

Stabilizer limits and tangent ideals

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Abstract. Let G be an algebraic group defined over $\mathbb{C}$ and V be a finite dimensional G -module. Let $y \in V$ and $O(y)$ be the G -orbit of y in V and $\overline{O(y)}$, its Zariski closure. Let $z \in \overline{O(y)} - O(y)$ and $\gamma(t) \subset G$ be a 1-parameter family such that $\gamma(t) \cdot y = z + t^e y_e + \dots$, where $y_e \neq 0$ and $e > 0$. Let $O(z)$ be the orbit of z , T_z its tangent space at z , and $\overline{N}$ be the quotient V/T_z . Let K, H be the stabilizers of y, z in G and $\mathfrak{k}, \mathfrak{h}$ be their corresponding Lie algebras. The purpose of this paper is to study the connections between K and H in the passage to limits.

Using $\gamma(t)$ we construct from $\mathfrak{k}$, a leading term Lie algebra $\widehat{\mathfrak{k}(t)}$ of the same dimension, and show that $\widehat{\mathfrak{k}(t)} \subseteq \mathfrak{h}_{\overline{y_e}}$, the stabilizer within $\mathfrak{h}$ of $\overline{y_e} \in \overline{N}$, the tangent of approach. When $\widehat{\mathfrak{k}(t)} \subsetneq \mathfrak{h}_{\overline{y_e}}$, the question arises if there is a G -variety W between $\overline{O(y)}$ and $\overline{O(z)}$. Towards this, let $I_y \subset I_z \subset \mathbb{C}[V]$ be the ideals defining $\overline{O(y)}$ and $\overline{O(z)}$ respectively, and R be the associated graded ring of the ideal I_z of $\mathbb{C}[V]$. Using $\gamma(t)$ and y_e , we construct a G -invariant ideal $\overline{J}$ of R between $\overline{I_y}$ and $\overline{I_z}$ and show that there is an open set $O \subseteq \text{Spec}(R)$, such that the dimension of $V(\overline{J}) \cap O$ is either $\dim(\mathfrak{g}) - \dim(\mathfrak{h}_{\overline{y_e}})$ or one more. This sheds light on the local structure as $O(y)$ meets $O(z)$ along $\gamma(t)$. In the Lie algebraic setting, our results expand on some of the work of Kempf and Luna in the vicinity of closed orbits $O(z)$ of stable points, to general z .

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1. Introduction

Let X be a finite dimensional vector space over $\mathbb{C}$ and let $G \subseteq GL(X)$ be an algebraic group defined over $\mathbb{C}$. Let V be a finite dimensional G -module via a rational map $\rho : G \rightarrow GL(V)$ defined over $\mathbb{C}$. For a $g \in G$ and $v \in V$, let $\rho(g) \cdot v$, or simply gv denote the action of g on v via ρ .

Let t be an indeterminate and $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of formal power series with coefficients in $\mathbb{C}$. Let $k = \mathbb{C}((t))$ denote its quotient field of formal Laurent series in t with coefficients in $\mathbb{C}$, whose elements may have a pole of finite order at $t = 0$.

For any algebraic subgroup $L \subseteq G$ defined over $\mathbb{C}$, let $L(k)$ be elements of L with entries in k . In other words, elements $h \in L(k)$ are matrices with entries in k and satisfy the algebraic relations (over $\mathbb{C}$) which define L .

Let $y \in V$ and $O(y)$ denote the G -orbit of y and $\overline{O(y)}$, its Zariski closure in V . Let $z \in \overline{O(y)}$ but $z \notin O(y)$. Moreover, let $\gamma(t) \in G(k)$ be such that:

$$y(t) = \gamma(t) \cdot y = \sum_{i \geq 0} t^i y_i = z + t^e y_e + \dots \quad (1)$$

where $z = y_0$ and $y_e \neq 0$. We denote this as $y \xrightarrow{\gamma} z$ and call z the leading term or limit of y under $\gamma(t)$ and y_e as the tangent of approach. We say $\gamma(t)$ is a one parameter family (1-PF) driving y to z . It is known that $z \in \overline{O(y)}$ implies the existence of such a $\gamma(t) \in G(k)$. When G is reductive, $\gamma(t)$ assumes a special form, see [8, 13].

