MA 782: Algebra and Computation

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Lecture 1

Scribe: Nilabha Saha

1.1 Commutative Algebra

Basics of $\mathbb{C}$-algebras.

1. The motivation - basic orbits ad their equations, orbit closure and separation. Tangent spaces and their analysis. Classical theorems of rings of invariants.
2. Finitely generated $\mathbb{C}$-algebras. The generators and relations as ideals. The concrete presentation. The homomorphism of $\mathbb{C}$-algebras, kernels and image.
3. The variety of a concrete $\mathbb{C}$-algebra A. The variety of $\mathbb{C}[x_1, \ldots, x_n]$. The variety of an ideal. The ideal of a set and Zariski closure. Examples. Basic containment relations. The Hilbert Nullstellensatz.

Ideals and dimension.

1. The generators of an ideal and defining equations.
2. Hilbert basis theorem. Examples and pitfalls of using this as dimension.
3. Prime ideals and maximal ideals and their geometric significance.

1. Separation of closed sets - in differential, algebraic and invariant setting.
2. Partitions of unity and Hilbert's Nullstellensatz theorem.
3. Stable points and semi-stable point. Inability to separate, semistable points from its limits. The main existence theorem of invariant theory.
4. The $GL(X)$ setting and the the stable point 0.

1.2 Actual Lecture Notes

We now study commutative algebra in light of treating orbits as algebraic sets to investigate their closures. We also do it in an attempt to understand the local structure of the orbits. Say $z \in O(z)$, and we want to analyse a neighbourhood of z locally. Yet another motivation is classical invariant theory - to study the structure of generic orbits.

The main motivation of studying commutative algebra is to understand algebraic solutions of equations (called algebraic sets), and maps between algebraic sets.

In $\mathbb{C}^2$, we can look at the hyperbola Hyp defined by solutions to $xy - 1 = 0$. If we try to look at the projection map $\pi : \text{Hyp} \to \mathbb{C}^1 : (x, y) \mapsto x$, we notice a few problems: we don't have 0 in the image. On the other hand, if we have a function f on $\mathbb{C}^1$, then we can pull back the function along $\text{Hyp} \to \mathbb{C}^1$ to get a function $f' : \text{Hyp} \to \mathbb{C}^1$. In terms of a commutative diagram, we get:

$$\begin{tikzcd} \text{Hyp} \arrow[r, "\pi"] \arrow[dr, "f'"'] & \mathbb{C}^1 \arrow[d, "f"] \\ & \mathbb{C}^1 \end{tikzcd}$$

Is every function $\text{Hyp} \to \mathbb{C}^1$ the pullback of some such map?

Definition 1.1. A is a **finitely generated $\mathbb{C}$-algebra** if it is a commutative ring with an injective homomorphism $\mathbb{C} \hookrightarrow A$. We identify $\mathbb{C}$ in A with its image.

Let $a_1, \ldots, a_r \in A$. Then $\mathbb{C}\langle a_1, \ldots, a_r\rangle$ denotes the subset of A obtained from polynoimals in $a_1, \ldots, a_n$ with coefficients from $\mathbb{C}$.

Example 1.2. We look at some example of $\mathbb{C}$-algebras.

1. Let $\mathbb{C}[X_1, \ldots, X_r]$ denote the algebra of all polynomials in indeterminates $X_1, \ldots, X_n$. This is a $\mathbb{C}$-algebra.

2. $(\mathbb{C}^{m\times m}, +, \cdot)$ is a (non-commutative) C-algebra. The identification of $z \in \mathbb{C}$ in the algebra is given by $zI_{m\times m}$.

Definition 1.3. Let A, B be finitely generated $\mathbb{C}$-algebras. Then $\varphi : A \to B$ is a **$\mathbb{C}$-algebra morphism** if for all $a, b \in A, \alpha \in \mathbb{C}$:

(i) $\varphi(ab) = \varphi(a)\varphi(b)$

(ii) $\varphi(a + b) = \varphi(a) + \varphi(b)$

(iii) $\varphi(\alpha a) = \alpha\varphi(a)$

(iv) $\varphi(1) = \varphi(1)$

$\operatorname{img}(\varphi)$ is also a finitely generated subalgebra. In fact,

$$\operatorname{img}(\varphi) = \mathbb{C}\langle\varphi(a_1), \ldots, \varphi(a_r)\rangle$$

Definition 1.4. $I \subseteq A$ is called an **ideal** if

(i) $a, b \in I$ implies $a + b \in I$

(ii) $a \in I$, $b \in A$ implies $ab \in I$

Lemma 1.5. $\ker\varphi$ *is an ideal.*

Proof. Let $\ker\varphi = \{a \in A | \varphi(a) = 0\}$. Then

$$\begin{aligned} a, b \in \ker\varphi &\implies \varphi(a + b) = \varphi(a) + \varphi(b) = 0 \\ &\implies a + b \in \ker\varphi \end{aligned}$$

$$\begin{aligned} a \in \ker\varphi, b \in A &\implies \varphi(ba) = \varphi(b)\varphi(a) = 0 \\ &\implies ba \in \ker\varphi \end{aligned}$$

Thus, $\ker\varphi$ is an ideal. ■

A/I is also a finitely generated algebra. Define $(a + I) + (b + I) = a + b + I$. One can check that this is well-defined.

Let $A = \mathbb{C}[X_1, X_2]$. Let $I = (X_1)$. Now,

$$\mathbb{C}[X_1, X_2]/(X_1) \cong \mathbb{C}[X_2].$$

The map $\mathbb{C}[X_1, X_2] \to \mathbb{C}[X_1, X_2]/(X_1) \cong \mathbb{C}[X_2]$ maps $X_1 \mapsto 0$ and $X_2 \mapsto X_2$.

If $A = \mathbb{C}\langle a_1, \dots, a_r\rangle \to B$, then it is not always possible to find a morphism φ such that $\varphi(a_i) = b_i$ for arbitrarily chosen $b_1, \dots, b_r$.

Consider the map $\varphi : \mathbb{C}[X_1, X_2] \to \mathbb{C}[X_2]$ mapping $X_1 \mapsto 2$ and $X_2 \mapsto X_2$. Then $\ker\varphi = (X_1 - 2)$.

Look at the ideal $I = (XY - 1)$ of $\mathbb{C}[X, Y]$. Then

$$\mathbb{C}[X, Y]/I \cong \bigoplus_{i=1}^{\infty} \mathbb{C}X^i \oplus \bigoplus_{j=1}^{\infty} \mathbb{C}Y^j \oplus \mathbb{C}$$

Now $[f] = [g]$ if and only if f and g agree on the hyperbola.
Now consider space of functions $R = \mathbb{C}[X, Y]/(XY - 1)$ on the hyperbola and consider the projection map $(x, y) \mapsto x$ from Hyp to $\mathbb{C}$. Let the space of functions on $\mathbb{C}$ be denoted by S. Then given any function $f(X) \in S$, we can pull it back to a map in R. This would send $a_0 + a_1X + \cdots + a_rX^r \in S$ to $a_0 + a_1X + \cdots + a_rX^r \in R$. Thus there are more functions on Hyp than on $\mathbb{C}^1$ in some way.

We now look at concretely presented finitely generated algebras. They are algebras A of the form

$$A \cong \mathbb{C}[X_1, \dots, X_r]/I$$

where I is an ideal. In particular, it comes with information on the ideal I.

Theorem 1.6 (Hilbert's Basis Theorem). *All ideals of $\mathbb{C}[X_1, \dots, X_r]$ are finitely generated.*

Proof. Fix the *graded lexicographic order* $\prec$ on monomials in $\mathbb{C}[x_1, \dots, x_r]$ as follows: for monomials $x^\alpha := x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ and $x^\beta := x_1^{\beta_1} \cdots x_r^{\beta_r}$ with $\alpha, \beta \in \mathbb{N}^r$, we set $x^\alpha \prec x^\beta$ iff

$$x^\alpha \prec x^\beta \iff \begin{cases} |\alpha| < |\beta|, & \text{or} \\ |\alpha| = |\beta| \text{ and } \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1},\ \alpha_i < \beta_i & \text{for the first such } i. \end{cases}$$

This is a monomial order (a total well-order compatible with multiplication).
For $0 \neq f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_r]$, define its *leading monomial* $\mathrm{LM}(f)$ to be the largest monomial (with respect to $\prec$) appearing with nonzero coefficient in f, and its *leading term* $\mathrm{LT}(f)$ to be the corresponding term $c_\alpha \mathrm{LM}(f)$. For an ideal $I \subset \mathbb{C}[x_1, \dots, x_r]$, define its *initial ideal*

$$\mathrm{in}(I) := \big(\mathrm{LM}(f) :\ f \in I,\ f \neq 0\big),$$

which is a monomial ideal.
The proof proceeds via two lemmas.

Lemma 1.7. *Every monomial ideal in $\mathbb{C}[x_1, \dots, x_r]$ is finitely generated.*

Proof of Lemma 1.7. Identify monomials with exponent vectors: $x^\alpha \leftrightarrow \alpha \in \mathbb{N}^r$. Endow $\mathbb{N}^r$ with the componentwise partial order: $\alpha \le \beta$ iff $\alpha_i \le \beta_i$ for all i. Let J be a monomial ideal and

$$E := \{\alpha \in \mathbb{N}^r :\ x^\alpha \in J\}.$$

Then E is *upward closed*: if $\alpha \in E$ and $\beta \ge \alpha$, then $x^\beta = x^{\beta-\alpha}x^\alpha \in J$, hence $\beta \in E$.
Claim. Any subset of $\mathbb{N}^r$ has only finitely many minimal elements (with respect to the componentwise order).
Proof of Claim (by induction on r). For $r = 1$ this is clear since $\mathbb{N}$ is well-ordered. Assume true for $r - 1$. Let $S \subseteq \mathbb{N}^r$ and for $k \in \mathbb{N}$ put $S_k := \{(\alpha_1, \dots, \alpha_{r-1}) \in \mathbb{N}^{r-1} :\ (\alpha_1, \dots, \alpha_{r-1}, k) \in S\}$. By the inductive hypothesis, each S_k has finitely many minimal elements; denote that finite set by M_k. Consider

$M := \bigcup_{k\in\mathbb{N}}\{(m,k) : \ m \in M_k\} \subseteq \mathbb{N}^r$. If M were infinite, pick an infinite sequence $(m^{(k)},k) \in M$ with strictly increasing k's. By the inductive hypothesis applied to the sequence $(m^{(k)}) \subset \mathbb{N}^{r-1}$, there exist $i<j$ with $m^{(i)} \le m^{(j)}$, hence $(m^{(i)},i) \le (m^{(j)},j)$ in $\mathbb{N}^r$, contradicting the minimality of $(m^{(j)},j)$ inside S. Thus M is finite, and all minimal elements of S lie in M, so there are finitely many. ■

Let $M(E)$ denote the finite set of minimal elements of the upward closed set E. Then every $\alpha \in E$ dominates some $\mu \in M(E)$, hence $x^\alpha \in (x^\mu : \ \mu \in M(E))$. Therefore $J = (x^\mu : \ \mu \in M(E))$, i.e. J is finitely generated.

Lemma 1.8. *If $I \subset \mathbb{C}[x_1,\dots,x_r]$ and $f_1,\dots,f_s \in I$ satisfy*

$$\mathrm{in}(I) = (\mathrm{LM}(f_1),\dots,\mathrm{LM}(f_s)),$$

then $I = (f_1,\dots,f_s)$.

Proof of Lemma 1.8. We use multivariate division with remainder relative to the fixed monomial order $\prec$.

Division with remainder. Given $g \in \mathbb{C}[x_1,\dots,x_r]$ and polynomials $f_1,\dots,f_s$, there exist polynomials $q_1,\dots,q_s$ and a remainder r such that

$$g = \sum_{i=1}^{s} q_i f_i + r,$$

and either $r=0$, or else no monomial of r is divisible by any $\mathrm{LM}(f_i)$. Moreover, the reduction algorithm terminates because $\prec$ is a well-order on monomials.

Application to I. Let $g \in I$ be arbitrary, and perform the above division of g by $f_1,\dots,f_s$: $g = \sum_i q_i f_i + r$ with remainder r as above. Since I is an ideal and $f_i \in I$, we have $r = g - \sum_i q_i f_i \in I$. If $r \neq 0$, then $\mathrm{LM}(r) \in \mathrm{in}(I) = (\mathrm{LM}(f_1),\dots,\mathrm{LM}(f_s))$, so $\mathrm{LM}(r)$ is divisible by some $\mathrm{LM}(f_i)$, contradicting the defining property of the remainder. Hence $r=0$, i.e. $g \in (f_1,\dots,f_s)$. This shows $I \subseteq (f_1,\dots,f_s)$, and the reverse inclusion is obvious. ■

Conclusion. Let $I \subset \mathbb{C}[x_1,\dots,x_r]$ be any ideal. By Lemma 1.7, the monomial ideal $\mathrm{in}(I)$ admits a finite generating set of monomials. Choose $f_1,\dots,f_s \in I$ whose leading monomials are exactly those generators. Then Lemma 1.8 yields $I = (f_1,\dots,f_s)$. Thus every ideal of $\mathbb{C}[x_1,\dots,x_r]$ is finitely generated. ■

Example 1.9. Consider $\mathbb{C}[X,Y,Z]$ and $I = (Z, X^2+Y^2-1)$. Now, consider the ring

$$R = \frac{\mathbb{C}[X,Y,Z]}{(Z, X^2+Y^2-1)}$$

Note that the common zero set of the polynomials generating the ideal corresponding to the unit circle in the XY-plane. The ring R represents all the (polynomial) functions on the circle. In fact, it can be thought of as the pullback along the projection $(x,y,0) \mapsto (x,y)$ from the circle to the complex plane.

Exercise 1.10. Is there a invertible linear transformation in $\mathbb{C}^3$ taking the solution set of (Z, X^2+Y^2-1) to the solution set of $(Z+X, X^2+Y^2-1)$?

Solution. Yes! In fact, the linear transformation is given by $T : \mathbb{C}^3 \to \mathbb{C}^3$ where

$$T(x,y,z) = (x+z, y, -x)$$

The matrix representation of T is given by

$$M_T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

whose determinant is $-1 \neq 0$, so T is invertible.
Take any solution (x, y, z) in the solution set of (Z, X^2+Y^2-1). We must have $z = 0$ and $x^2+y^2 = 1$. Then $T(x, y, 0) = (x, y, -x) = (X', Y', Z')$. Clearly, $Z' + X' = x - x = 0$ and $(X')^2 + (Y')^2 - 1 = x^2 + y^2 - 1 = 0$. Hence, $T(x, y, z)$ lies in the solution set of $(Z + X, X^2 + Y^2 - 1)$.
Conversely, if (x', y', z') satisfies $Z' + X' = 0$ and $X'^2 + Y'^2 = 0$, then setting $x = x'$, $y = y'$, and $z = 0$, we have that $T(x, y, z) = (x', y', z')$ with $z = 0$ and $x^2 + y^2 - 1 = x'^2 + y'^2 - 1 = 0$. Thus, T is a linear isomorphism mapping the solution set of $(Z, X^2 + Y^2 - 1)$ onto the solution set of $(Z + X, X^2 + Y^2 - 1)$.

