CS728

An Introduction to Geometric Complexity Theory

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Module 1: Group Actions and orbits.

About the course.

1. What are objectives of the course. Prerequisites: Groups, Linear Algebra. Commutative Algebra. Basics of Ideals and Varieties. Artin -Algebra covers all. Lectures and conduct. Taking notes. Exams and quizzes. Reading work.
2. Algorithms and computational complexity. The function as a subroutine. The reduction in computer science. Examples - The flow problem, LP, Hamiltonian circuit. The classes P and NP.
3. The reduction by algebraic substitutions. The singular substitution and the problem of orbit closure.
4. Formula size and Valiant's result of the universality of the determinant, i.e., Combinatorial $\Rightarrow$ Algebraic. What is GCT.

Simple examples of the orbit closure problem.

1. Examples of configurations and actions. 15-puzzle, the elevator, the milk distribution problem, n-ball problems. Some are group problems, most are not. Invertibility and universal applicability.
2. The rubik cube -actions and configurations and a typical question. The necklace problem under the rotation and dihedral group. The definition of the orbit.

Review of groups

1. The basic axioms. The basic examples including $\mathbb{Z}_n, S_n, D_n$ and GL_n, O_n. Homomorpshisms.
2. The group of substitutions $x \to ax + b$. Its subgroup $x \to x + b$.
3. Subgroups generated by elements - the finite and the infinite case. Groups as functions on sets.

Group actions and Orbits

1. Group acting on sets - definition $\rho : G \to Bij(S)$. Examples - D_n and S_n. Definition of the orbit. The equivalence $\sim$ and the partition of S into orbits.
2. Associated actions on $S \times S, 2^S$. The action of S_3 on $S \times S \times S$. Diagonal action on S_1 and S_2. Whyis it important.
3. Stabilizers and examples -D_n, S_n and GL_n.
4. What is the orbit and how does it connect with cosets. The symmetries of the cube and its accounting in different ways.

Orbits and Vector Spaces

1. The vector space $\mathbb{C} \cdot S$ and the action of G as permutations $\rho(g)$. Necklaces as vectors. Orbits for D_n and S_n.
2. Motivating invariants. How do I check if the two necklaces are the "same"? How many distinct necklaces exist? How do I quickly check if two necklaces are the "same"? Functions on $\mathbb{C} \cdot S$ and the need for invariants. $\mathbb{C} \cdot S$ and its dual $X = \{x_1, \ldots, x_n\}$. The combined action on the dual and the primal. Polynomials as functions on necklaces and the action of G on polynomials.

3. The Fourier functions $f_k(x) = \sum_{i=0}^{n-1} x_i \alpha^{ik}$ and their properties. The FFT and proof that the Fourier functions separate orbits for Z_n.

4. The action of G on $\mathbb{C}[S]$ the ring of polynomials. The symmetric group on subsets of a set. And on vectors. The symmetric polynomials. Proof that the symmetric polynomials separate orbits.

5. Polynomials in one variable with the substitution $x \to ax + b$. Why is this a subgroup of GL_2? How does this act on the vector space of polynomials? Writing matrices for this action. Invariants as elimination of variables.

6. The Galilean group on colored points in $\mathbb{R}^2$. Writing matrices for this action. The distance invariants and the determinant for orientation. Exact match and almost exact match. Nearby orbits and the importance of invariants. The question of 2D photographs of 3D objects.

Functions and Invariants

1. The general group action $\rho : G \to GL(V)$, where V is an n-dimensional vector space with coordinate functions $X = \{x_1, \ldots, x_n\}$. Examples from last class. New examples: The adjoint action of GL_2 on 2×2-matrices.

2. Getting the action on $x_1, \ldots, x_n$. The action on a space and its dual. Definition of the group action on functions and its associativity. The action so that the diagonal action of G on $\mathbb{C}[V] \times V$ such that $g(f, v)$ is invariant, in other words $g(f(v)) = f^g(g \cdot v)$. The associativity of $G \times \mathbb{C}[V] \to \mathbb{C}[V]$.

3. The notion of a general invariant $f^g = \phi(g) \cdot f$. Ring of invariants. What does it signify?

4. The big question: Can orbits be separated by invariants? The finite group case. The averaging operation. The separation of points sets by Lagarange interpolation.

Summary

- Group actions. Orbits and stabilizers. Quotients and Invariants.
- The Z_n fourier invariants and the usual invariants. Is it possible to determine the fourier invariants from the usual?

Module 2: Vector Spaces and Maps

Basic definitions

1. The field $\mathbb{C}, \mathbb{R}$ etc. The basic definitions - linear independence and subspaces. The existence of a basis. Dimension. Isomorphism and Homomorphism. Kernel, Image and quotient. The dimension theorem.

2. Choice of basis and the matrix representation. Change of basis and the invertible matrix. The matrix, its row space and column space. The nullspace of a matrix and the rank-nullity theorem. The row echelon form and the expression of a matrix $M = TR$, where R is the row echelon form. The structure of T and R. The equality of dimensions of the row space and column space.

3. Examples: $\mathbb{R}^n$, polynomials - various bases and their importance. Matrices and subspaces with special properties. Lie algebras.