The motivation of this paper is to study the following question. Given elements $z, y \in V$ with stabilizers $H, K \subseteq G$, and the condition $y \xrightarrow{\gamma} z$, utilize $\gamma(t)$ to connect the stabilizers K and H .

2. Our Contributions

Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of G . Note that $\mathfrak{g}$ is a G -module under the adjoint action. Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of $H = G_z$ and $K = G_y$. Let $T_z O(z) = \mathfrak{g} \cdot z$ (or simply T_z), denote the tangent space of the orbit $O(z)$ at z . Note that T_z is an H -module. Let $\overline{N}$ be the H -module (as well as the $\mathfrak{h}$ -module) V/T_z which corresponds to a “normal” slice at z to the tangent space T_z .

Definition 2.1. Let $V' = V \otimes k$ and $\mathfrak{g}' = \mathfrak{g} \otimes k$. Then

1. For any $v' \in V'$ with $v' \neq 0$, let $v' = \sum_{i \geq a} t^i v_i$ with $v_a \neq 0$. We set $\text{ord}(v') = a$ and call it the order of v' , and $\widehat{v'}$ as the vector $v_a \in V$. Similarly, for any $\mathfrak{g}' \in \mathfrak{g}'$, with $\mathfrak{g}' \neq 0$, let $\mathfrak{g}' = \sum_{i \geq b} t^i \mathfrak{g}_i$ with $\mathfrak{g}_b \neq 0$. We set the order of $\mathfrak{g}'$ as $\text{ord}(\mathfrak{g}') = b$ and $\widehat{\mathfrak{g}'} = \mathfrak{g}_b \in \mathfrak{g}$. By convention, for $\bar{0}$ in V' or in $\mathfrak{g}'$, we set $\text{ord}(\bar{0}) = \infty$.
2. We denote by $(\dot{\cdot})$ the formal derivative with respect to t . For $v' \in V'$ written as $v' = \sum_i t^i v_i$, we set $\dot{v}' = \sum_i i t^{i-1} v_i$. For $g = (g_{ij}(t)) \in G(k)$, we set $\dot{g} = \left(\frac{d}{dt} g_{ij}(t) \right)$, the matrix obtained by differentiating each entry with respect to t .
3. For any $v \in V$ or $\mathfrak{g} \in \mathfrak{g}$, we set $v(t) = \gamma(t) \cdot v \in \widehat{V'}$ and $\mathfrak{g}(t) = \gamma(t) \cdot \mathfrak{g} = \gamma(t) \mathfrak{g} \gamma^{-1}(t) \in \mathfrak{g}'$. By abuse of notation, we set $\widehat{v} = \widehat{v(t)}$ and $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}(t)}$ and call them the respective leading terms. Moreover, we set $\text{ord}_\gamma(v) = \text{ord}(v(t))$ and $\text{ord}_\gamma(\mathfrak{g}) = \text{ord}(\mathfrak{g}(t))$.

Definition 2.2. Let y, z, γ be as above. We define

1. $\widehat{\mathfrak{k}} \subseteq \mathfrak{g}$ as the $\mathbb{C}$ -vector space generated by $\{\widehat{\mathfrak{k}} | \mathfrak{k} \in \mathfrak{k}\}$, the collection of leading terms of $\mathfrak{k}(t)$, with $\mathfrak{k} \in \mathfrak{k}$.
2. $\mathfrak{k}(t) \subseteq \mathfrak{g}'$ as the k -vector space generated by all vectors $\mathfrak{k}(t)$, for $\mathfrak{k} \in \mathfrak{k}$.

3. $\widehat{\mathfrak{k}(t)} \subseteq \mathfrak{g}$ as the $\mathbb{C}$ -vector space generated by $\widehat{\mathbf{k}'}$, where $\mathbf{k}' \in \mathfrak{k}(t)$.