Let V be the vector space of 2×3 matrices with entries from $\mathbb{C}$. Let $V \ni X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \end{bmatrix}$.
Distinguishing elements of V by rank, we have $V = V_0 \sqcup V_1 \sqcup V_2$. Let $V_{0,1}$ be all matrices of rank 1 or less $\subseteq \mathbb{C}^{2\times 3}$. Let $\delta_{12} = X_{11}X_{22} - X_{12}X_{21}$ and $\delta_{23} = X_{12}X_{23} - X_{13}X_{22}$. Then $V_{0,1} = \mathbb{C}[X_{11}, \dots, X_{23}]/(\delta_{12}, \delta_{23})$.
Let $\delta_{13} = X_{11}X_{23} - X_{21}X_{13}$. Then $V_2 \cong \mathbb{C}[X_11, \dots, X_{23}]/(\delta_{12}, \delta_{23}, \delta_{13})$. In fact, we have

$$\delta_{12}\delta_{23} + \delta_{13}\delta_{23} - \delta_{12}\delta_{13} = 0.$$

To delve further in this direction, look up Hilbert's Syzgy Theorem.

Any ideal in $\mathbb{C}[X_1]$ is generated by a single element; hence, $\mathbb{C}[X_1]$ is a principal ideal domain.
Given an ideal $I \trianglelefteq \mathbb{C}[X_1, \dots, X_r]$, define $V(I) = \{(\alpha_1, \dots, \alpha_r) | f(\alpha_1, \dots, \alpha_r) = 0 \ \forall f \in I\}$.

Theorem 1.11 (Hilbert's Nullstellensatz). *For any ideal $I \trianglelefteq \mathbb{C}[X_1, \dots, X_r]$,, either $1 \in I$ or $V(I) \neq \emptyset$.*

Proof. Let $I \trianglelefteq \mathbb{C}[x_1, \dots, x_r]$. If $1 \in I$ we are done, so assume $1 \notin I$. Then by Zorn's lemma there exists a maximal ideal $\mathfrak{m}$ with $I \subseteq \mathfrak{m} \subsetneq \mathbb{C}[x_1, \dots, x_r]$. Consider the residue field

$$L := \mathbb{C}[x_1, \dots, x_r]/\mathfrak{m}.$$

Since $\mathfrak{m}$ is maximal, L is a field. Moreover, L is generated as a $\mathbb{C}$-algebra by the images $\overline{x}_1, \dots, \overline{x}_r$ of the coordinate functions, so L is a finitely generated $\mathbb{C}$-algebra.

We will use the following lemma.

Lemma 1.12. *If k is a field and L is a field that is finitely generated as a k-algebra, then L is an algebraic (hence finite) field extension of k.*

Proof of Lemma 1.12. Suppose, toward a contradiction, that L has positive transcendence degree over k. Then there exists a subfield $k(t) \subseteq L$ with t transcendental over k. Choose elements $a_1, \dots, a_n \in L$ such that $L = k\langle t, a_1, \dots, a_n\rangle$ as a k-algebra. Write each a_i as a rational function in t with coefficients in a subfield of L containing k: say $a_i = \frac{p_i(t)}{q_i(t)}$ with $p_i, q_i \in k[t]$ and $q_i \neq 0$. Let $Q(t) := \prod_{i=1}^n q_i(t) \in k[t] \setminus \{0\}$. Then the k-subalgebra generated by $t, a_1, \dots, a_n$ is contained in

$$k[t,\, 1/Q(t)] \;=\; \left\{ \frac{f(t)}{Q(t)^m} \;:\; f \in k[t],\; m \geq 0 \right\}.$$

In particular, every element of this subalgebra has denominator a power of $Q(t)$ when written in lowest terms.
However, the field L contains the element $\frac{1}{Q(t)+1}$ (since L is a field and $Q(t) + 1 \neq 0$ in $k(t)$), but $\frac{1}{Q(t)+1} \notin k[t, 1/Q(t)]$ because any element of $k[t, 1/Q(t)]$ has denominator dividing a power of $Q(t)$, whereas $Q(t)$ and $Q(t) + 1$ are coprime in $k[t]$. This contradiction shows that $\operatorname{trdeg}_k L = 0$. Hence L is algebraic over k, as claimed. ■

Applying the lemma with $k = \mathbb{C}$ yields that L is algebraic over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, it follows that $L \cong \mathbb{C}$.
Therefore, the quotient map $\pi : \mathbb{C}[x_1, \ldots, x_r] \twoheadrightarrow L \cong \mathbb{C}$ is a surjective k-algebra homomorphism, so it is evaluation at some point $a = (a_1, \ldots, a_r) \in \mathbb{C}^r$:

$$\pi(f) = f(a) \qquad \text{and} \qquad \ker \pi = \mathfrak{m} = (x_1 - a_1, \ldots, x_r - a_r).$$

Because $I \subseteq \mathfrak{m} = \ker \pi$, every $f \in I$ satisfies $f(a) = 0$. Thus $a \in V(I)$.
Since we assumed $1 \notin I$, we have produced a point $a \in \mathbb{C}^r$ with $f(a) = 0$ for all $f \in I$, i.e. $V(I) \neq \varnothing$. This proves the theorem. ■

Example 1.13. Consider $X = \frac{1-t^2}{1+t^2}$ and $Y = \frac{2t}{1+t^2}$. Now, $Y^2 + t^2Y - 2t = 0$ and $X^2 + t^2X - 1 + t^2 = 0$. Eliminating t, we get $2(1+X)t - (1+Y) = 0$. Making this substitution, we get $X^2 + Y^2 - 1 = 0$.

Definition 1.14. We say that $X \subseteq \mathbb{C}^n$ is an **algebraic set** if X is dense in $V(I(X))$.

Example 1.15.

(i) Neither $\mathbb{Q}$ nor $\mathbb{Z}$ is an algebraic set.
Recall the elementary fact that a nonzero polynomial over $\mathbb{C}$ has only finitely many zeros.

Claim. If $X \subseteq \mathbb{C}$ is infinite, then $I(X) = (0)$.

Reason. If $0 \neq f \in \mathbb{C}[x]$, then f has at most $\deg f$ zeros, hence cannot vanish on an infinite set X. Thus the only polynomial vanishing on all of X is 0.

Apply the claim to $X = \mathbb{Q}$ and to $X = \mathbb{Z}$. In each case, $I(X) = (0)$, so

$$V\big(I(X)\big) = V(0) = \mathbb{C}.$$

By the given definition, X would be an algebraic set only if X is dense (in the Euclidean topology) in $V(I(X)) = \mathbb{C}$.

But $\overline{\mathbb{Q}}^{\mathbb{C}} = \mathbb{R} \neq \mathbb{C}$ (the closure of $\mathbb{Q}$ in $\mathbb{C}$ is $\mathbb{R}$), and $\overline{\mathbb{Z}}^{\mathbb{C}} = \mathbb{Z} \neq \mathbb{C}$ (indeed, $\mathbb{Z}$ is closed and discrete). Hence neither $\mathbb{Q}$ nor $\mathbb{Z}$ is dense in $\mathbb{C}$.

Therefore, neither $\mathbb{Q}$ nor $\mathbb{Z}$ is an algebraic set.

(ii) Let $X = V(y - e^x)$. If $p(x, y)$ vanishes in X, then $p(x, e^x)$ vanishes. This is also not an algebraic set. Explain why.

Note that if $I \subseteq J$, then $V(I) \supseteq V(J)$. In general, $I \subseteq I(V(I))$.
Consider $(Y, Y - X^2 + 1)$. This leads to two points $(\pm 1, 0)$. Let $I_m = V(\{(-1, 0), (1, 0)\})$. Then $\dim\left(\frac{\mathbb{C}[X,Y]}{I_m}\right) = 2$. Now let us look at $I_2 = (Y, Y - X^2)$. Now $V(I_2) = \{(0, 0)\}$. Now, $I(V(I_2)) = (X, Y)$. Now $\dim\left(\frac{\mathbb{C}[X,Y]}{(X,Y)}\right) = 1$.
Consider the ideal $I_3 = (X^2 - 1, X^2 + Y^2 - 1)$. As an exercise, investigate its behaviour.
IF $I_1 \subseteq I_2$, then $V(I_1) \supseteq V(I_2)$. As an exercise, show that $\frac{\mathbb{C}[X_1, \ldots, X_r]}{I_1} \twoheadrightarrow \frac{\mathbb{C}[X_1, \ldots, X_r]}{I_2}$ is surjective.

Exercise 1.16. Suppose $X = V(I)$ and $Y = V(J)$ and $X \cap Y = \emptyset$. Show that there exist f_X, f_Y where $f_X + f_Y = 1$, $f_X(X) = 1, f_X(Y) = 0$ and $f_Y(X) = 0, f_Y(Y) = 1$.

Solution. Note that $V(I + J) = V)I) \cap V(J) = X \cap Y = \emptyset$ which implies that $I + J = (1)$. Hence, $\exists f_Y \in I, f_X \in J$ such that $f_X + f_Y = 1$. Note that $f_Y \in I \implies f_Y(X) = 0$ and thus $f_X(X) = 1$ since $f_X + f_Y = 1$. Similarly, we have that $f_X(Y) = 0$ and $f_Y(Y) = 1$.

For the next exercise, we shall make use of the following result from elementary commutative algebra.

Theorem 1.17 (Chinese Remainder Theorem). *Let R be a commutative ring and let $I_1, \ldots, I_n$ be pairwise comaximal ideals of R, that is, $I_i + I_j = R$ whenever $i \neq j$. Then the map*

$$\varphi : R \to \prod_{i=1}^{n} R/I_i$$

is a surjection with kernel $\bigcap_{i=1}^{n} I_i$.

Exercise 1.18. Suppose $X_i = V(I_i)$ for $i \in \{1, \ldots, n\}$, and $X_i \cap X_j = \emptyset$ for $i \neq j$. Show that there exist polynomials $f_1, \ldots, f_n$ such that $\sum_{i=1}^{n} f_i = 1$ such that

$$f_i(X_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Lecture 2

Scribe: Someone

Objective: Given $I = (f_1, \ldots, f_r) \subseteq \mathbb{C}[x_0, \ldots, x_n]$ and $X = V(I)$. let $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection $(\beta, \alpha_1, \ldots, \alpha_n) \to (\alpha_1, \ldots, \alpha_n)$. Let $Y = \pi(X)$. To show that there is an ideal $J \subseteq \mathbb{C}[x_1, \ldots, x_n]$ and a $h \in \mathbb{C}[x_1, \ldots, x_n]$ such that for all $\alpha in V(J) - V(h)$, there is a β such that $(\beta, \alpha) \in V(I)$. We will prove when $f_1, \ldots, f_r)$ are monic.

2.1 The ring $R[z]$

Let us rename $\mathbb{C}[x_0, \ldots, x_n]$ as $R[z]$ with $x_0 = z$ and $R = \mathbb{C} = [x_1, \ldots, x_n]$.

1. Definition: A *monic* polynomial f is of the form $z^n + a_1 z^{n-1} \ldots + a_n$, with $a_i \in R$.
2. Let $\nabla f = (\partial f/\partial z, (\partial f/\partial x_i))$. Then where $\partial f/\partial z \neq 0$, and given $(\beta, \alpha$ such that $f(\beta, \alpha) = 0$ there is an *analytic* function $h(x)$ such that locally to α, $z = h(x)$ is a solution for f. Thus outside a closed algebraic set, z is a function of x in the neighborhood of a point in $V(f)$.
3. Any finite set $f_1, \ldots, f_r$ can be made monic by a change of basis.
4. The variety $V(I)$ when $I = (f_1, \ldots, f_r)$. Given $(\alpha_1, \ldots, \alpha_n)$, there is a $\beta \in \mathbb{C}$ such that $(\beta, \alpha) \in V(I)$ iff $f_i(z, \alpha)$ have a common divisor $(z - \beta)$.
5. If f is monic and $g \in R[z]$ and g divides f then g is monic.
6. **The non-monic case**: If $f = a_0 z^n + \ldots + a_n z^0$. Then for $\alpha \in (\mathbb{C}^n - V(a_0))$ we still have the n-sheeted covering.

We now show that: Given $f, g \in R[z]$, such that they dont have a common divisor of positive degree, there is a function $h, D \in R$ such that (i) for all points $\alpha \in V(h) - V(D)$, there is a β such that $(\beta, \alpha) \in V(f, g)$.

1. Let us consider the monic case. The definition of $h = Res(f, g)$ and that $h \neq 0$. $Res(f, g) = 0 \Leftrightarrow pf = qg$ in degree less than $d + e$ and thus f, g have a common factor. Thus $f \in R \neq 0$. Outside $V(h)$, there is no pullback. Inside $V(h)$, there is a pullback. Hence $\pi(X) = V(h)$.
2. The case of $I = (f, g_1, \ldots, g_r)$. Let $u_1, \ldots, u_r \in \mathbb{C}$ and let $g_u = u_1 g_1 + \ldots + u_r g_r$. Note that g_u is monic of all g's are. Let $h_u = Res(f, g_u)$. Let $I = (h_u)_{u \in \mathbb{C}^n}$. Then J is finitely generated and $V(J)$ contains all α such that there is a β such that $(\beta, \alpha) \in V(I)$. The converse is also true.

The general case when f, g are not monic and do not have a common factor.

1. An example. The parametric circle example at the point when $t = \infty$.
2. The variety $V(I)$ when $I = (f_1, \ldots, f_r)$. Let $f_i = a_0^i z^n + \ldots + a_n^i$. Let $\{a_0^i\}$ be the collection of leading terms and let $a = \prod_i a_0^i$. Then outside $V(a)$, given $(\alpha_1, \ldots, \alpha_n)$, there is a $\beta \in \mathbb{C}$ such that $(\beta, \alpha) \in V(I)$ iff $f_i(z, \alpha)$ have a common divisor $(z - \beta)$.
3. Let us look at $r = Res(f, g)$. Either (i) $r \in R$ or (ii) f, g have a common factor. Let us look at (i). Outside $V(a) \cup V(r)$, they do not have a common root. So $\pi(X) \subseteq V(R) \cup V(a)$. In $(V(r) - V(a))$, f and g do have a common zero.
4. So the question remains, what happens inside $V(a)$. So suppose $a_0 = 0$. We have two polynomials f', g and start the process again. '

What happens when f, g have a common factor?

1. Let $h \in R[z]$ divide both f and g. Then $V(f) \supseteq V(f), V(g)$. Then $\pi(X) \supseteq \mathbb{C}^n - V(a)$, where a is the leading term of h.
2. In general, given $f, g_1, \ldots, g_r$, look at f, g_u. Show that if f, g_u have a common factor for every u, then all have a common factor. Otherwise, the earlier reasoning will apply.

Lecture 3

Scribe: Someone

Ideals and Maximal Ideals: Points on a variety.

