_1, w_1)$ and $L_2 = (v_2, w_2)$, determine if these lines are identical or parallel. (ii) Outline a test to determine if a point p lies on L.

4. Show that if two lines are neither parallel nor intersecting, then there are two parallel planes each containing one of the lines. Is such a pair unique?

5. A plane is given by the equation $ax + by + cz + d = 0$. Let p be a point in $\mathbb{R}^3$. Give an algorithm to find the closest point q on the plane to p.

6. Determine if a line $L = (v, w)$ lies on a plane of the above form.

7. Let P and Q be two planes given by the equations $ax+by+cz+d = 0$ and $a'x+b'y+c'z+d' = 0$. Compute the intersection of P and Q.

8. Consider the vector $w = [1, 2, 3]$. Compute a basis for the space $\{v|(v, w) = 0\}$, of all vectors orthogonal to w.

9. Let $P = \{p_1, \ldots, p_r\}$ be a collection of points. Let $Q = \{q_1, \ldots, q_s\}$ be another collection so that *each* q_i is a convex combination of elements of P. Now let r be a convex combination of elements of Q. Show that r is also a convex combination on P.

10. Let p_1, p_2, p_3 be distinct points. Let T be the collection of all convex combinations of these points. Show that T is a triangle with vertices p_1, p_2, p_3. Show that every point in T has a *unique* expression as a convex combination of the points p_1, p_2 and p_3.

11. Repeat the above problem with 4 points. *Caution*: Some things are different!

12. A 3×3 matrix X is called a **rotation matrix** if $XX^T = I$, the identity matrix. We say that v_1, v_2, v_3 form an **orthonormal** system if all are unit vectors and orthogonal to each other.

    Show that if X is a rotation matrix, then its rows form an orthonormal system. Show the converse. Show that if X, Y are rotation matrices, then so are $XYX^T = XYX^{-1}$ and $X^TYX = X^{-1}YX$.

13. Show that the following matrix $Z(\theta)$ is a rotation matrix.

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Recall that a matrix defines a linear transformation on $\mathbb{R}^3$ treated as row-vectors. Show that $[0,0,1]Z(\theta) = [0,0,1]$, and thus the Z-axis is left invariant by $Z(\theta)$.

14. Construct matrices $X(\theta)$ and $Y(\theta)$ with the appropriate properties. Let R be a rotation matrix with rows v_1, v_2, v_3. Show that $v_3 R^T Z(\theta) R = v_3$. Describe geometrically, this matrix.

15. Let v be a unit vector. Construct a rotation matrix R such that $vR = [1,0,0]$.

Polynomials

1. Show that $T_a = \{1, (t-a)^1, \ldots, (t-a)^n\}$ is a basis for $P^n[t]$.

2. Let p be a polynomial. Suppose that $p(a) = 0$, then argue that $(t-a)$ divides p.

3. Evaluate $\int B_i^n(t)dt$ and $\frac{dB_i^n(t)}{dt}$. What is the maximum value of $B_i^n(t)$ on $[0,1]$?

4. Prove the degree elevation and subdivision formulae given in the class.

5. Construct a cubic polynomial p such that $p(0) = p'(0) = p(1) = 0$ and $p'(1) = 1$. This polynomial is one of the Hermite polynomials.

6. Argue for the linear independence of the Bernstein and the Lagrange Basis. *Hint* for the lagrange basis: show the invertibility of the van der Waerden matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$$

7. Construct $B^3(f)$, the 3-rd degree bernstein approximation to the function $f(t) = t^2$. How close is $B^3(f)$ to f?

8. Do the same for the Lagrange interpolator $L^3(f)$ with $f = t^2$.

Curves and Surfaces

1. Let $f : [-1,1]$ be defined piece-wise as follows:

$$f(t) = \begin{cases} t & t \in [-1,0] \\ \sin t & t \in [0,1] \end{cases}$$

Compute the order of continuity of f at 0.

2. Consider the map $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$t \to \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

Show that image $f(t)$ of any point t lies on the unit circle. Compute $f(0), f(\pm 1), f(\pm 2)$. Is there any point on the unit circle which is **not** an image of f?

3. Let S be a given by an equation $f(X,Y,Z) = 0$. The **gradient** of the function f is given by the sequence of functions
$$\nabla(f) = (\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z})$$
Compute $\nabla(f)$ for the function $X^2 + Y^2 + Z^2 - 1$ at the point $[1,0,0]$.