We prove:

Theorem 2.3. *Let y, z, γ be as in Equation 1. Then $\widehat{\mathfrak{k}} \subseteq \widehat{\mathfrak{k}(t)} \subseteq \mathfrak{h}$ and $\dim(\widehat{\mathfrak{k}(t)}) = \dim(\mathfrak{k})$. Moreover, $\widehat{\mathfrak{k}(t)} \subseteq \mathfrak{h}_{\overline{y_e}}$, the Lie algebra stabilizer within $\mathfrak{h}$ of $\overline{y_e} \in \overline{N}$, where y_e is the tangent of approach.*

Definition 2.4. For the data y, z, γ as in equation 1, and $y(t) = \gamma(t)y$. We say that an element $\mathbf{v}(t) \in \mathfrak{g}'$ satisfies the **path condition** if $\mathbf{v}(t)y(t) = \dot{y}(t)$.

Note that $\mathbf{v}(t)$ in the above definition is a Laurent power series with coefficients in $\mathfrak{g}$. It is easily shown that a suitable $\mathbf{v}(t)$ satisfying the path condition may be constructed from $\gamma(t)$ itself.

Proposition 2.5. *Let y, z, γ be as in equation 1 and $y(t) = \gamma(t) \cdot y$. Moreover, suppose that $y_e \notin T_z$. Let $\mathbf{v}(t) \in \mathfrak{g}'$ satisfy the path condition. Then there is a $\mathbf{v}'(t)$ so that (i) it satisfies the path condition, and (ii) if $\ell = \widehat{\mathbf{v}'(t)}$, then $\ell \in \mathfrak{h}$, ℓ normalizes $\mathfrak{h}_{\overline{y_e}}$, i.e., $[\ell, \mathfrak{h}_{\overline{y_e}}] \subseteq \mathfrak{h}_{\overline{y_e}}$ and yet $\ell \notin \widehat{\mathfrak{k}(t)}$.*

The following proposition ensures that the tangent of approach is outside T_z at the cost of changing the 1-PF $\gamma(t)$.

Proposition 2.6. *Let $y \xrightarrow{\gamma} z$ and let N be a complement to T_z in V . Then there is a $\gamma'(t) \in G(k)$ such that $y'(t) = \gamma'(t) \cdot y \in N \otimes \mathcal{O}$ and $y \xrightarrow{\gamma'} z$.*

This has the following important corollary.

Corollary 2.7. *Let $z \in \overline{O(y)}$ be such that $\dim(O(z)) = \dim(O(y)) - 1$. Then there is a γ with a tangent of approach $y_e \notin T_z$, such that $y \xrightarrow{\gamma} z$ and $\mathfrak{h} = \widehat{\mathfrak{k}(t)} \oplus \mathbb{C}\ell$, where ℓ is obtained as $\mathbf{v}(t) \in \mathfrak{g}'$ satisfying the path condition. Moreover, either (i) $\widehat{\mathfrak{k}(t)} = \mathfrak{h}_{\overline{y_e}}$ and is normalized by ℓ , or (ii) $\mathfrak{h}_{\overline{y_e}} = \mathfrak{h}$.*

Note that $\dim(\mathfrak{h}) - \dim(\mathfrak{h}_{\overline{y_e}})$ is the dimension of the orbit of the tangent of approach $\overline{y_e}$ in $\overline{N}$, which may be identified by a transversal slice N at z to T_z . Since $O(z)$ is of dimension $\dim(\mathfrak{g}) - \dim(\mathfrak{h})$, this indicates a “local dimension” of $O(y)$ in the vicinity of z to be $(\dim(\mathfrak{g}) - \dim(\mathfrak{h})) + (\dim(\mathfrak{h}) - \dim(\mathfrak{h}_{\overline{y_e}})) = \dim(\mathfrak{g}) - \dim(\mathfrak{h}_{\overline{y_e}})$. Thus, when $\widehat{\mathfrak{k}(t)} \subsetneq \mathfrak{h}_{\overline{y_e}}$, this local dimension is less than $(\dim(\mathfrak{g}) - \dim(\mathfrak{k}))$, the dimension of $O(y)$. The question arises if there is an intermediate G -variety $\overline{O(z)} \subset W \subset \overline{O(y)}$ of dimension $(\dim(\mathfrak{g}) - \dim(\mathfrak{h}_{\overline{y_e}}))$ which is tangent to $O(y)$. Our next result concerns that.