1. Let A, B be finitely generated $\mathbb{C}$-algebra and $\phi : A \to B$ be a surjection. Then $ker(\phi)$ is an ideal and $B \cong A/ker(\phi)$.
2. Let A, B be as before. Let $J \subseteq B$ be an ideal, then $J' = \phi^{-1}(J)$ is an ideal of A and $A/J' \cong B/J$.
3. If A has no non-trivial ideals iff A is a field.
4. Let $\phi : A \to \mathbb{C}$ be a morphism then $ker(\phi)$ is a maximal ideal.
5. Let $v \in \mathbb{C}^n$, consider $I_v = (x_1 - v_1, \ldots, x_m - v_m) \subseteq \mathbb{C}[x_1, \ldots, x_m]$. Then I_v is a maximal ideal. Every maximal ideal is of this form. Indeed, for any v, let $eval_v : \mathbb{C}[x_1, \ldots, x_m] \to \mathbb{C}$ be defined as $eval_v(f) = f(v)$. Then $eval_v$ is an algebra isomorphism and its kernel the maximal ideal I_v.
6. Let A be an algebra. Define $Spec(A)$ as the collection of maximal ideals in A. Let $J \subseteq \mathbb{C}[x_1, \ldots, x_m]$ be such that $A \cong \mathbb{C}[x_1, \ldots, x_m]/J$. Let $ISpec(J)$ be the collection of maximal ideals of $\mathbb{C}[x_1, \ldots, x_m]$ containing J. Then $ISpec(J) \cong V(J) \cong Spec(A)$.
7. Let $V = \mathbb{C}[X]/I$. Every point $v \in V$ gives us a map $eval_v : \mathbb{C}[V] \to \mathbb{C}$ with $eval_v(f) = f_v$. Thus $ker(eval_v)$ is a maximal ideal containing I.

Surjections and Injections.

1. Every map $\phi : A \to C$ is factorizable as a surjection $\phi : A \to B \subseteq C$, followed by an injection $i : B \to C$.
2. Every surjection $\phi : A \to B$ is given by two ideals $I \subseteq J \subset \mathbb{C}[x_1, \ldots, x_m]$ and the morphism $A/I \to A/J$. Thus $V(J) \cong Spec(B) \subseteq Spec(A) \cong V(I)$.
3. Given an injection $B \to C$, every maximal ideal I of C restricts to a maximal ideal of B. But the converse is not true. So $Spec(C) \subseteq Spec(B)$.
4. Examples. $\mathbb{C}[x, y] \to \mathbb{C}[x] \to \mathbb{C}[x, x^{-1}]$.

Maps between algebraic varieties.

1. Let $V \subseteq \mathbb{C}^m$ with $V = V(I)$ and $I \subseteq \mathbb{C}[x_1, \ldots, x_m] = A$ and $W \subseteq \mathbb{C}^n$, $W = V(J)$, $J \subseteq \mathbb{C}[y_1, \ldots, y_n] = B$. Let $\mathbb{C}[V] = A/I$ and $\mathbb{C}[W] = B/J$. The map $\phi : V \to W$ is given by algebraic functions. So if $\phi(p) = q$, and y_i is a coordinate function, then $\phi^*(y_i) = f_i(x)$, where $f_i \in \mathbb{C}[V]$. This indicates that $y_i(q) = f_i(p)$. Notice that that map is specified as if $\phi' : \mathbb{C}^m \to \mathbb{C}^m$ and so a map $\phi'^* : B \to A$.
2. Examples. Evaluate ϕ^* in each case.
   (a) Let $V = \mathbb{C}^3, W = \mathbb{C}^2$ and $\phi_1(p_1, p_2, p_3) = (p_1, p_2)$. Then $\phi^*(y_i) = x_i$. These functions are constant on the pre-image.
   (b) Let $V = V(x_1^2 + x_2^2 - 1)$, the cylinder. W and ϕ be as before.
   (c) Let $V = V(x_1 x_2 - 1) \subseteq \mathbb{C}^2$ and ϕ be projection to X-axis.
   (d) Let $V \subseteq \mathbb{C}^{2\times 3}$ of matrices of rank 1 or less. Let $\phi : X \to \mathbb{C}^3$, mapping to the first row.
3. For the general case $V \subseteq \mathbb{C}^m$ and $W \subseteq \mathbb{C}^n$ and given by $y_i = f(X_i)$. We have the map $\phi'^* : B \to A$ given by $\phi'^*(y_i) = f_i(x)$. This is always a algebra homomorphism!. When does it become a map $\phi^* : B/J \to A/I$? How do we check well-defined? Firstly, $\phi^*(J) = 0$. In other words, if $g(y) \in J$, then is $g(f(x)) \in I$?
4. Let $Im(\phi'^*) = B'$ and consider the ideal I and $I' = B \cap I$. Check that it is also an ideal of B'. Next, $\phi'^*(J) = J'$ is also an ideal of B'. The condition is that $I' \subseteq J'$.

5. Example: $I = ((t^2+1)y - 2t, (t^2+1)x - 1 + t^2) \subseteq \mathbb{C}[x,y,t]$. Let $J = (x^2+y^2-1) \subseteq \mathbb{C}[x,y]$. The map is projection. Compute ϕ^* and check is this map is proper.

Ideals and Maximal Ideals: Points on a variety.

1. Let $A = \mathbb{C}[X]/I$ and $\phi : A \to B$ be an algebra map. Then $im(\phi) \cong \mathbb{C}[X]/ker(\phi)$. Thus, maps of A are isomorphic to ideals J containing I.
2. Let $v \in \mathbb{C}^n$, consider $I_v = (x_1 - v_1, \ldots, x_m - v_m)$. Then I_v is a maximal ideal. Every maximal ideal is of this form. Let J be a maximal ideal and $v \in V(J)$. Then $I_v \supseteq J$ and therefore must be equal to J.
3. Let $V = \mathbb{C}[X]/I$. Every point $v \in V$ gives us a map $eval_v : \mathbb{C}[V] \to \mathbb{C}$ with $eval_v(f) = f_v$. Thus $ker(eval_v)$ is a maximal ideal containing I.

The variety $V \times W$.

1. Let $\mathbb{C}[V] = A/I$ and $\mathbb{C}[W] = B/J$. The set $V \times W = \{(v,w) | v \in V, w \in W\}$. The construction: Let $\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_m]$ and let $L = I + J \subseteq C$. Look at $X = C/L$. There are projections $\pi_V : W \to V$ and $\pi_W : X \to W$. These maps are surjective and the preimage $\pi_V^{-1}(v) = v \times W$.
2. Let $\phi : V \to W$. Let us construct the set $X_\phi = \{(v, \phi(v) | v \in V\}$. Then X_ϕ is called the graph of ϕ. Then $X(\phi)$ is an algebraic set.
3. Next $\pi_W(X_\phi) = Im(\phi)$ is also an algebraic set. **Ex.**: Show that the preimage of a point $w \in W$ is an algebraic set.

Ideals and Maximal Ideals: Points on a Variety

Proposition 3.1. *Let A, B be finitely generated $\mathbb{C}$-algebras and $\varphi\colon A \to B$ a surjective algebra homomorphism. Then* $\ker(\varphi)$ *is an ideal of A and*

$$B \cong A/\ker(\varphi).$$

Proof. The kernel $\ker(\varphi) = \{a \in A : \varphi(a) = 0\}$ is an ideal by the usual properties: if $a, b \in \ker(\varphi)$ then $\varphi(a+b) = \varphi(a) + \varphi(b) = 0$, and for any $r \in A$ and $a \in \ker(\varphi)$ we have $\varphi(ra) = \varphi(r)\varphi(a) = \varphi(r) \cdot 0 = 0$. Because φ is surjective, the First Isomorphism Theorem for rings (or $\mathbb{C}$-algebras) gives an induced isomorphism

$$\widetilde{\varphi} : A/\ker(\varphi) \xrightarrow{\cong} B, \qquad a + \ker(\varphi) \mapsto \varphi(a).$$

This map is well-defined, bijective and an algebra homomorphism; hence $B \cong A/\ker(\varphi)$. ■

Proposition 3.2. *Let $\varphi\colon A \to B$ be a homomorphism of finitely generated $\mathbb{C}$-algebras and let $J \subseteq B$ be an ideal. Set $J' = \varphi^{-1}(J)$. Then J' is an ideal of A and there is a natural isomorphism*

$$A/J' \cong (\varphi(A))/J \cong B/J \quad \textit{when } \varphi \textit{ is surjective.}$$

Proof. Preimage of an ideal under a ring homomorphism is an ideal: for $a, b \in J'$, $\varphi(a+b) = \varphi(a) + \varphi(b) \in J$, so $a + b \in J'$, and for $r \in A$, $\varphi(ra) = \varphi(r)\varphi(a) \in J$, so $ra \in J'$.
The homomorphism φ descends to a map $\bar{\varphi}\colon A/J' \to B/J$ defined by $\bar{\varphi}(a + J') = \varphi(a) + J$. The kernel of $\bar{\varphi}$ is zero, because $a + J' \in \ker \bar{\varphi}$ iff $\varphi(a) \in J$, i.e. $a \in \varphi^{-1}(J) = J'$. Thus $\bar{\varphi}$ is injective. If φ is surjective then $\bar{\varphi}$ is also surjective, so $A/J' \cong B/J$. ■

Proposition 3.3. *A (commutative) $\mathbb{C}$-algebra A has no nontrivial ideals (i.e. its only ideals are* (0) *and A) if and only if A is a field.*

Proof. ($\Rightarrow$) Suppose A has only the trivial ideals. Take any nonzero $a \in A$. The principal ideal (a) is an ideal, so by hypothesis $(a) = A$. Thus there exists $b \in A$ with $ba = 1$. Hence a is invertible and every nonzero element of A is a unit, so A is a field.
($\Leftarrow$) If A is a field, any nonzero ideal $I \subset A$ contains some nonzero element a; since a is invertible, $1 = a^{-1}a \in I$, so $I = A$. Thus no proper nonzero ideals exist. ∎

Proposition 3.4. *If $\varphi\colon A \to \mathbb{C}$ is a $\mathbb{C}$-algebra homomorphism, then $\ker(\varphi)$ is a maximal ideal of A.*

Proof. The image $\varphi(A) \subseteq \mathbb{C}$ is a sub-$\mathbb{C}$-algebra of the field $\mathbb{C}$, hence a domain. In fact any subalgebra of $\mathbb{C}$ that contains $\mathbb{C}$-scalars is a field (closed under inverses for nonzero elements), so $\varphi(A)$ is a field. By the First Isomorphism Theorem,

$$A/\ker(\varphi) \cong \varphi(A)$$

which is a field; hence $\ker(\varphi)$ is maximal. ∎

Proposition 3.5 (Maximal ideals in polynomial rings). *For $v = (v_1, \ldots, v_m) \in \mathbb{C}^m$ set*

$$I_v = (x_1 - v_1, \ldots, x_m - v_m) \subseteq \mathbb{C}[x_1, \ldots, x_m].$$

Then I_v is a maximal ideal. Conversely, every maximal ideal $\mathfrak{m} \subseteq \mathbb{C}[x_1, \ldots, x_m]$ is of this form.

Proof. The evaluation map $\mathrm{eval}_v : \mathbb{C}[x_1, \ldots, x_m] \to \mathbb{C}$ given by $f \mapsto f(v)$ is a surjective algebra homomorphism whose kernel is precisely I_v. Thus by the previous proposition I_v is maximal.
For the converse, let $\mathfrak{m}$ be a maximal ideal. Consider the quotient ring $R = \mathbb{C}[x_1, \ldots, x_m]/\mathfrak{m}$. Since $\mathfrak{m}$ is maximal, R is a field which is a finitely generated $\mathbb{C}$-algebra. By a standard consequence of Hilbert's Nullstellensatz (the weak Nullstellensatz), any finitely generated $\mathbb{C}$-algebra which is a field is isomorphic to $\mathbb{C}$ itself. Therefore $R \cong \mathbb{C}$, and the composite map

$$\mathbb{C}[x_1, \ldots, x_m] \twoheadrightarrow \mathbb{C}[x_1, \ldots, x_m]/\mathfrak{m} \cong \mathbb{C}$$

is an evaluation at some point $v \in \mathbb{C}^m$. The kernel of this composite map is $\mathfrak{m}$, so $\mathfrak{m} = I_v$.
(For full detail: one shows that each x_i has image in R which is algebraic over $\mathbb{C}$; since $\mathbb{C}$ is algebraically closed, each image is in $\mathbb{C}$, hence defines coordinates v_i.) ∎

Proposition 3.6. *Let V be an affine variety with coordinate ring $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_m]/I$. Each point $v \in V$ gives an evaluation map $\mathrm{eval}_v : \mathbb{C}[V] \to \mathbb{C}$ and the kernel of eval_v is a maximal ideal containing I.*

Proof. If $v \in V$ then any polynomial in I vanishes at v, so evaluation descends to a well-defined map on the quotient $\mathbb{C}[x_1, \ldots, x_m]/I$, i.e. $\mathrm{eval}_v : \mathbb{C}[V] \to \mathbb{C}$. Its kernel consists of those classes of polynomials vanishing at v, which is a maximal ideal of $\mathbb{C}[V]$ by the previous proposition. ∎

Surjections and Injections

Proposition 3.7. *Every algebra map $\varphi\colon A \to C$ factors canonically as a surjection onto its image followed by the inclusion of the image into C:*

$$A \xrightarrow{\pi} \varphi(A) \hookrightarrow C,$$

where π is surjective and the second map is injective.

Proof. Define $\pi : A \to \varphi(A)$ by $\pi(a) = \varphi(a)$. By construction π is surjective and the inclusion $\iota : \varphi(A) \hookrightarrow C$ is injective, and $\varphi = \iota \circ \pi$. ∎

Proposition 3.8. *Every surjection $A \twoheadrightarrow B$ of finitely generated $\mathbb{C}$-algebras may be written (up to identification) as the quotient map*

$$\mathbb{C}[x_1, \dots, x_m]/I \twoheadrightarrow \mathbb{C}[x_1, \dots, x_m]/J$$

for ideals $I \subset J$, and geometrically this corresponds to the inclusion of varieties

$$V(J) \hookrightarrow V(I).$$

In particular $\mathrm{Spec}(B) \cong V(J) \subseteq V(I) \cong \mathrm{Spec}(A)$.

Proof. Choose a presentation $A \cong \mathbb{C}[x_1, \dots, x_m]/I$. A surjection $A \twoheadrightarrow B$ corresponds to some quotient of A, hence there exists an ideal $J \supseteq I$ with $B \cong A/(J/I) \cong \mathbb{C}[x_1, \dots, x_m]/J$. Passing to varieties by taking maximal spectra (or zero loci) yields the inclusion $V(J) \subseteq V(I)$ as required. ■

Proposition 3.9. *If $\iota \colon B \hookrightarrow C$ is an injective map of finitely generated $\mathbb{C}$-algebras then the contraction of a maximal ideal of C is a maximal ideal of B, i.e. if $\mathfrak{M} \subset C$ is maximal then $\iota^{-1}(\mathfrak{M}) \subset B$ is maximal. The converse need not hold, so in general* $\mathrm{Spec}(C) \subsetneq \mathrm{Spec}(B)$.

Proof. Let $\mathfrak{M} \subset C$ be maximal. The quotient $C/\mathfrak{M}$ is a field, and the induced map

$$B/\iota^{-1}(\mathfrak{M}) \hookrightarrow C/\mathfrak{M}$$

is injective. Since the codomain is a field, the domain is a (sub)domain which is a finite type $\mathbb{C}$-algebra; by Nullstellensatz the domain must be a field, hence $\iota^{-1}(\mathfrak{M})$ is maximal.
However, given a maximal ideal $\mathfrak{m} \subset B$, there may be no maximal ideal of C lying over it (or there may be several); the extension of ideals need not preserve maximality. Thus contraction gives an inclusion of spectra in the indicated direction. ■

Example. The maps

$$\mathbb{C}[x, y] \xrightarrow{\pi} \mathbb{C}[x] \xrightarrow{\iota} \mathbb{C}[x, x^{-1}]$$

where π sends $y \mapsto 0$ and ι is localisation at x, illustrate: π is surjective (quotient by (y)) and ι is injective.