4. Tangent spaces: Two definitions - small movements and derivations (but what are functions on solutions of equations?). Computation of tangent spaces from polynomial equations. The gradient nullity form.

   The dimension of tangent spaces and their significance. Non-singular and singular points as examples. The sphere, SL_n, the orthogonal group, the elliptic curve and the singular cubic.

   Maps between manifolds and the tangent map. The parametrization of the surface of the sphere. The map from rank 1 2×3-matrices to the 3 determinants.

5. The membership problem. Given a subspace $W \subseteq V$ in terms of a basis of V, to answer : Is $w \in W$?. Simplification of a basis of a subspace in terms of another. The LU decomposition. Nested subspaces and dimensions. Classification of subspaces.

The Linear Map

1. $V = M^{n \times r}$ as a left GL_n-module. The orbits and orbit closures. The REF as a section. SL_n and GL_n-orbits. Stabilizer and their dimensions.

2. The basic orbit structure of an SL_n-module. The stable and semi-stable points and the hull-cone. The ring of invariants and what they can separate. Example: binary forms. Writing matrices for binary forms and Sym_n-modules. Writing matrices for the adjoint action.

3. How things change with the GL_n-orbits. The $M^{n\times r}$ again. GL_n-orbit closure and modules as separators.

4. The determinant, basic operations and invariance and its universality. The product law. The proof of invertibility $\Leftrightarrow$ non-singularity. The rank condition for general matrices. The quadratic relations. The matrices for the $\wedge_n$-modules.

5. The orbit structure of $Hom(V, W)$. Analysis of a single linear transformation ϕ. Change of basis and the conjugation action. The Cayley Hamilton theorem and the Jordan representation. The rank closure conditions and filtration of Jordon blocks. Orbits, Hilbert'ss 1-PS criteria (without proof) and Closures. Invariants. Stabilizers and their dimension. The GIT of the conjugate action.

Module 3: Geometric Invariant Theory and Geometric Complexity Theory

The Gordan-Hilbert historical problem

1. The homogeneous action of $G \subseteq GL(X)$ on a general space V and on $\mathbb{C}[V]$. The basic questions - What is the space of orbits? Does the ring of invariants separate orbits? Is the ring of invariants finitely generated. The importance of finite generation.

2. Hilbert's solution for SL_n. The null-cone and the extent of non-separation. The definition of unstable points and Hilbert's 1-PS solution. Stable points and their extent. The GIT structure the of $Sym^k(\mathbb{C}^n)$ and $End(V)$ and the core invariants.

3. Later developments. The semi-stable and the open stable points. The set $null(z)$ of all points which close onto z. The Mumford-Kempf criteria, optimal 1-PS and stabilizers.

Rings and ideals

1. The Ring $\mathbb{C}[V]$. Algebraic sets, ideals and varieties. Correspondence between radical ideals and varieties. maximal ideals. The coordinate ring and $\mathbb{C}[V]/I$. Finite generation and Hilbert basis theorem.

2. The resultant in $R[x]$ and its cases. An example. The easier version of Hilbert's Nullstelensatz. Its consequences. The harder version. The orbit is an almost algebraic set.

3. Dimension and the Jacobian condition. Singularity. The dimension of orbits and the complementarity of stabilizers. Examples.

4. Group actions and the map $\rho^* : \mathbb{C}[V] \to \mathbb{C}[V] \otimes \mathbb{C}[G]$. Its consequences - homogeneity and finite dimensionality of modules.

5. Lie alegbras, their definitions and examples. Lie algebra actions. The computation of $\rho_1 : \mathcal{G} \times V \to V$. Stabilizer conditions. Examples.

Groups and reductivity

1. Algebraic groups as subgroups of $GL(X)$. Computation of the coordinate rings. Examples of GL_n, SL_n, O_n and computation of the ρ^*-map for Sym^d and $\wedge^d$.

2. Definition of reductivity of groups and irreducible modules. Basic categorical properties.

3. The action of reductive groups on $\mathbb{C}[V]$ and the Π^G Reynolds operator.

Geometric Invariant Theory and Geometric Complexity Theory

1. GIT: the fundamental theorems. Finite generation and separation of closed sets. The Nagata equivalence relation. Closed orbits and The statement of the Hilbert-Mumford 1-PS condition. Examples.

2. GCT I: The affine pull-back problem of g from f. The homogenization and the 1-PS formulation. The orbit closure and the witness formulation. The Peter-Weyl condition and the Obstruction Cojecture.

3. GCT II: The action of λ on V and $\mathcal{G}$. The basic equation:
$$\lambda(t)y = t^d y_d + t^e y_e + \ldots + t^D y_D$$
The weight spaces and leading terms. The first theorem: leading term Lie algebra $\hat{\mathcal{H}}$ of $\mathcal{H}$ and module $\hat{N}$ of N. The second theorem: $\lim_{t\to} t^{-d} y(t) = z$ and its implication $\widehat{\mathcal{G}_y} \to \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$ (where $\mathcal{H} = \mathcal{G}_z$).

4. Alignment and the dichotomy result. Consequences of alignment - rectangular decompositon. The absence of alignment and intermediate G-varieties.