Definition 2.8. Let $R = \oplus_{i \geq 0} R_i$ be the associated graded ring for the ideal $I_z \subseteq \mathbb{C}[V]$. In other words, $R_i = I_z^i / I_z^{i+1}$ (with $I_z^0 = \mathbb{C}[V]$).

Note that, for all i , R_i is a G -module. For any ideal $I \subseteq I_z$, set $I_i = I_z^i \cap I$ and $\bar{I}_i = (I_z^{i+1} + I_i)/I_z^{i+1}$ and let $\bar{I} = \bigoplus_{i \geq 0} \bar{I}_i$. Then $\bar{I}$ is an ideal of R . Note that for the G -invariant ideal $I_y \subseteq I_z$, the ideal $\bar{I}_y$ is a graded G -submodule of R .

Next, for an $f \in \mathbb{C}[V]$ and an indeterminate ϵ , we may evaluate f at the point $z + \epsilon y_e$ to obtain a polynomial $p_f = \sum_{i \geq 0} p_i \epsilon^i$.

Definition 2.9. For $f \in \mathbb{C}[V]$ we define $D_{z,y_e}^r(f)$, or simply $D^r(f)$ as the r -th coefficient p_r above.

The first coefficient D^1 in the above definition, is of course, the derivation associated with $y_e \in T_z V$. For any $f \in I_z^r$ and $f' \in I_z^{r+1}$ we have $D^r(f') = 0$, and $D^r(f + f') = D^r(f)$ making D^r a functional on I_z^r/I_z^{r+1} . We now come to our second result.

Theorem 2.10. Suppose that $y \xrightarrow{\gamma} z$ with y_e as the tangent of approach. Moreover suppose that $y_e \notin T_z O(z)$ and $\widehat{\mathfrak{k}(t)} \subsetneq \mathfrak{h}_{\overline{y_e}}$. Then there is a G -invariant ideal $\bar{J}_{z,y_e} \subseteq R$ or simply $\bar{J} = \bigoplus_{i \geq 1} \bar{J}_i$ such that:

1. For any $\bar{f} \in \bar{J}_r$ and any representative $f \in I_z^r$, we have $D^r(f) = 0$.
2. $\bar{I}_y \subseteq \bar{J} \subsetneq \bar{I}_z$.
3. Let $V(\bar{J}) \subseteq \text{Spec}(R)$ be the variety of $\bar{J}$. Then there is an open subset $O \subseteq \text{Spec}(R)$ such that the dimension of $V(\bar{J}) \cap O$ is $(\dim(G) - \dim(H_{\overline{y_e}})) + \kappa$, where $\kappa = 0$ if there is an $\mathfrak{h} \in \mathfrak{h}$ such that $\mathfrak{h}\overline{y_e} = \alpha\overline{y_e}$ with $\alpha \neq 0$, and $\kappa = 1$ otherwise.

Thus, the variety of $\bar{J}$ extends $O(z)$ in the direction y_e . We call $\bar{J} = \bar{J}_{z,y_e}$ as the **tangent ideal** for the data (z, y_e) .