3.1 Maps between algebraic varieties

Let $V \subseteq \mathbb{C}^m$ be an algebraic variety defined by the ideal $I \subseteq \mathbb{C}[x_1, ..., x_m] = A$, and let $W \subseteq \mathbb{C}^n$ be defined by $J \subseteq \mathbb{C}[y_1, ..., y_n] = B$. The coordinate rings are $\mathbb{C}[V] = A/I$ and $\mathbb{C}[W] = B/J$. A map $\phi : V \to W$ is given by algebraic functions. If $\phi(p) = q$, and y_i is a coordinate function, then $\phi^*(y_i) = f_i(x)$, where $f_i \in \mathbb{C}[V]$. This means $y_i(q) = f_i(p)$. The map ϕ is specified by an underlying map $\phi' : \mathbb{C}^m \to \mathbb{C}^n$, which induces an algebra homomorphism $\phi'^* : B \to A$.
We can evaluate ϕ^* in several examples:

1. Let $V = \mathbb{C}^3$, $W = \mathbb{C}^2$ and $\phi_1(p_1, p_2, p_3) = (p_1, p_2)$. The map ϕ is the projection to the first two coordinates. Since V and W are affine spaces, $I = (0)$ and $J = (0)$. The pre-image map on coordinate rings is:

$$\phi^* : \mathbb{C}[y_1, y_2] \to \mathbb{C}[x_1, x_2, x_3]$$

$$\phi^*(y_1) = x_1, \quad \phi^*(y_2) = x_2$$

   These functions are constant on the pre-image.

2. Let $V = V(x_1^2+x_2^2-1)$, the cylinder. W be as before ($\mathbb{C}^2$). Let $\phi(x_1, x_2, x_3) = (x_1, x_3)$. $I = (x_1^2+x_2^2-1)$. $J = (0)$. The coordinate ring of V is $\mathbb{C}[V] = \mathbb{C}[x_1, x_2, x_3]/I$. The map is:
$$\phi^* : \mathbb{C}[y_1, y_2] \to \mathbb{C}[V]$$
$$\phi^*(y_1) = \overline{x}_1, \quad \phi^*(y_2) = \overline{x}_3$$
where $\overline{x}_i$ denotes the coset $x_i + I$.

3. Let $V = V(x_1x_2-1) \subseteq \mathbb{C}^2$ and be projection to the X-axis. $V \subseteq \mathbb{C}^2$ with coordinates (x_1, x_2), $W \subseteq \mathbb{C}^1$ with coordinate y_1. $I = (x_1x_2 - 1)$. $J = (0)$. The map is $\phi(x_1, x_2) = x_1$. The coordinate ring $\mathbb{C}[V] \cong \mathbb{C}[x_1, x_1^{-1}]$. The map is:
$$\phi^* : \mathbb{C}[y_1] \to \mathbb{C}[V]$$
$$\phi^*(y_1) = \overline{x}_1$$

4. Let $V \subseteq \mathbb{C}^{2\times 3}$ of matrices of rank 1 or less. Let $\phi : X \to \mathbb{C}^3$, mapping to the first row. $V \subseteq \mathbb{C}^6$ (coordinates x_{ij}). $W \subseteq \mathbb{C}^3$ (coordinates y_1, y_2, y_3). $J = (0)$. Let $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}$. ϕ maps X to its first row (x_{11}, x_{12}, x_{13}). The map is:
$$\phi^* : \mathbb{C}[y_1, y_2, y_3] \to \mathbb{C}[V]$$
$$\phi^*(y_1) = \overline{x}_{11}, \quad \phi^*(y_2) = \overline{x}_{12}, \quad \phi^*(y_3) = \overline{x}_{13}$$

In the general case where $V \subset \mathbb{C}^m$ and $W \subseteq \mathbb{C}^n$ are related by $y_i = f(X_i)$, the map $\phi'^* : B \to A$ given by $\phi'^*(y_i) = f_i(x)$ is always an algebra homomorphism. The key question is when this lifts to a map $\phi^* : B/J \to A/I$. This requires checking if ϕ^* is well-defined, which is equivalent to checking if $\phi^*(J) = 0$. In other words, if $g(y) \in J$ then must $g(f(x)) \in I$?

Proposition 3.10. *The map $\phi^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is well-defined if and only if $\phi'^*(J) \subseteq I$.*

Proof. The map $\phi^* : B/J \to A/I$ is defined by $\phi^*(\overline{g}) = \overline{\phi'^*(g)}$.
Necessity ($\Longrightarrow$): Assume ϕ^* is well-defined. If $g \in J$, then $\overline{g} = \overline{0}$ in B/J. For the map to be well-defined, we must have $\phi^*(\overline{g}) = \overline{0}$ in A/I. This means $\overline{\phi'^*(g)} = \overline{0}$, which is equivalent to $\phi'^*(g) \in I$. Since this holds for all $g \in J$, we must have $\phi'^*(J) \subseteq I$.
Sufficiency ($\Longleftarrow$): Assume $\phi'^*(J) \subseteq I$. If $\overline{g_1} = \overline{g_2}$ in B/J, then $g_1 - g_2 \in J$. By assumption, $\phi'^*(g_1 - g_2) \in I$. Since ϕ'^* is a homomorphism, $\phi'^*(g_1) - \phi'^*(g_2) \in I$. This implies $\overline{\phi'^*(g_1)} = \overline{\phi'^*(g_2)}$ in A/I, so $\phi^*(\overline{g_1}) = \phi^*(\overline{g_2})$. Thus, ϕ^* is well-defined. ∎

Let $\mathrm{Im}(\phi'^*) = B'$. The well-definedness condition can be rephrased in terms of B' as well. We consider the ideal I and $I' = B \cap I$, which is an ideal of B'. Next, $\phi'^*(J) = J'$ is also an ideal of B'. The condition for ϕ^* to be well-defined is that $I' \subseteq J'$.
Finally, we consider the specific example: $I = ((t^2+1)y - 2t, (t^2+1)x - 1 + t^2) \subseteq \mathbb{C}[x, y, t]$. Let $J = (x^2 + y^2 - 1) \subseteq \mathbb{C}[x, y]$. The map is projection $\phi(x, y, t) = (x, y)$.

1. Well-Definedness: The map is well-defined if $\phi'^*(J) \subseteq I$. J is generated by $g = x^2 + y^2 - 1$. We show $x^2 + y^2 - 1 \in I$. The generators of I imply $(t^2 + 1)x \equiv 1 - t^2 \pmod{I}$ and $(t^2 + 1)y \equiv 2t \pmod{I}$. Squaring and adding these relations modulo I:
$$(t^2+1)^2(x^2+y^2) \equiv (1-t^2)^2 + (2t)^2 \equiv 1 - 2t^2 + t^4 + 4t^2 \equiv (t^2+1)^2 \pmod{I}$$
$$(t^2+1)^2(x^2+y^2-1) \equiv 0 \pmod{I}$$
Since the variety $V(I)$ is empty in $\mathbb{C}^3$ (as I is the unit ideal), $x^2 + y^2 - 1 \in I$ is true.

2. Compute ϕ^*: Since $J \subseteq I$, ϕ^* is well-defined. The map on coordinate rings is:
$$\phi^* : \mathbb{C}[x, y]/(x^2 + y^2 - 1) \to \mathbb{C}[x, y, t]/I$$
$$\phi^*(\overline{x}) = \overline{x}, \quad \phi^*(\overline{y}) = \overline{y}$$

3. Properness: The variety V is $V(I) = \emptyset$. A map from the empty set is vacuously a proper map.

The Variety $V \times W$ and Graphs

Proposition 3.11. *Let $\mathbb{C}[V] = A/I$ and $\mathbb{C}[W] = B/J$ be coordinate rings of affine varieties V, W. Then the coordinate ring of the product $V \times W$ is*

$$\mathbb{C}[V \times W] \cong \mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]/(I + J) \cong (A/I) \otimes_{\mathbb{C}} (B/J).$$

The projections $\pi_V : V \times W \to V$ and $\pi_W : V \times W \to W$ are algebraic (dual to the inclusions on coordinate rings) and surjective; the fibre $\pi_V^{-1}(v) = \{v\} \times W$.

Proof. On coordinate rings, functions on $V \times W$ are polynomials in the x- and y-variables modulo the relations from I and J. Thus the ideal of relations is $I + J$, and the coordinate ring is the stated quotient. Equivalently the coordinate ring is the tensor product $\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[W]$.
The projections are given on rings by the canonical maps $\mathbb{C}[V] \to \mathbb{C}[V] \otimes \mathbb{C}[W]$ and similarly for W. On points these dualize to the usual projections. Surjectivity follows since for any $v \in V$ and any $w \in W$ the pair $(v, w) \in V \times W$, and for fixed v varying w we recover $\{v\} \times W$ as the fibre. ■

Proposition 3.12. *Let $\varphi\colon V \to W$ be a morphism of affine varieties. The graph*

$$\Gamma_\varphi = \{(v, \varphi(v)) \mid v \in V\} \subset V \times W$$

is an algebraic subset (its ideal is generated by the pullback relations), and the projection $\pi_W(\Gamma_\varphi) = \mathrm{Im}(\varphi)$ is an algebraic set. Moreover, for any $w \in W$ the preimage $\varphi^{-1}(w)$ is an algebraic set in V.

Proof. On coordinate rings the map φ corresponds to an algebra homomorphism $\varphi^* : \mathbb{C}[W] \to \mathbb{C}[V]$. The graph corresponds to the image of the map

$$\mathbb{C}[V] \otimes \mathbb{C}[W] \to \mathbb{C}[V], \qquad f \otimes g \mapsto f \cdot \varphi^*(g),$$

and concretely one can write equations cutting out the graph: if $y_1, \dots, y_n$ are coordinates on W and $\varphi^*(y_i) = F_i(x)$ are functions on V, then the graph is defined inside $V \times W$ by the equations $y_i - F_i(x) = 0$ for all i. Hence Γ_φ is algebraic.
The image $\pi_W(\Gamma_\varphi) = \mathrm{Im}(\varphi)$ is the projection of an algebraic set; in the affine algebraic category projections of algebraic sets are constructible, and the Zariski-closure of the image is algebraic. For many standard situations (e.g. when φ is a morphism of affine varieties) one shows directly that $\mathrm{Im}(\varphi)$ is the variety corresponding to the kernel of the map on coordinate rings, hence algebraic.
Finally, the fibre $\varphi^{-1}(w)$ is the intersection of the graph with $V \times \{w\}$ which is algebraic, so the fibre is algebraic as well. ■

Lecture 4

Scribe: Someone

Algebraic nature of Orbits.

Given $\rho : G \times V \to$. Let us assume for simplicity, that V is a vector space and the action is *homogeneous.* By standard algebraic geometry, $\mathbb{C}[G \times V] \cong \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[V]$, the tensor product algebra.
Then, the map $\rho : G \times V \to V$ gives us a map $\rho^* : \mathbb{C}[V] \to \mathbb{C}[G] \otimes \mathbb{C}[V]$. We thus have for any $q \in \mathbb{C}[V]$:

$$\rho^*(q) = \sum_{j=1}^{K} p_j(M) q_j(V)$$

where $p_{ij} \in \mathbb{C}[G]$ and $q_j \in \mathbb{C}[V]$. Thus we have the important proposition:
Proposition 4.1. *Recall the action of G on $\mathbb{C}[V]$, where if $q \in \mathbb{C}[V]$ and $g \in G$, then we define $g \cdot q$ or simply $f^G \in \mathbb{C}[V]$ as the function:*

$$q^g(v) = q(g^{-1}v)$$

Then there are functions $q_1, \ldots, q_k \in \mathbb{C}[V]$ and $p_1, \ldots, p_k \in \mathbb{C}[G]$ such that for all g, we have:

$$q^g = \sum_{i=1}^{k} p_i(g) q_i$$

Remark 4.2. *Show, using this, that there is a finite dimensional G-invariant vector space containing $\{q^g | g \in G\}$.*

Given that ρ is linear on V, gives us the special case:

$$\rho^*(V_i) = \sum_{j=1}^{N} p_{ij}(M) V_j$$

We may write this in the matrix form:

$$\begin{bmatrix} \rho^*(V_1) \\ \vdots \\ \rho^*(V_n) \end{bmatrix} = \begin{bmatrix} p_{11}(M) & \ldots & p_{1N}(M) \\ \vdots & & \vdots \\ p_{N1}(M) & \ldots & p_{NN}(M) \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

It is important to understand the above equation. If $v \in V$ and $g \in G$ such that $w = g \cdot v$, then we have the concrete matrix product:

$$\begin{bmatrix} V_1(w) \\ \vdots \\ V_N(w) \end{bmatrix} = \begin{bmatrix} p_{11}(g) & \ldots & p_{1N}(g) \\ \vdots & & \vdots \\ p_{N1}(g) & \ldots & p_{NN}(g) \end{bmatrix} \begin{bmatrix} V_1(v) \\ \vdots \\ V_N(v) \end{bmatrix}$$

Example 4.3. $Sym^2(\mathbb{C}^2)$.

Thus, the orbit of v is better seen as the **graph** of the group action on v, i.e., the set $\{(g, w) | w = g.v \, and \, g \in G\} \subseteq G \times V$ and the projection from $\pi : G \times V \to V$ with $\pi(g, x) = x$. Thus, if we use a fresh set of coordinates for V, viz $W_1, \ldots, W_N$, then the graph Y of v is given by the equations:

$$\begin{bmatrix} W_1 \\ \vdots \\ W_N \end{bmatrix} = \begin{bmatrix} p_{11}(M) & \ldots & p_{1N}(M) \\ \vdots & & \vdots \\ p_{N1}(M) & \ldots & p_{NN}(M) \end{bmatrix} \begin{bmatrix} V_1(v) \\ \vdots \\ V_N(v) \end{bmatrix}$$

These are N equations plus the equations needed to define I_G in $n^2 + N$ variables (M_{ij}) and (W_k). Then $O(v) = \pi(Y) \subseteq V$. Therefore, we have:

Corollary 4.4. Let G be an algebraic group and V be a G-module via a rational map $\rho : G \to GL(V)$. Let $v \in V$. Then the closure of orbit $O(v)$ within V is an algebraic variety. The stabilizer G_v of v is an algebraic group.

Proof: The first assertion follows from Prop. **??**. For the second, we consider the algebraic subset of G defined by the equations:

$$\begin{bmatrix} V_1(v) \\ \vdots \\ V_N(v) \end{bmatrix} = \begin{bmatrix} p_{11}(g) & \dots & p_{1N}(g) \\ \vdots & & \vdots \\ p_{N1}(g) & \dots & p_{NN}(g) \end{bmatrix} \begin{bmatrix} V_1(v) \\ \vdots \\ V_N(v) \end{bmatrix}$$

Separation and Invariants

Lemma 4.5. *If X, Y and two closed G-invariant sets, then there is a function $f \in \mathbb{C}[V]$ such that $(X) = 0$ and $f(Y) = 1$. If $v, w \in V$ are points such $\overline{O(v)} \cap \overline{O(w)} = \phi$ then they are separated by an $f \in \mathbb{C}[V]$.*

Lemma 4.6. *Let $f \in \mathbb{C}[V]$ and let M_f be the vector space generated by $\{f^g \mid g \in G\}$. Then M_f is finite dimensional. Let N_f be module generated by $\{f^g - f^{g'} | g, g' \in G\}$. Then*

1. *N_f is a G-submodule of N_f and M_f/N_f is of dimension 0 or 1.*
2. *Let f be a non-zero constant on an orbit, $O(v)$. Say $f(v) = 1$. Then $dim(M_f/N_f) = 1$ and let f^G be a representative. Then for any $g \in G$ we have $f^G(v) = f^G(g^{-1}v) = 1$.*

Definition 4.7. G is called reductive if every inclusion of finite-dimensional G-modules $N \subseteq M$ has a G-invariant complement $N^\perp$ such that $M = N \oplus N^\perp$.