4. Let ϵ be so small a quantity such that $\epsilon^2 = 0$. Let a surface S be given by $f(X,Y,Z) = 0$ and let $p = (x_0, y_0, z_0)$ be a point on S. Let $q = (x,y,z)$ be another vector. Let us consider all vectors q so that $f(p + \epsilon q) = 0$. Such q will be called **tangent vectors** at p.

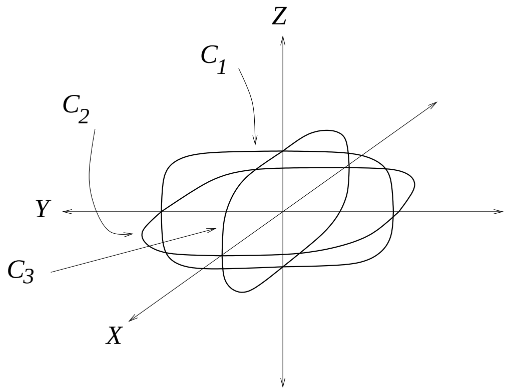
   Compute the space of tangent vectors for the point $[1,0,0]$ on the unit sphere. As an example, we see that $[0,0,1]$ is a tangent since we have $p + \epsilon q = [1,0,\epsilon]$. Substituting this in $X^2 + Y^2 + Z^2 - 1 = 0$, we see that $p + \epsilon q$ does indeed satisfy the equation.

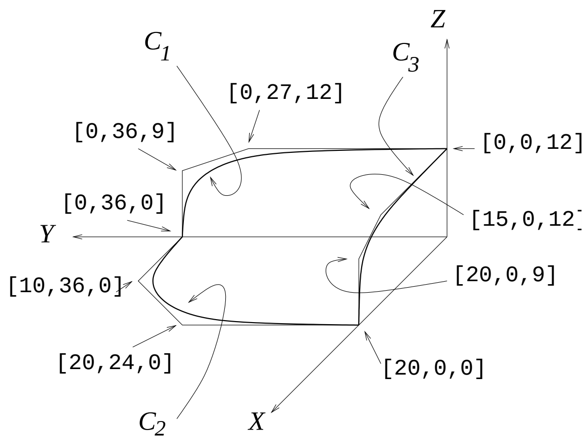
   Compute the tangent plane on a generic point on the cone $X^2 + Y^2 - Z^2 = 0$. Is there any relationship between the tanegnt space at a point and the gradient there?

5. Compute the implicit form for the torus (as a polynomial).

6. Let C be a Bezier Curve with control polygon $P = [p_0, \ldots, p_n]$. Show that an affine transformation (i.e., a translation and/or a homogenous linear transformation) of the curve is obtained by applying the same transformation to the control points.

7. A soap tablet has been specified by its cross-sections





   Construct bezier surface patches to match these specifications.

8. Let P be the set of control points for a cubic bezier curve C as shown below:

p_0	p_1	p_2	p_3
$[0,0,0]$	$[0,1,0]$	$[1,1,0]$	$[2,0,0]$

   (i) Evalute $C(0.5)$ and subdivide C to 0.5.
   (ii) Elevate the degree of C to 4.

9. Consider a quadratic bezier surface given by the following control points:

$[2,0,0]$	$[2,1,0]$	$[2,1,3]$
$[1,0,0]$	$[1,1,0]$	$[1,1,2]$
$[0,0,0]$	$[0,1,0]$	$[0,1,1]$

Compute $S(0.5, 0.5)$. Also elevated the u degree to 3.

10. We are given the knot vector $[0, 0, 0, 2, 3, 3, 3]$, and control points:

p_1	p_2	p_3	p_4	p_5
$[0, 0, 0]$	$[0, 1, 0]$	$[1, 1, 0]$	$[2, 0, 0]$	$[3, 0, 0]$

Evaluate this B-spline at $t = 1$.

Constructions and Operations

1. Let $f(u, v) = u^2 + u + 2v$ and $g(u, v) = v^2 + 2u + v$. Starting from the initial guess of $(1, 1)$, use the Newton-Raphson technique to compute the next two iterations.
2. Formulate a procedure for creating surfaces of revolution.
3. Consider the situation of a drafted exterude where the profile has sharp corners. Describe the geometry/topology near these sharp corners.
4. Argue why the surface line on the belnd surface discussed in the class is indeed so.
5. Given two pints on a unit sphere, derive the parametrization of the great circle passing through it.
6. Let S be the unit cube and let e_1, e_2, e_3 be the edges incident at a vertex. Suppose e_1, e_2 are blended first with radius r and e_3 subsequently with radius R. Describe the geometry of all the surfaces created. Cover the cases when $r < R$ and $r > R$ separately. Describe what happens when this sequence is reversed.