This work contributes to the overall understanding of stabilizers in the vicinity of general points $z \in V$. Let us define the *null cone* of z as $N_G(z) = \{y \in V \mid z \in \overline{O(y)}\}$. The classical null cone $N(V)$ is of course, $N_G(0)$. When G is reductive, the Hilbert-Mumford-Kempf theory shows that for all points $y \in N_G(0)$, there is an “optimal” 1-PS $\lambda(t)$ such that $y \xrightarrow{\lambda} 0$. The stabilizer $K = G_y$ is contained within the parabolic subgroup $P(\lambda)$, whose Lie algebra is precisely the elements of $\mathfrak{g}$ of non-negative order (see Remark 3.3). The leading terms y_e of such optimal limits (i.e., the tangents of approach) have interesting properties, see [6]. They are eigenvectors with non-zero eigenvalues of semisimple elements in G and are also shown to be $L(\lambda)$ -semistable (where $L(\lambda)$ is a “twisted” version of the reductive subgroup of all elements in $P(\lambda)$ which commute with λ), see [15]. Luna [10] works with stable points $z \in V$ whose orbit $O(z)$ is closed. Such points must have reductive stabilizers $H = G_z$. Using an H -complement $N \subseteq V$ to T_z , he shows that the stabilizer K of $y \in N_G(z)$ is then a subgroup (up to conjugation) of H . $N_G(z)$ is then intimately connected to $N_H(0)$, the null cone for the H -action on N . The tangent again is an eigenvector of a semisimple element in H .

Our results extend some of this to a general z and $y \in N(z)$. We show that the Lie algebra stabilizer $\mathfrak{k} = \mathfrak{g}_y$ passes to a Lie subalgebra of the stabilizer $\mathfrak{h} = \mathfrak{g}_z$

in the limit. However, tangents of approach may well be eigenvectors of nilpotent elements in $\mathfrak{h}$. The H -module $\overline{N} = V/T_z$ plays an analogous role in the analysis of $N(z)$.

The rest of the paper is organized as follows. Theorem 2.3, on stabilizer limits and Proposition 2.5 are proved in Section 3. Examples of such limits are provided in Section 4. In Section 5 we construct the ideal $\overline{J}$, compute its dimension and prove Theorem 2.10. Examples of the construction of $\overline{J}$ are provided in Section 6. In Section 7, we outline the determinant vs. permanent problem, a central question in computational complexity theory, which provided the main motivation for this research. We also outline the connection with the Bruhat-Tits theory on bounded subgroups of Lie groups.

3. Stabilizer Limits

We recall some well known facts on Lie group and Lie algebra actions. The reader is referred to Humphreys [7] for details. Let G act on a vector space V via $\rho : G \rightarrow GL(V)$. Let $\mathfrak{g} = Lie(G)$ be the Lie algebra of G . Note that $\mathfrak{g}$ is a subalgebra of $gl(X) = gl_n(\mathbb{C})$, the Lie algebra of $n \times n$ matrices over $\mathbb{C}$. Then $\mathfrak{g}$ acts on V via the differential of ρ . G acts on $\mathfrak{g}$ by the adjoint action $ad : G \rightarrow GL(\mathfrak{g})$, $ad(g) \cdot \mathfrak{g} = gg g^{-1}$. Given ρ , the induced Lie algebra action $\rho_1 : \mathfrak{g} \rightarrow End(V)$ is compatible with the adjoint action: $\rho_1(ggg^{-1}) = \rho(g)\rho_1(\mathfrak{g})\rho(g^{-1})$.

For any $v \in V$ and $\mathfrak{g} \in \mathfrak{g}$ (resp. $g \in G$), gv (resp. gv) will denote the action of $\mathfrak{g}$ (resp. g) on the element v via ρ_1 (resp. ρ). For any $v \in V$, the *stabilizer* of v will refer to either $G_v := \{g \in G | gv = v\}$, the subgroup of G fixing v , or to $\mathfrak{g}_v := \{\mathfrak{g} \in \mathfrak{g} | gv = 0\}$, the Lie subalgebra of $\mathfrak{g}$ sending v to zero. In fact, $\mathfrak{g}_v$ is the Lie algebra of the group G_v . Of course, $\mathfrak{g}_v$ is determined by the identity connected component of G_v .