Lemma 4.8. *Let W be a G-module. Let W' be the vector space generated by all elements $\{w - gw | w \in W, g \in G\}$, then W' is G-invariant. Moreover, there is a unique G-complement $W^G = \{w \in W | gw = w\}$.*

Proof: Suppose that $dim(W) = n$ and $dim(W') = r$. It is easy to check that W' is G-closed and thus there is a G-complement W''. For a $w'' \in W''$, we see that $gw'' = w''$ for all g. This is simply because $w'' - gw'' \in W'' \cap W' = 0$. Thus $W'' \subseteq W^G$ and $dim(W^G) \geq n - r$. Conversely, if $\pi : W \to W^G$ is a projection then for any w, g, $\pi(w - gw) = \pi(w) - \pi(gw) = 0$. Thus $W' \subseteq ker(\pi)$, and $r \leq n - dim(W^G)$.

Corollary 4.9. Let $V \subseteq W$ be G-modules. Let $\pi_W^G : W \to W^G$ be the canonical projection implementing $W = W' \oplus W^G$. Let $V \subseteq W$ be a G-module. Then $\pi_V^G = \pi_W^G |_V$.

Proof: It is clear that $V' \subseteq W'$ and $V^G \subseteq W^G$. The result follows. $\square$

Proposition 4.10. *Let V be a G-module and G be reductive. Let $f \in \mathbb{C}[V]$. Then there is an $f^G \in M_f$ such that $(f^G)^g = f^G$. In other words, f^G is an invariant. Moreover, this f^G is unique.*

Proof: If this were not so, then $dim(M_f^G) \geq 2$ and $dim(N_f) \leq dim(M_f) - 2$. But we know that is not possible. $\square$

Corollary 4.11. If f, f' are invariants and $h \in \mathbb{C}[V]$ then (i) $(f + f')^G = f^G + f'^G$, and (ii) $(fh)^G = fh^G$.

Proof: Let $W = M_f + M_{f'}$ and $V = M_{f+f'} \subseteq W$. Then by additivity $\pi_W^G(f_1 + f_2) = \pi_W^G(f_1) + \pi_W^G(f_2) = f_1^G + f_2^G$. On the other hand, applying π_V^G, we get $\pi_V^G(f_1 + f_2) = (f_1 + f_2)^G$. The other assertion is similarly proved. □

Ring of invariants and the fundamental theorem.

Proposition 4.12. *Consider the algebra $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$. Then there is a canonical projection $\pi : \mathbb{C}[V] \to \mathbb{C}[V]^G$. Moreover, it preserves degree. Further, if $f \in \mathbb{C}[V]_d^G$ and $f' \in \mathbb{C}[V]_e$, the $\pi(ff') = f\pi(f')$.*

Warning: Note that π is NOT an algebra homomorphism.
Proof: The only part which needs proof is the second part that $\pi(ff') = f\pi(f')$ when $f \in \mathbb{C}[V]^G$. We may easily reduce to the case when f and f' are homogeneous. Let $M \subseteq \mathbb{C}[V]$ be the G-module generated by f'. Suppose now that $deg(f) = d$ and $deg(f') = e$. Consider the module $M' = \mathbb{C}f \otimes M \subseteq \mathbb{C}[V]_{d+e}$. It is clear that $M'^G = f \otimes M^G$. □
Proposition 4.13. *The ring $\mathbb{C}[V]^G$ is finitely generated.*

We now come to some results on the structure of orbits.

Lemma 4.14.
1. *Let $v \in V$ and $O(v)$ be the G-orbit of V and $\overline{O(v)}$ be its closure. Let $w \in \overline{O(v)}$, then there is no invariant which separates v from w.*
2. *If $v' \in V$ is such that $\overline{O(v')} \cap \overline{O(v)} = \phi$ then there is an invariant $f \in \mathbb{C}[V]^G$ which separates them.*

Proposition 4.15.
1. *Let $v \in V$ and $O(v)$ be the G-orbit of V and $\overline{O(v)}$ be its closure. Then there is a unique orbit $O(w)$ of minimum dimension, and it is closed.*
2. *Let us define $\sim$ on V as $v \sim v'$ is $\overline{O(v)} \cap \overline{O(v')} \neq \phi$. Then $\sim$ is an equivalence relation. Moreover, $[v]_\sim$ is closed. If $v' \notin [v]_\sim$ then there is an invariant which separates them.*

Definition 4.16. A $v \in V$ is called stable if its G-orbit is closed.

Warning: The presence of closed orbits may be very different for a $GL(X)$-module V when restricted to $SL(X) \subset GL(X)$.

Theorem 4.17. *Suppose G is reductive and $\lambda(t) \subseteq G$ is a 1-PS. Let $\lambda(t)v = \sum_i t^i v_i$. Then if $v_i = 0$ for all < 0 then v is not stable. Conversely, if v is not stable, then there is a 1-PS $\lambda(t)$ and a $w \in V$ such that $\lim_{t\to 0} \lambda(t)v = w$. Note that $w = 0$ is also permitted.*

The converse part of the theorem is the Hilbert-Mumford theorem. Note that proving something as unstable is shown by a suitable $\lambda(t)$. However, very few techniques exist to prove stability.

Proposition 4.18 (11.10). *Let V be a G-module and G be reductive. Let $f \in \mathbb{C}[V]$. Then there is an $f^G \in M_f$ such that $(f^G)^g = f^G$. In other words, f^G is an invariant. Moreover, this f^G is unique.*

Proof. The module M_f generated by $\{f^g \mid g \in G\}$ is a finite-dimensional G-module (by Lemma 11.6).
The set of G-invariants in M_f is $M_f^G = \{h \in M_f \mid h^g = h \text{ for all } g \in G\}$.
Since G is **reductive** and M_f is a finite-dimensional G-module, by Lemma 11.8 (applied to $W = M_f$), we have a unique decomposition $M_f = M'_f \oplus M_f^G$, where M'_f is the vector space generated by $\{h - h^g \mid h \in M_f, g \in G\}$.
Since $f \in M_f$, it can be uniquely written as $f = f' + f^G$, where $f' \in M'_f$ and $f^G \in M_f^G$.
Existence: The component f^G is, by definition, an invariant in M_f.

Uniqueness: Suppose $h \in M_f$ is another invariant, i.e., $h \in M_f^G$. The unique decomposition implies that the projection $\pi : M_f \to M_f^G$ is unique. Since f is projected to f^G and f^G is an invariant, f^G must be this unique invariant.
Alternatively, using the structure suggested by the notes: $N_f = \text{span}\{f^g - f^{g'} \mid g, g' \in G\}$ is precisely M'_f. Lemma 11.6 (2) states $\dim(M_f/N_f) \leq 1$. Since $M_f/M'_f \cong M_f^G$, we have $\dim(M_f^G) \leq 1$. If f^G exists, $\dim(M_f^G) = 1$, which proves the uniqueness of the invariant (up to a scalar, which is fixed by the projection). ■

Corollary 4.19 (11.11). If f, f' are invariants and $h \in \mathbb{C}[V]$, then (i) $(f + f')^G = f^G + f'^G$, and (ii) $(fh)^G = fh^G$.

Proof. **(i) Additivity:** We want to show that $\pi(f + f') = \pi(f) + \pi(f')$, where $\pi(h) = h^G$. The map $\pi : \mathbb{C}[V] \to \mathbb{C}[V]^G$ is the composition of the inclusion $\mathbb{C}[V] \to \bigoplus_f M_f$ and the G-invariant projection $\pi^G_{M_f}$. Since the projection $\pi^G_W : W \to W^G$ onto the invariant part of any G-module W is a linear map, and $M_{f+f'} \subseteq M_f + M_{f'}$, the result follows from the linearity of π:

$$(f + f')^G = \pi(f + f') = \pi(f) + \pi(f') = f^G + f'^G$$

(ii) Multiplication by an Invariant: We want to show $(fh)^G = fh^G$ where $f \in \mathbb{C}[V]^G$. The G-module generated by fh is M_{fh}. Since f is G-invariant, we have $(fh)^g = f \cdot h^g$. Thus, $M_{fh} = f \cdot M_h$.
The element $(fh)^G$ is the unique invariant component of fh in M_{fh} (Prop 11.10). The element fh^G is also an invariant, since $f \in \mathbb{C}[V]^G$ and $h^G \in \mathbb{C}[V]^G$. Since $h^G \in M_h$, we have $fh^G \in f \cdot M_h = M_{fh}$.
As fh^G is an invariant element belonging to M_{fh}, by the uniqueness of the invariant component, we must have $(fh)^G = fh^G$. ■

Proposition 4.20 (11.12). *Consider the algebra $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$. Then there is a canonical projection $\pi : \mathbb{C}[V] \to \mathbb{C}[V]^G$. Moreover, it preserves degree. Further, if $f \in \mathbb{C}[V]^G_d$ and $f' \in \mathbb{C}[V]_e$, the $\pi(ff') = f\pi(f')$.*

Proof. **Existence of π:** The map $\pi(f) = f^G$ is defined by Proposition 11.10 and is linear by Corollary 11.11 (i). If $f \in \mathbb{C}[V]^G$, then $f^G = f$, so π is a **projection** onto the subring $\mathbb{C}[V]^G$.
Degree Preservation: The action $f \mapsto f^g$ preserves the degree of homogeneous polynomials because the action of G on V is assumed to be homogeneous (linear)[cite: 6]. Thus, the G-module M_f is graded. The invariant component f^G must respect this grading, meaning that if f is a sum of homogeneous polynomials $\sum f_d$, then $f^G = \sum f_d^G$, where f_d^G has the same degree as f_d. Hence, π preserves degree.
The Special Multiplication Property: The property $\pi(ff') = f\pi(f')$ when $f \in \mathbb{C}[V]^G$ is the content of Corollary 11.11 (ii). ■

Proposition 4.21 (11.13). *The ring $\mathbb{C}[V]^G$ is finitely generated.*

Proof. This is **Hilbert's Finiteness Theorem for Invariants** for reductive groups.
The ring of invariants $\mathbb{C}[V]^G$ is a subring of the polynomial ring $\mathbb{C}[V]$. We will show that every ideal in $\mathbb{C}[V]^G$ is finitely generated (i.e., $\mathbb{C}[V]^G$ is Noetherian).
Let $\mathcal{I}$ be any ideal in $\mathbb{C}[V]^G$. Since $\mathcal{I}$ is also an ideal in the polynomial ring $\mathbb{C}[V]$, by Hilbert's Basis Theorem, the extended ideal $\mathcal{I} \cdot \mathbb{C}[V]$ is finitely generated:

$$\mathcal{I} \cdot \mathbb{C}[V] = (f_1, f_2, \ldots, f_k)_{\mathbb{C}[V]}, \quad \text{with } f_i \in \mathbb{C}[V].$$

Now, apply the projection $\pi : \mathbb{C}[V] \to \mathbb{C}[V]^G$ (Proposition 11.12) to the generators f_i, obtaining $f_i^G = \pi(f_i)$. Let $\mathcal{I}^G = (f_1^G, f_2^G, \ldots, f_k^G)_{\mathbb{C}[V]^G}$ be the ideal in $\mathbb{C}[V]^G$ generated by the invariant components.

Take any $h \in \mathcal{I}$. Since $h \in \mathcal{I} \cdot \mathbb{C}[V]$, we have $h = \sum_{i=1}^{k} f_i h_i$ for some $h_i \in \mathbb{C}[V]$. Since $h \in \mathbb{C}[V]^G$, we have $\pi(h) = h$.

$$h = \pi(h) = \pi\left(\sum_{i=1}^{k} f_i h_i\right)$$

Since $f_i \in \mathcal{I} \subseteq \mathbb{C}[V]^G$, we use the special multiplication property (Proposition 11.12) and linearity:

$$h = \sum_{i=1}^{k} \pi(f_i h_i) = \sum_{i=1}^{k} f_i \pi(h_i) = \sum_{i=1}^{k} f_i h_i^G$$

Since $f_i \in \mathcal{I} \subseteq \mathbb{C}[V]^G$ and $h_i^G \in \mathbb{C}[V]^G$, this expression shows $h \in \mathcal{I}^G$. Thus, $\mathcal{I} \subseteq \mathcal{I}^G$. Since $f_i^G \in \mathcal{I} \cdot \mathbb{C}[V]$ and $\mathcal{I} \cdot \mathbb{C}[V] \cap \mathbb{C}[V]^G = \mathcal{I}$, we have $f_i^G \in \mathcal{I}$, so $\mathcal{I}^G \subseteq \mathcal{I}$.
Hence, $\mathcal{I} = \mathcal{I}^G$, meaning every ideal $\mathcal{I}$ in $\mathbb{C}[V]^G$ is finitely generated. Since $\mathbb{C}[V]^G$ is Noetherian, it must be finitely generated as a $\mathbb{C}$-algebra. ■

Proposition 4.22 (11.15). *1. Let $v \in V$ and $O(v)$ be the G-orbit of v and $\overline{O(v)}$ be its closure. Then there is a unique orbit $O(w)$ of minimum dimension, and it is closed.*

2. Let us define $\sim$ on V as $v \sim v'$ is $\overline{O(v)} \cap \overline{O(v')} \neq \phi$. Then $\sim$ is an equivalence relation. Moreover, $[v]_\sim$ is closed. If $v' \notin [v]_\sim$ then there is an invariant which separates them.

Proof. **Part 1: Orbit of Minimum Dimension.** The closure $\overline{O(v)}$ is an algebraic variety (Corollary 11.4). In the closure of any orbit of an algebraic group, there is a unique orbit $O(w)$ that is closed and has the minimum dimension. This is a standard result in the theory of algebraic group actions. The point w is called a **stable point** in $\overline{O(v)}$ (Definition 11.16).
Part 2: The Equivalence Relation. The relation is defined as $v \sim v'$ if $\overline{O(v)} \cap \overline{O(v')} \neq \phi$.

- **Reflexivity:** $\overline{O(v)} \cap \overline{O(v)} = \overline{O(v)} \neq \phi$. Thus $v \sim v$.
- **Symmetry:** If $\overline{O(v)} \cap \overline{O(v')} \neq \phi$, then clearly $\overline{O(v')} \cap \overline{O(v)} \neq \phi$. Thus $v \sim v' \implies v' \sim v$.
- **Transitivity:** This relies on the result from Part 1. The key fact is that two orbits $O(v)$ and $O(v')$ are related by $\sim$ if and only if they share the same unique closed orbit of minimum dimension in their closures. Let $O(w)$ be the unique closed orbit in $\overline{O(v)}$. If $v \sim v'$, then $O(w)$ is the unique closed orbit in $\overline{O(v')}$. If $v' \sim v''$, then $O(w)$ is the unique closed orbit in $\overline{O(v'')}$. Since $O(w)$ is contained in both $\overline{O(v)}$ and $\overline{O(v'')}$, their intersection is non-empty, so $v \sim v''$.

Closure and Separation: The equivalence class $[v]_\sim$ is the union of all orbits $O(v')$ that have the same minimal closed orbit $O(w)$. This set is known to be closed. The value of any invariant $f \in \mathbb{C}[V]^G$ is constant on an orbit $O(v)$ (Lemma 11.14 (1) is often stated this way in this context) and constant on its closure $\overline{O(v)}$. If $v' \notin [v]_\sim$, then $\overline{O(v')}$ and $\overline{O(v)}$ are disjoint closed G-invariant sets. By Lemma 11.14 (2), disjoint closed G-invariant sets are separated by an invariant $f \in \mathbb{C}[V]^G$. ■

Lecture 5

Scribe: Someone

Basic definitions.

1. Infinitesimal motions and the ∇ condition.

2. Tangent space T_pX at a point p on X as the null space of the gradients of the defining equations. Examples. Cone. Line on a cone. Singular cubic. Rank 1 matrices.

3. The rank argument and the openness of the rank. Joint equations of rank of $[\nabla f_1, \ldots, \nabla f_r]$ and the $(f_1, \ldots, f_r)$. The openness conclusion. Definition of tangent space and dimension. Singularities of X. Examples.

4. Maps $\phi : \mathbb{C}^m \to \mathbb{C}^n$, where $y_i = \phi_i(x)$ and at the tangent spaces $T_p\mathbb{C}^m \to T_q\mathbb{C}^n$. The Jacobian computation. What when the image is within $V(g)$. Using the chain rule and restriction to T_pX to obtain $\phi_* : T_pX \to T_q\mathbb{C}^m \subseteq T_qY$. The kernel and the image. The matrix equivalent of the image under the Jacobian of the kernel of $\nabla f]$. Examples. Sphere to plane projection. Line on a cone.

5. The tangent space of $GL(X)$ and $SL(X)$. The orbit action at e, the identity, and the map of tangent spaces. GL_2 on Sym^2 and GL_n by conjugation. The surjection of the map onto the tangent space of the orbit. The openness condition in case of groups and orbits. The kernel and its dimension. Example of non-trivial stabilizers.

The algebra of tangent spaces.

1. Connection of proof of dimension preservation under projection. $X \subseteq \mathbb{C}^{m+1}$ and $\pi : \mathbb{C}^{m+1} \to \mathbb{C}^m$, where $\pi(x_0, \ldots, x_m) = (x_1, \ldots, x_m)$. Reduction to the case when $X = V(f_1, \ldots, f_r)$, monic polynomials.

2. $T_p\mathbb{C}^m$ and derivations $\mathcal{D}_p$ at $p \in \mathbb{C}^m$. Equivalence of the two definitions. The map $\phi : X \to Y$ and the map $\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ and therefore $\phi_* : \mathcal{D}_{X,p} \to \mathcal{D}_{Y,q}$

Lecture 6

Scribe: Nilabha Saha

6.0.1 Tangent spaces of groups and orbits

1. Revision: The gradient condition, rank condition and dimension. Maps and tangent map as a linear map between $\phi_* : T_pX \to T_pY$.

2. Exercise: Stereographic projection:

$$(X, Y) = (\frac{x}{1-z}, \frac{y}{1-z})$$

$$(x, y, z) = (\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2})$$

3. The tangent space of $GL(X)$ and $SL(X)$ and O_n. The "finite" part which is missed.

4. The orbit action at e, the identity, and the map of tangent spaces. GL_2 on Sym^2 and GL_n by conjugation. The surjection of the map onto the tangent space of the orbit. The openness condition in case of groups and orbits. The kernel and its dimension. Example of non-trivial stabilizers.

6.0.2 Lie Algebras

Definition 6.1. Let $G \subseteq GL_n$ be an algebraic group. The Lie algebra of G, denoted by $\mathcal{G}$ of $Lie(G)$ is the tangent space at the identity element I. In other words, $\mathcal{G} = T_I G$, the space of all $X \in M_n$ such that $I + \epsilon X \in G$, when $\epsilon^2 = 0$. $\mathcal{G}$ is closed under the lie bracket: if $X_1, X_2 \in \mathcal{G}$, then so is $[X_1, X_2] = X_1X_2 - X_2X_1$.

Proposition 6.2. *Consider the map $exp : M_n \to GL_n$, given by $exp(X) = e^X$. Then exp has the following properties:*

1. *$exp(X)exp(-X) = I$, the identity. For any $A \in GL_n$, we have $Aexp(X)A^{-1} = exp(AXA^{-1})$.*

2. *If the eigenvalues of X are λ_1, λ_r with multiplicities $d_1, \ldots, d_r$, then $e^{\lambda_1}, \ldots, e^{\lambda_r}$ are the eigenvalues of e^X with the same multiplicities.*

3. *If X, Y commute, then $exp(X + Y) = e^X e^Y$.*

Why does the Lie bracket work?

1. The Lie algebra gl_n and the exponential map $exp : gl_n \to GL_n$ in the vicinity of 0.

2. $Lie(G) = \{\mathfrak{g} | exp(\mathfrak{g}) \in G\}$.

3. Equality of $Lie(G)$ with T_eG. The identity $e^X e^Y = e^{\gamma(X,Y)}$, where $\gamma(X, Y) = X + Y + [X, Y] +$ higher terms. Using this to compute $e^{tX}e^{uY}e^{-tX}e^{-uY}$ for $t^2 = u^2 = 0$ to obtain this as $e^{tu[X,Y]}$. This proves closure under Lie bracket.

Proposition 6.3. *For any group $G \subseteq GL(X)$, with Lie alegbra $\mathcal{G}$, there is an open neighborhood $O \subseteq \mathcal{G}$ around 0, and map $exp : O \to G$ which is a diffeomorphism on its image. Moreover, for any $X \in \mathcal{G}$, the image of the line tX, denoted by $\gamma_X(t) \subseteq G$, is a curve such that $\gamma'(0) = X$. Conversely, if $\gamma(t) \subseteq G$ is a smooth curve then (i) $X = \gamma'(0) \in Lie(G)$ and e^{tX} and $\gamma(t)$ osculate.*

Example 6.4. SL_n and O_n.

Proposition 6.5. *G acts on $\mathcal{G}$ via the* **adjoint** *representation. If $g \in G$ and $\mathfrak{g} \in \mathcal{G}$, then $adj(g)(\mathfrak{g}) = g\mathfrak{g}g^{-1}$, where the multiplication happens in GL_n.*

Lemma 6.6. *If $G \subseteq GL(X)$ is a group, then $dim(G) = dim(Lie(G))$. Moreover, if H is a subgroup of G, then $Lie(H) \subseteq Lie(G)$ and $dim(Lie(H)) = dim(H)$.*

Proposition 6.7. *G acts on $\mathcal{G}$ via the* **adjoint** *representation. If $g \in G$ and $\mathfrak{g} \in \mathcal{G}$, then $adj(g)(\mathfrak{g}) = g\mathfrak{g}g^{-1}$, where the multiplication happens in GL_n.*

Example 6.8. Exponential map for SL_n and O_n.

Now, given the map $\rho : G \times V \to V$, we may pick points g, v and $w = \rho(g)(v)$ or simply gv and have the tangential map:

$$(\rho_*)_{g,v} : T_g G \times T_v V \to T_w V$$

More specifically, at $g = I$, the identity matrix, we have $T_I(G) = \mathcal{G}, v = w$ and $T_v \cong V$. Therefore, we have a map, which we denote by ρ_1:

$$\rho_1 : \mathcal{G} \times V \to V$$

This we call the *Lie algebra representation V*.

Definition 6.9. For a Lie algebra $\mathcal{G}$, we say that V is a $\mathcal{G}$-representation of a $\mathcal{G}$-module, if we have a map $\rho_1 : \mathcal{G} \to End(V)$ such that for any $X_1, X_2 \in \mathcal{G}$, we have $\rho_1([X_1, X_2]) = [\rho_1(X_1), \rho_1(X_2)]$.

Note that the Lie bracket on the right happens in $End(V)$ space.

Proposition 6.10. *Let $\rho : G \times V \to V$ be a representation. Let $\rho_1 : \mathcal{G} \to End(V)$ its tangent map. Then ρ_1 is a Lie algebra representation. Moreover, for any $g \in G$ and $\mathfrak{g} \in \mathcal{G}$:*

$$\rho(g)\rho_1(\mathfrak{g})\rho(g)^{-1} = \rho_1(adj(g) \cdot \mathfrak{g}) = \rho_1(g\mathfrak{g}g^{-1})$$

Finally, for a point v with stabilizer G_v, $Lie(G_v) = \{\mathfrak{g} \in \mathcal{G} | \rho_1(\mathfrak{g})(v) = 0\}$.

Note that $\rho_1 : \mathcal{G} \to End(V)$ is a linear map! Thus the Lie algebra of the stabilizer of a point v is a linear algebra computation, once representation ρ_1 is computed. How is $\rho_1(X)$ to be computed? The simplest is to use the matrix for ρ and differentiate the matrix $\rho(g)$ in the direction X. We will illustrate with two examples.

Example 6.11. Let us look at the form $f = Ax^2 + Bxy + Cy^2$ under the infinitesimal Lie element:

$$I + \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 + \epsilon a & b \\ c & 1 + \epsilon d \end{bmatrix}$$

Its action on f is given by:

$$\begin{array}{ll} & A(x + \epsilon ax + \epsilon by)^2 + B(x + \epsilon ax + \epsilon by)(y + \epsilon cx + \epsilon dy) + C(y + \epsilon cx + \epsilon dy)^2 \\ = & (Ax^2 + Bxy + Cy^2) + \epsilon\{A(2ax^2 + 2bxy) + B(cx^2 + dxy + axy + by^2) + C(2cxy + 2dy^2) \\ = & \begin{bmatrix} x^2 & xy & y^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} + \epsilon \begin{bmatrix} x^2 & xy & y^2 \end{bmatrix} \begin{bmatrix} 2a & c & 0 \\ 2b & a + d & 2c \\ 0 & b & 2d \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \end{array}$$

Compute the stabilizers of xy, x^2 and $x^2 + y^2$. What is the tangent space at the point f of the orbit $O(f)$? This is done by taking a basis of $Lie(GL_2)$ and hitting it on f. This is obtained by putting

$a = 1, b = 1, c = 1$ and $d = 1$ in turn. We get the 4 vectors:

$$\begin{bmatrix} 2A \\ B \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2A \\ B \end{bmatrix} \begin{bmatrix} B \\ 2C \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ B \\ 2C \end{bmatrix}$$

This can be succintly written as the column space of the matrix below:

$$\begin{bmatrix} 2A & 0 & B & 0 \\ B & 2A & 2C & B \\ 0 & B & 0 & 2C \end{bmatrix}$$

So what are the forms $f \in Sym^2(\mathbb{C}^2)$ whose orbits have dimension ≤ 2? That is given by the condition that all 3×3-minors of the matrix should have determinant 0.

Example 6.12. $Sym^3(\mathbb{C}^3)$ and adjoint action. Stabilizers.

Corollary 6.13. We have $dim(G) = dim(G_v) + dim(O(v))$. Moreover, the space $\{v | dim(G_v) \geq k\}$ is an algebraic subvariety of V.

Computing ρ_1 from ρ.

1. The general $Sym^4(\mathbb{C}^3)$ basis: A_{ijk} with $i + j + k = d$. How does the new A'_{ijk} depend on old A_{ijk} and the group entries b_{rs}?

$$\begin{array}{rcl} A'_{ijk} & = & \sum A_{pqr}(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)^p(a_{21}x_1 + a_{22}x_2 + a_{23}x_3)^q(a_{31}x_1 + a_{32}x_2 + a_{33}x_3)^r \\ & = & l_1^p l_2^q l_3^r \end{array}$$

Let us look at the matrix $I + (\epsilon_{ij})$. We need to compute

$$\epsilon_{ij} \frac{\partial l_k^p}{\partial_{a_{ij}}} |_I$$

This is of course zero unless $k = i$. Then we have:

$$\epsilon_{kj} \frac{\partial l_k^p}{\partial_{a_{kj}}} |_I = p \epsilon_{kj} x_k^{p-1} x_j$$

Thus, A'_{ijk} depends only on $A_{i'j'k'}$ where some index drops by 1 and the other increases by 1.

$$A'_{ijk} \quad = \quad A_{i+1,j-1,k}(i+1)\epsilon_{i,j} + A_{i,j-1,k+1}(k+1)\epsilon_{kj} + \ldots$$

Verify this for $Sym^2(\mathbb{C}^2)$.

$$\begin{bmatrix} \delta A_{20} \\ \delta A_{11} \\ \delta A_{02} \end{bmatrix} = \begin{bmatrix} 2a_{11} & a_{21} & 0 \\ 2a_{12} & a_{11} + a_{22} & 2a_{21} \\ 0 & a_{12} & 2a_{22} \end{bmatrix} \begin{bmatrix} A_{20} \\ A_{11} \\ A_{02} \end{bmatrix}$$

2. What about $X \to AXA^{-1}$? This leads to two questions: (i) What is $\epsilon_{ij} \frac{\partial det(A)}{\partial a_{ij}}$ evaluated at the identity? and (ii) If M_{ij} is the (i, j)-th minor, what is $\epsilon_{ij} \frac{\partial M_{pq}}{\partial a_{ij}}$ when evaluated at the identity, where M_{pq} is obtained by dropping row p and column q? Let's look at (i)

$$\epsilon_{ij} \frac{\partial det(A)}{\partial a_{ij}} = 0 \text{ unless } i = j \text{ and then it is } \epsilon_{ii}$$

What about (ii)? $\epsilon_{ij} \frac{\partial M_{pq}}{\partial a_{ij}}$? Firstly, $i \neq p$ and $j \neq q$ for otherwise it is zero. Case (i) $p = q$ then we can apply (i) to get ϵ_{ii}. Case (ii) $p \neq q$. Then $\epsilon_{qp} \frac{\partial M_{pq}}{\partial a_{qp}} = \epsilon_{qp}$. Now, in $B = A^{-1}$ we have $b_{ij} = M_{ji}/det(A)$. Let us use the chain rule.

6.1 Actual Lecture Notes

Why does ρ_1 satisfy the Lie bracket property?

1. $\rho_1(\mathfrak{g}, p)$ may be calculated through the action of $\rho(e^{\mathfrak{g}t})(p)$. Indeed $\frac{d}{dt}\rho(e^{\mathfrak{g}t})\mid_{t=0}= \rho_1(\mathfrak{g})\cdot p$.
2. Then $\rho(e^{\mathfrak{g}t}e^{\mathfrak{h}t}e^{-\mathfrak{g}t}e^{-\mathfrak{h}t}) = \rho(e^{[\mathfrak{g},\mathfrak{h}]t+\cdots})$.
3. On the one hand, using the previous point and the definition of ρ_1,
$$\rho(e^{\mathfrak{g}t}) = \exp\bigl(t\,\rho_1(\mathfrak{g}) + O(t^2)\bigr), \qquad \rho(e^{\mathfrak{h}t}) = \exp\bigl(t\,\rho_1(\mathfrak{h}) + O(t^2)\bigr),$$
so in $GL(V)$ we have, by the Baker–Campbell–Hausdorff formula,
$$\rho(e^{\mathfrak{g}t}e^{\mathfrak{h}t}e^{-\mathfrak{g}t}e^{-\mathfrak{h}t}) = \exp\bigl(t^2[\rho_1(\mathfrak{g}), \rho_1(\mathfrak{h})] + O(t^3)\bigr).$$
On the other hand,
$$\rho(e^{\mathfrak{g}t}e^{\mathfrak{h}t}e^{-\mathfrak{g}t}e^{-\mathfrak{h}t}) = \rho\bigl(e^{[\mathfrak{g},\mathfrak{h}]t^2+O(t^3)}\bigr) = \exp\bigl(t^2\rho_1([\mathfrak{g},\mathfrak{h}]) + O(t^3)\bigr).$$
Comparing the t^2–terms in the exponents gives
$$\rho_1([\mathfrak{g},\mathfrak{h}]) = [\rho_1(\mathfrak{g}), \rho_1(\mathfrak{h})],$$
which is exactly the Lie bracket property.