Note that via ρ_1 , V' is also a $\mathfrak{g}'$ module. Thus, if $v' = \sum_{i \geq a} t^i v_i$ and $\mathfrak{g}' = \sum_{j \geq b} t^j \mathfrak{g}_j$ then $\mathfrak{g}'v' = \sum_{i \geq a, j \geq b} t^{i+j} \mathfrak{g}_j v_i$. Moreover, for $\mathfrak{g}', \mathfrak{h}' \in \mathfrak{g}'$ with $\mathfrak{g}' = \sum_{i \geq a} t^i \mathfrak{g}_i$ and $\mathfrak{h}' = \sum_{j \geq b} t^j \mathfrak{h}_j$, then $[\mathfrak{g}', \mathfrak{h}'] = \sum_{i \geq a, j \geq b} t^{i+j} [\mathfrak{g}_i, \mathfrak{h}_j]$. Thus, we have:

Lemma 3.1. (A) For $\mathfrak{g}' \in \mathfrak{g}'$ and $v' \in V'$ we have either (i) $ord(\mathfrak{g}'v') = ord(\mathfrak{g}') + ord(v')$ and $\widehat{\mathfrak{g}'v'} = \widehat{\mathfrak{g}'} \cdot \widehat{v'}$, or (ii) $ord(\mathfrak{g}'v') > ord(\mathfrak{g}') + ord(v')$ and $\widehat{\mathfrak{g}'v'} = 0$. (B) If $\mathfrak{g}', \mathfrak{h}' \in \mathfrak{g}'$, then either (i) $\widehat{[\mathfrak{g}', \mathfrak{h}']} = [\widehat{\mathfrak{g}'}, \widehat{\mathfrak{h}'}]$ and $ord([\mathfrak{g}', \mathfrak{h}']) = ord(\mathfrak{g}') + ord(\mathfrak{h}')$ or (ii) $\widehat{[\mathfrak{g}', \mathfrak{h}']} = 0$ and $ord([\mathfrak{g}', \mathfrak{h}']) > ord(\mathfrak{g}') + ord(\mathfrak{h}')$.

We fix y, z, γ such that $y \xrightarrow{\gamma} z$. We also fix $H = G_z, K = G_y$ and their Lie algebras $\mathfrak{h} = Lie(H), \mathfrak{k} = Lie(K)$.

We then have the basic dichotomy lemma which extends Lemma 3.1. Recall that for $\mathfrak{g} \in \mathfrak{g}$ or $v \in V$, we have $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}(t)}$, where $\mathfrak{g}(t) = \gamma(t) \cdot \mathfrak{g}$ and $\widehat{v} = \widehat{v(t)}$, where $v(t) = \gamma(t) \cdot v$.

Lemma 3.2. If $\mathfrak{g} \in \mathfrak{g}$ and $v \in V$ are arbitrary elements then either (i) $\widehat{\mathfrak{g}v} = \widehat{\mathfrak{g}}\widehat{v}$ and $ord_\gamma(\mathfrak{g}v) = ord_\gamma(\mathfrak{g}) + ord_\gamma(v)$ or (ii) $\widehat{\mathfrak{g}v} = 0$ and $ord_\gamma(\mathfrak{g}v) > ord_\gamma(\mathfrak{g}) + ord_\gamma(v)$. For $\mathfrak{g}, \mathfrak{h} \in \mathfrak{g}$, either (i) $ord_\gamma([\mathfrak{g}, \mathfrak{h}]) = ord_\gamma(\mathfrak{g}) + ord_\gamma(\mathfrak{h})$ and $\widehat{[\mathfrak{g}, \mathfrak{h}]} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}}]$, or (ii)

$\text{ord}_\gamma([\mathbf{g}, \mathbf{h}]) > \text{ord}_\gamma(\mathbf{g}) + \text{ord}_\gamma(\mathbf{h})$ and $[\widehat{\mathbf{g}}, \widehat{\mathbf{h}}] = 0$.