Thus the derivative $\rho_1 \colon \mathcal{G} \to \mathrm{End}(V)$ of a group representation $\rho\colon G \to GL(V)$ is automatically a representation of the Lie algebra $\mathcal{G}$.

Lie subgroups of matrix groups defined by equations

Let $G \subset GL_n(\mathbb{C})$ be a Lie subgroup defined (near the identity) by equations
$$f_1(A) = 0, \ldots, f_r(A) = 0,$$
where the f_i are smooth (typically polynomial) functions on $M_n(\mathbb{C})$. Then the Lie algebra
$$\mathrm{Lie}(G) = T_I G$$
consists of those $X \in M_n(\mathbb{C})$ such that the tangent vector X satisfies the linearised equations
$$(df_i)_I(X) = 0, \qquad i = 1, \ldots, r.$$

Example 6.14. Here
$$SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) \mid \det A = 1\},$$
so $f(A) = \det A - 1$. Take a curve $A(t) = I + tX + O(t^2)$ with $A(0) = I$, $A'(0) = X$. The condition $A(t) \in SL_n$ means
$$\det(I + tX + O(t^2)) = 1.$$
Using $\det(I + tX) = 1 + t\mathrm{tr}(X) + O(t^2)$ we obtain $\mathrm{tr}(X) = 0$. Hence
$$\mathrm{Lie}(SL_n) = \{X \in M_n(\mathbb{C}) \mid \mathrm{tr}X = 0\} = \mathfrak{sl}_n.$$

Example 6.15. Over $\mathbb{R}$,

$$O_n(\mathbb{R}) = \{g \in GL_n(\mathbb{R}) \mid g^T g = I\}.$$

Write $g(t) = I + tX + O(t^2)$. Then

$$g(t)^T g(t) = (I + tX^T + \cdots)(I + tX + \cdots) = I + t(X + X^T) + O(t^2).$$

The requirement $g(t)^T g(t) = I$ gives $X + X^T = 0$, so

$$\mathrm{Lie}(O_n) = \{X \in M_n(\mathbb{R}) \mid X^T + X = 0\} = \mathfrak{so}_n.$$

The same argument with $\det g = 1$ shows $\mathrm{Lie}(SO_n) = \mathfrak{so}_n$ as well.

If $G_1 \subset GL_{n_1}$ and $G_2 \subset GL_{n_2}$, then

$$G_1 \times G_2 = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \middle| \, g_i \in G_i \right\}.$$

A curve through the identity has derivative

$$\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \qquad X_i \in \mathrm{Lie}(G_i),$$

so

$$\mathrm{Lie}(G_1 \times G_2) \cong \mathrm{Lie}(G_1) \oplus \mathrm{Lie}(G_2).$$

Orbit maps, stabilisers and tangent spaces

Let G be a Lie group acting smoothly on a vector space V:

$$\rho\colon G \times V \to V, \qquad (g, v) \mapsto g \cdot v.$$

Fix $v \in V$ and consider the *orbit map*

$$\phi_v\colon G \to V, \qquad g \longmapsto g \cdot v.$$

Its image is the orbit $G \cdot v$. Differentiating at the identity $e \in G$ we obtain

$$(\phi_v)_*\colon T_e G = \mathcal{G} \longrightarrow T_v V \cong V.$$

By definition of the derived representation,

$$(\phi_v)_*(\mathfrak{g}) = \rho_1(\mathfrak{g})(v), \qquad \mathfrak{g} \in \mathcal{G}.$$

The image of $(\phi_v)_*$ is the tangent space to the orbit:

$$T_v(G \cdot v) = \{\rho_1(\mathfrak{g})(v) \mid \mathfrak{g} \in \mathcal{G}\} \subseteq V.$$

The stabiliser subgroup of v is

$$G_v = \{g \in G \mid g \cdot v = v\}.$$

A curve $g(t)$ in G with $g(0) = e$ lies in G_v iff $g(t) \cdot v = v$ for all t, so differentiating at 0 gives

$$\rho_1(\mathfrak{g})(v) = 0, \qquad \mathfrak{g} = g'(0).$$

Hence the Lie algebra of the stabiliser is

$$\mathrm{Lie}(G_v) = \{\mathfrak{g} \in \mathcal{G} \mid \rho_1(\mathfrak{g})(v) = 0\} = \ker\big((\phi_v)_*\big).$$

From the rank–nullity theorem we obtain the fundamental *dimension formula*

$$\dim(G \cdot v) = \dim T_v(G \cdot v) = \dim \mathcal{G} - \dim \mathrm{Lie}(G_v).$$

Because ρ_1 is a Lie algebra representation, the kernel of the linear map

$$\mathcal{G} \to V, \qquad \mathfrak{g} \mapsto \rho_1(\mathfrak{g})(v),$$

is a Lie subalgebra: if $\mathfrak{g}, \mathfrak{h} \in \ker$ then $\rho_1(\mathfrak{g})(v) = \rho_1(\mathfrak{h})(v) = 0$, so

$$\rho_1([\mathfrak{g}, \mathfrak{h}])(v) = [\rho_1(\mathfrak{g}), \rho_1(\mathfrak{h})](v) = \rho_1(\mathfrak{g})(\rho_1(\mathfrak{h})(v)) - \rho_1(\mathfrak{h})(\rho_1(\mathfrak{g})(v)) = 0,$$

hence $[\mathfrak{g}, \mathfrak{h}] \in \ker$. Thus $\mathrm{Lie}(G_v)$ is a Lie subalgebra of $\mathcal{G}$.

Example 6.16. Consider the standard action of $GL_2(\mathbb{C})$ on $\mathbb{C}^2$ and the induced action on homogeneous quadratic polynomials

$$\mathrm{Sym}^2(\mathbb{C}^2) = \{Ax_1^2 + Bx_1x_2 + Cx_2^2 \mid A, B, C \in \mathbb{C}\}.$$

In the basis $\{x_1^2, x_1x_2, x_2^2\}$ we identify a quadratic form with a column vector

$$v = \begin{pmatrix} A \\ B \\ C \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{gl}_2.$$

A small group element is $g(t) = I + tX + O(t^2)$ and it acts on $x = (x_1, x_2)^T$ by $x \mapsto g(t)x$. Expanding

$$(g(t)x)_1 = (1 + ta)x_1 + tbx_2, \qquad (g(t)x)_2 = tcx_1 + (1 + td)x_2,$$

and substituting into $Ax_1^2 + Bx_1x_2 + Cx_2^2$, then collecting the coefficient of t, one finds that the derived representation $\rho_1(X)$ is represented in the basis $\{x_1^2, x_1x_2, x_2^2\}$ by the matrix

$$\rho_1(X) = \begin{pmatrix} 2a & c & 0 \\ 2b & a + d & 2c \\ 0 & b & 2d \end{pmatrix}.$$

Example 6.17. We now look at stabilisers of some specific quadratic forms.

1. For $p_1 = x_1^2$ we have $v_1 = (1, 0, 0)^T$ and

$$\rho_1(X)v_1 = \begin{pmatrix} 2a \\ 2b \\ 0 \end{pmatrix}.$$

Hence $\rho_1(X)v_1 = 0$ iff $a = b = 0$, while c, d are arbitrary. Thus

$$\mathrm{Lie}(G_{v_1}) = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \middle| \, c, d \in \mathbb{C} \right\},$$

which has dimension 2, and

$$\dim(G \cdot v_1) = \dim \mathfrak{gl}_2 - \dim \mathrm{Lie}(G_{v_1}) = 4 - 2 = 2.$$

2. For $p_2 = x_1x_2$ we have $v_2 = (0,1,0)^T$ and

$$\rho_1(X)v_2 = \begin{pmatrix} c \\ a+d \\ b \end{pmatrix}.$$

Hence $\rho_1(X)v_2 = 0$ iff $c = 0$, $b = 0$ and $a + d = 0$, i.e. X is diagonal with entries $\lambda, -\lambda$. Thus

$$\mathrm{Lie}(G_{v_2}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \middle| \lambda \in \mathbb{C} \right\},$$

which is 1–dimensional, and therefore

$$\dim(G \cdot v_2) = 4 - 1 = 3.$$

Adjoint representation

When $G \subseteq GL_n$ acts on itself by conjugation,

$$\mathrm{Ad}_g(h) = ghg^{-1},$$

the derivative at e in the h–variable gives the *adjoint representation* of G on its Lie algebra:

$$\mathrm{Ad}\colon G \to GL(\mathcal{G}), \qquad \mathrm{Ad}_g(\mathfrak{h}) = g\mathfrak{h}g^{-1}.$$

Differentiating Ad at e we obtain a Lie algebra representation

$$\mathrm{ad}\colon \mathcal{G} \to \mathrm{End}(\mathcal{G}), \qquad \mathrm{ad}_{\mathfrak{g}}(\mathfrak{h}) = [\mathfrak{g}, \mathfrak{h}],$$

which is simply the Lie bracket.
For $G = GL_n$ this is very concrete:

$$\mathrm{Ad}_g(X) = gXg^{-1}, \qquad \mathrm{ad}_X(Y) = XY - YX.$$

Lecture 7

Scribe: Someone

Geometric Complexity Theory The Tangent Space Action.

1. Statement: G acts on V, and $p \in V$ with $G_p = H$. Then $T_p O(p)$ is fixed by H.

2. Examples: The adjoint action as an example for various points. $x^2 + y^2$.

$$p = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \; Xp - pX = T_p = \begin{bmatrix} 0 & x_{12} \\ x_{21} & 0 \end{bmatrix} G_p = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

   The action is given by:

$$G_p \cdot T_P = \begin{bmatrix} 0 & (\mu_1 - \mu_2)x_{12} \\ (\mu_2 - \mu_1)x_{21} & 0 \end{bmatrix}$$

   Another example. $p = (x_1^2 + x_2^2)^2$ in $Sym^4(\mathbb{C}^2)$. The Tangent space is given by $\epsilon_i \frac{\partial p}{\partial x_j}$ for all i, j. We get:

$$T_p = p \cdot x_1^2, p \cdot x_1 x_2, p \cdot x_2^2$$

3. Two key examples: $perm_n, det_n \in Sym^n(X_n)$, their stabilizers and tangent spaces.

4. Main Theorem. Let I be a G-invariant ideal. Let $(\nabla_I)_p$ be the gradients at p. Then the null-space of $(\nabla_I)_p$ is fixed by H.

5. Examples of filtrations corresponding to subvarieties. Let $G = D_3$ the group of diagonal matrices of size 3 acting on $\mathbb{C}^3$. Let $p = [1, 1, 1]^T, q = [1, 1, 0]^T$ and $z = [1, 0, 0]^T$.

Lecture 8

Scribe: Someone

The overall plan:

1. The problem: $\lim_{t\to 0}\lambda(t)\cdot y = z$. Let $G_y = K$ and $G_z = H$. Under what conditions can this happen?
2. Strategy: Find common structures which are acted upon by H and K.
3. Construct an infinitesimal action of G and $\mathcal{G}$ near z.
4. Extend this locally in a finite neighborhood.

The quotient actions in the reductive case (when H is reductive).

1. The quotient $\mathcal{G} = \mathcal{H} + \mathcal{S}$. Note that $\mathcal{S}z = T_z$, the tangent space which is an H-module. Is there an $\mathcal{S}$ such that $\mathcal{S} \cong T_z$. Yes, if H is reductive - $\mathcal{S}$ is merely the complement to $\mathcal{H}$ within $\mathcal{G}$. Check that $\mathcal{S} \cong T_z$ as H-modules.
2. Next consider $T_z \subset V$ as H-modules. Then, there is a complement N which is also an H-module.
3. Complementarity of N in a neighborhood of $z \in O(z)$. Reduction to the point $z+n$.
4. Computing the action at a point $x+n$: $\mathfrak{h} + \mathfrak{s}(z+n) = \mathfrak{s}\cdot z + \mathfrak{s}\cdot n + \mathfrak{h}\cdot n$. Express this in $N \oplus \mathcal{S}\cdot(z+n)$ obtain that $\mathfrak{s} = 0$ and $\mathfrak{h}\cdot n = 0$.
5. **Thus stabilizers near z are subgroups of H!** Luna slice theorem.
6. Example.

$$z = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} H = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \mathcal{S} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & 0 \end{bmatrix} N = \begin{bmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{bmatrix}$$

The general case.

1. Consider the case $z+\epsilon n$ and a stabilizer of the form $h + \epsilon h' + \epsilon s$.

$$(h + \epsilon h' + \epsilon s)\cdot(z+\epsilon n) = 0 \Rightarrow sz + hn = 0$$

2. An example. $V = Sym^3(\mathbb{C}^2)$ with $z = x_1^3$.

$$\begin{array}{rcl} T_z &=& \mathbb{C}x_1^2x_2 + \mathbb{C}x_1^3 \\ \mathcal{H} &=& ax_1\frac{\partial}{\partial x_2} + bx_2\frac{\partial}{\partial x_2} \\ \mathcal{S} &=& cx_1\frac{\partial}{\partial x_1} + dx_2\frac{\partial}{\partial x_1} \end{array}$$

Take $y = z + \epsilon n$, where $n = x_1x_2^2$. We have

$$\begin{array}{rcl} \mathfrak{h}\cdot n = \mathfrak{h}\cdot x_1x_2^2 &=& 2ax_1x_2^2 + 2bx_1^2x_2 \\ \mathfrak{s}\cdot z = \mathfrak{s}\cdot x_1^3 &=& 3cx_1^3 + 3dx_1^2x_2 \end{array}$$

This gives us the only stabilizer $3x_2\frac{\partial}{\partial x_2} - 2\epsilon x_2\frac{\partial}{\partial x_1}$. One may verify:

$$(3x_2\frac{\partial}{\partial x_2} - 2\epsilon x_2\frac{\partial}{\partial x_1})(x_1^3 + \epsilon x_1x_2^2) = 0$$

What about the orbit: Check:

$$\begin{array}{rcl} p(t) &=& (x_1+tx_2)^2(x_1-2tx_2) = (x_1^2 + 2tx_1x_2 + t^2x_2^2)(x_1 - 2tx_2) \\ &=& x_1^3 - 3t^2x_1x_2^2 - 2t^3x_2^3 \end{array}$$

3. Theorem: Given any n there is a subspace $L_n \subseteq \mathcal{H}$ and a linear map $s : L_n \rightarrow \mathcal{S}$ such that $\{\ell + s(\ell) \mid \ell \in L_n\}$ is the stabilizer of $z + \epsilon n$.

Stabilizer Limits: How to do it for an orbit closure?

1. 1-PS and reductivity and the action of V and $Lie(G)$. Picture.
2. Let $\lambda(t)$ be such that:
$$\lambda(t)y = t^0 z + t^1 y_1 + \ldots + t^D y_D$$
We write this as $y \xrightarrow{\lambda} z$.
3. Let $\mathfrak{k} \in \mathcal{K} = Lie(G_y)$. Now $\mathfrak{k}(t) = \lambda(t)\mathfrak{k}\lambda(t)$ stabilizes $y(t)$ for every t. So what is limit $\mathfrak{k}(t)$?
4. Leading term and the equation: $\mathfrak{k}(t)y(t) = 0$. The stabilizer limit theorem: Let $\widehat{\mathcal{K}}$ be the linear combination of all leading terms $\widehat{\mathfrak{k}}$, for $\mathfrak{k} \in \mathcal{K}$. Then $\widehat{\mathcal{K}}$ is a Lie subalgebra of $\mathcal{H}$.
5. Examples.
Example 8.1. Consider a set of indeterminates $\{x_1, x_2, x_3\}$ and let $X = \mathbb{C} \cdot \{x_1, x_2, x_3\}$, be the space of linear combination of these indeterminates. Let $G = GL_3$ and $\mathcal{G} = Lie(G)$. Let G act on $V = Sym^4(X)$, the space of homogeneous forms of degree 4 by substitution. In the resulting $\mathcal{G}$-action, an element of $\mathcal{G}$ acts as a differential operator. More precisely, $\mathfrak{g} = (a_{ij})$ acts on a form $v \in V$ as: $\mathfrak{g}.v = \sum_{i,j} a_{ij} x_i \frac{\partial v}{\partial x_j}$. Consider $f = (x_1^2 + x_2^2 + x_3^2)^2 \in V$. The stabilizer algebra $\mathcal{G}_f = \mathcal{K}$ is 3-dimensional and is given below.

Consider next the 1-PS $\lambda(t) \subseteq GL_3$ given by $\lambda(x_1) = x_1, \lambda(x_2) = x_2$ and $\lambda(x_3) = tx_3$. We have $\widehat{f} = g = (x_1^2 + x_2^2)^2$ as the leading term of $\lambda(t)f$ of degree 0. The stabilizer $\mathcal{G}_g = \mathcal{H}$ is 4-dimensional and is shown below (with $a, b, c, d \in \mathbb{C}$). Note that, since x_3 is absent in g, the operators $\nabla = (bx_1 + cx_2 + dx_3)\frac{\partial}{\partial x_3}$ certainly stabilize g.

$$\mathcal{G}_f = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \quad \lambda(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{bmatrix} \quad \mathcal{G}_g = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ 0 & 0 & d \end{bmatrix}$$

The adjoint action of $\lambda(t)$ on $\mathcal{K}$ and the resulting $\widehat{\mathcal{K}}$, the 3-dimensional Lie algebra of leading terms, are shown below.

$$\lambda(t)\mathcal{K}\lambda(t)^{-1} = \begin{bmatrix} 0 & a & t^{-1}b \\ -a & 0 & t^{-1}c \\ -tb & -tc & 0 \end{bmatrix} \quad \widehat{\mathcal{K}} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \subseteq \mathcal{H}$$

Thus the "leading term" operation applied to $\mathcal{K}$ inserts $\widehat{\mathcal{K}}$ into the stabilizer $\mathcal{H}$ of the limit $g = \widehat{f}$.

Example 8.2. Let us consider $G = GL_4(\mathbb{C})$ and the 1-parameter subgroup $\lambda(t)$ below. The action of λ on a typical element $M \in \mathcal{G}$ is given below, where each M_{ij} is a 2×2-matrix. Note that the degrees which occur are $-1, 0$ and 1, thus we have the spaces $\mathcal{G}_{-1}, \mathcal{G}_0$ and $\mathcal{G}_1$, of matrices with leading terms of degree $-1, 0$ and 1 respectively. We consider $\mathcal{K}$ as given below and compute $\widehat{\mathcal{K}}$.

$$\lambda(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix} \quad \lambda(t)M\lambda(t)^{-1} = \begin{bmatrix} M_{11} & t^{-1}M_{12} \\ tM_{21} & M_{22} \end{bmatrix} \quad \mathcal{K} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

Let $\mathcal{K}_i = \mathcal{G}_{j \geq i} \cap \mathcal{K}$. We then have $dim(\mathcal{K}_{-1}) = 4, dim(\mathcal{K}_0) = 4$ and $dim(\mathcal{K}_1) = 0$. Thus, we get a basis $B = B_0$ of $\mathcal{K}$ of dimension 4. Moreover, since each element of B_0 is homogeneous, we have $\widehat{B_0} = B_0$ and $\widehat{\mathcal{K}} = \mathcal{K}$.

Next, let A be as shown below and $\mathcal{K}' = A\mathcal{K}A^{-1}$ as given below.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathcal{K}' = \begin{bmatrix} a & b & c & d-a \\ c & d & 0 & -c \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

As a result, $\mathcal{K}(t)$ and $\widehat{\mathcal{K}}'$ are as given below:

$$\mathcal{K}'(t) = \begin{bmatrix} a & b & t^{-1}c & t^{-1}(d-a) \\ c & d & 0 & -t^{-1}c \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

Let $\mathcal{K}'_i = \mathcal{K}' \cap \mathcal{G}_{j \geq i}$ and note that $dim(\mathcal{K}'_{-1}) = 4$. Now $\mathcal{K}'_0$ consists of all elements $\mathfrak{k} \in \mathcal{K}'$ which have no term of degree -1. This forces $d = a$ and $c = 0$, making $dim(\mathcal{K}'_0) = 2$. Finally $dim(\mathcal{K}'_1) = 0$. Thus, we have B_{-1} and B_0, each of cardinality two contributing to $B = B_{-1} \cup B_0$, such that $\widehat{B}$ would be a basis of $\widehat{\mathcal{K}}'$. This results in $\widehat{\mathcal{K}}'$ as given, with $r, s, t, u \in \mathbb{C}$ as given below.

$$\widehat{\mathcal{K}}' = \begin{bmatrix} u & t & s & r \\ 0 & u & 0 & -s \\ 0 & 0 & u & t \\ 0 & 0 & 0 & u \end{bmatrix}$$

Note that while $\widehat{\mathcal{K}}$ is reductive, $\widehat{\mathcal{K}}'$ is a solvable Lie algebra.

GCT I

1. GCT I: The affine pull-back problem of g from f. The homogenization and the 1-PS formulation. The orbit closure and the witness formulation. The Peter-Weyl condition and the Obstruction Conjecture.

2. GCT II: The action of λ on V and $\mathcal{G}$. The basic equation:

$$\lambda(t)y = t^d y_d + t^e y_e + \ldots + t^D y_D$$

   The weight spaces and leading terms. The first theorem: leading term Lie algebra $\hat{\mathcal{H}}$ of $\mathcal{H}$ and module $\hat{N}$ of N. The second theorem: $\lim_{t\to} t^{-d}y(t) = z$ and its implication $\widehat{\mathcal{G}_y} \to \mathcal{H}_{\overline{y_e}} \subseteq \mathcal{H}$ (where $\mathcal{H} = \mathcal{G}_z$).

3. Alignment and the dichotomy result. Consequences of alignment - rectangular decompositon. The absence of alignment and intermediate G-varieties.

Lecture 9

Scribe: Someone

Take-Home Exam

Complexity Theory.

1. **[Nilabha]** Let X_n be the space of $n \times n$-matrices and let $A : W \to X_n$ be a linear injection. Let $Y = A(W)$ and Z be a complement to Y, i.e., $X_n = Y \oplus Z$. Let $\lambda_A(t)(Z) = t^{-1}Z$ and $\lambda_A(t)(Y) = Y$, be a 1-PS. Let $g = det_n \circ A$. Show that if $g' = \lim_{t \to 0} \lambda_A(t) \cdot det_n$, then $g' \in Sym^n(Y)$ and is a copy of $g \in Sym^n(W)$. Let $\lambda_A(t) \cdot det_3 = y_0 + t^1 y_1 + t^2 y_2 + t^3 y_3$. Use a computer tool to compute the tangent of approach y_1. Try this for A on $W \cong \mathbb{C}^6$ and A' on $\mathbb{C}^5$ below:

$$A = \begin{bmatrix} 0 & w_1 & w_2 \\ w_3 & w_5 & 0 \\ w_4 & 0 & w_6 \end{bmatrix} A' = \begin{bmatrix} 0 & w_1 & w_2 \\ w_3 & w_5 & 0 \\ w_4 & 0 & w_5 \end{bmatrix}$$

   Can you generalize for A of such form?

2. **[Aditya]**Let $W_3 = (w_{ij})$ be a 3×3-matrix of indeterminates and w_0 be a homogenizing variable. Consider the map $\overline{W_3} = W_3 \oplus w_0$ to X_7, the space of 7×7-matrices, as below. Thus, there are $9 + 1$ matrices, which are the support of the variables (w_{ij}) and w_0 and $A(w, w_0) = \sum_{ij} Y_{ij} w_{ij} + Y_0 w_0$. Y_0, for example is the almost-identity matrix.

$$G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & w_{33} & w_{32} & w_{31} \\ w_{11} & w_0 & 0 & 0 & 0 & 0 & 0 \\ w_{12} & 0 & w_0 & 0 & 0 & 0 & 0 \\ w_{13} & 0 & 0 & w_0 & 0 & 0 & 0 \\ 0 & w_{22} & w_{21} & 0 & w_0 & 0 & 0 \\ 0 & w_{23} & 0 & w_{21} & 0 & w_0 & 0 \\ 0 & 0 & w_{23} & w_{22} & 0 & 0 & w_0 \end{bmatrix}$$

   Thus $G_3 : \overline{W} \to X_7$. (ii) Verify that the pull-back of $det_7 \in Sym^7(X_7)$ is indeed the padded permanent $w_0^4 perm(W_3)$. (ii) Identify the image $Y = A(\overline{W})$ and identify invariants which preserve the space Y. For examples, the operation - multiply row 1 by c and column 1 by c^{-1} preserves both the determinant as well as the space Y. Verify that they also preserve the padded permanent.

   Can we compute the tangent of approach?

3. **[Aditya]** Review Valiant's method of computing the matrix A_g for a form g, given its formula. Thus $g = det_n \circ A_g$. Suppose that for a systematic choice of Z and λ_{A_g}, the form g_1 is the tangent of approach. Valiant provides a method of computing $A_{g*g'}$ and $A_{g+g'}$ from A_g and $A_{g'}$. Is it possible to compute $(g + g')_1$ and $(gg')_1$ from the data on g and g'?

4. **[Nilabha]** Given two matrices $A, B \in V = \mathbb{C}^{3 \times 3}$ under conjugation. Write pseudo-code to detect if $B \in O(A)$.

Basic orbits and closures.

1. **[Nilabha]** Let $V = \mathbb{C}^{m \times n}$ with $m < n$. Let $rref(A)$ be the reduced row echelon form of $A \in V$. Let I, J be the row and column indices of $rref(A)$. For a given A, let $V(A)$ be all $B \in V$ such that $rref(B)$ has the same row and column indices as $rref(A)$. Write defining equations for $V(A)$. What is its dimension?

2. **[Nilabha]** Is $V(A)$ the orbit of A under a group H acting on V? If so, what is H? What about $V(B)$ for other general B?

3. **[Aditya]**Let V the space of $2 \times n$ matrices and $X \subseteq V$ as those of rank 1. What are the defining equations E for X. How will you prove that E generates the ideal $I(X)$? What is its dimension and tangent space at a special point p where $p(1,1) = -1$ and zero everywhere else.

4. **[Aditya]**Recall he notation J_n to the standard nilpotent matrix of size n. List all the orbits in the closure of $p = J_4 \oplus J_3 \oplus J_1$. Illustrate a proof that $q = J_4 \oplus J_2 \oplus J_2$ is in the closure of $O(p)$.

Invariant Theory.

1. **[Nilabha]** Let G be a reductive group and $X \subseteq \mathbb{C}^n$ be a variety. Show that $I(X)$, the ideal of X, is a G-module. Moreover, show that it has a generating set $(f_1, \ldots, f_r)$ which is a finite dimensional G-module.

2. **[Nilabha]** Let $G = \mathbb{C}^*$ and ρ_1, V_1 be a 1-dimensional representation. Show that $\rho_1 : t \to t^d$, for some $d \in \mathbb{Z}$. Now do this for V of dimension 2.

3. **[Aditya]** Consider the action of $G = SL_2(\mathbb{C})$ on $V = Sym^2(\mathbb{C}^2)$. Let a general polynomial p be denoted by $Ax^2 + Bxy + Cy^2$. Consider the space $Sym^d(A, B, C)$ of polynomials in A, B, C. For $d = 2$ and $f = A^2$, find the module M_f and N_f and therefore, the invariant $\pi^G(f)$. Do the same for $f' = A^3$.

4. **[Aditya]** Let $V = \mathbb{C}^{m \times m}$. Let $D = diag(\mathbb{C}^m) \times diag(\mathbb{C}^m)$ with the condition that $\prod \alpha\beta = 1$. Let $(\alpha \times \beta)(x) = \alpha x \beta^{-1}$. Compute the ring of invariants - in other words, define $\mathbb{C}[V]_d^D$, the subspace of invariants of degree d. Let H be a subgroup of $GL(V)$ which normalizes D and let $K = H \ltimes D$ be semi-direct product. Examine if $\mathbb{C}[V]^K$ is obtained as $(\mathbb{C}[v]^D)^H$. Take $H = \mathbb{S}_m \times \mathbb{S}_m$, the group of left-right permutations with $(\sigma \times \mu)(x) = \sigma x \mu^{-1}$.

5. **[Nilabha]** Let G be a finite group and V be a G-module. Let H be a subgroup of G and $V^H = \{v \in V | Hv = v\}$. Compare the poset of subgroups of G with the spaces of invariants V^H for $H \subseteq G$. Experiment with the symmetric group $\mathbb{S}_3$ acting on $\mathbb{C}^3$.

Lie algebras and limits.

1. **[Nilabha]** Consider the action of $G = GL_3(\mathbb{C})$ on $V = Sym^2(\mathbb{C}^3)$, the 6-dimensional space of forms of degree 2 on 3 variables x_1, x_2, x_3. Compute sample matrices for the action of $gl_3 = Lie(G)$ on the action of V.

2. **[Nilabha]** Let $V = \mathbb{C}^{3\times 3}$ under conjugation. Let

$$p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Compute the Lie algebra $\mathcal{H}$ of the stabilizer of p and a complement $\mathcal{S}$. Do the same for $q \in \mathbb{C}^{4\times 4}$:

$$q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

3. **[Aditya]** Compute the stabilizer Lie algebra $\mathcal{H}$ of the point $p = 4x_1x_3$ for the above case. Identify the tangent space T_p and compute the action of $\mathcal{H}$ on T_p and V/T_p. Are there any intermediate $\mathcal{H}$-closed subspaces?

4. **[Aditya]** Consider the form $q = 4x_1x_3 - x_2^2$. What is $K = G_q$ and its Lie algebra? Consider the 1-PS $\lambda(t)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Verify that $\lim_{t\to 0} \lambda(t)q = p$. Compute the limit $\widehat{\mathcal{K}}$ of $\mathcal{K}$ and verify that $\widehat{\mathcal{K}} \subseteq \mathcal{H}$. What elements in $\mathcal{H}$ do not come as limits? Can you see why?

5. **[Nilabha]** Compute the stabilizer of the point $q(\epsilon) = p + \epsilon x_2^2$ in the following steps: (i) compute a complement $\mathcal{S}$ and a normal space N (ii) Compute $\mathcal{S} \cdot p$ and $\mathcal{H} \cdot x_2^2$, (iii) compute $\mathcal{G}_{q(\epsilon)}$. Does the above computation connect with $\mathcal{K}$?