Proof. For the first assertion, note that if $w = \mathbf{g} \cdot v$, then:

$$w(t) = \gamma(t) \cdot w = \gamma(t)\mathbf{g}\gamma(t)^{-1}\gamma(t)v = \mathbf{g}(t)v(t)$$

The assertion then follows from Lemma 3.1. For the second assertion, we note that:

$$[\mathbf{g}, \mathbf{h}](t) = \gamma(t)([\mathbf{g}, \mathbf{h}])\gamma(t)^{-1} = [\gamma(t)\mathbf{g}\gamma(t)^{-1}, \gamma(t)\mathbf{h}\gamma(t)^{-1}] = [\mathbf{g}(t), \mathbf{h}(t)]$$

The second assertion also follows from Lemma 3.1. ■

Remark 3.3. Let $\mathfrak{g}_i^+ = \{\mathbf{g} \in \mathfrak{g} \mid \text{ord}_\gamma(\mathbf{g}) \geq i\}$. Then (i) $(\mathfrak{g}_i^+)$ is a nested sequence of subspaces of $\mathfrak{g}$ with $[\mathfrak{g}_i^+, \mathfrak{g}_j^+] \subseteq \mathfrak{g}_{i+j}^+$, and (ii) $\mathfrak{g}_0^+ \subseteq \mathfrak{g}$ is a subalgebra and $\mathfrak{g}_1^+ \subseteq \mathfrak{g}_0^+$, a nilpotent ideal.

Proposition 3.4. Let M be a finite-dimensional G -module over $\mathbb{C}$, and let $N \subseteq M$ be a subspace. Let $M' = M \otimes k$ and $N(t) \subseteq M'$ be the k -vector space generated by the elements $\{\gamma(t) \cdot n \mid n \in N\}$. Let $\widehat{N(t)} = \{\widehat{n'} \mid n' \in N(t)\}$. Then $\widehat{N(t)} \subseteq M$ is a vector space over $\mathbb{C}$ of dimension $\dim(N)$. Next, let $\widehat{N}$ be the vector space generated by elements $\{\widehat{n(t)} \mid n \in N\}$. Then $\widehat{N} \subseteq \widehat{N(t)}$.

Proof. Firstly, observe that $\gamma(t)$ is an invertible k -linear transformation on $M' = M \otimes k$. Thus, for a basis $n_1, \dots, n_r$ of N , these elements continue to be independent in M' and so do the elements $\gamma(t) \cdot n_1, \dots, \gamma(t) \cdot n_r$. Thus $\dim_k(N(t)) = \dim_{\mathbb{C}}(N)$.

Next, let $N[[t]] = N(t) \cap (M \otimes_{\mathbb{C}} \mathcal{O})$, which is an $\mathcal{O}$ -module. Let $N_i[[t]] = \{n(t) \in N[[t]] \mid \text{ord}(n(t)) \geq i\}$, then $N_i[[t]]$ is also an $\mathcal{O}$ -module. Since, for any $n(t) \in N(t)$, there is an a such that $t^a n(t) \in N_0[[t]]$, we have $\widehat{N_0[[t]]} = \widehat{N[[t]]} = \widehat{N(t)}$.

Let $C = \{c_1(t), \dots, c_p(t)\} \subset N_0[[t]]$ be such that $\widehat{C}$ is a basis for $\widehat{N_0[[t]]}$. Without loss of generality, we assume that $\text{ord}(c_i(t)) = 0$ for all $1 \leq i \leq p$. We claim that C is an $\mathcal{O}$ -basis for $N_0[[t]]$ and therefore for $N[[t]]$.

For any $m \in \widehat{N[[t]]}$, if $m = \sum_{i=1}^p \alpha_i \widehat{c_i(t)}$, with $\alpha_i \in \mathbb{C}$, then let $c(m) = \sum_i \alpha_i c_i(t) \in N_0[[t]]$. Note that $\widehat{c(m)} = m$. Let $\bar{\alpha}(m) = (\alpha_1, \dots, \alpha_p)$ as above.

Now let $n(t) = \sum_{i \geq 0} t^i m_i \in N_0[[t]]$, where $m_i \in M$. Let us define the following recursive sequence: $n'_0(t) = n(t)$, $m'_0 = m_0$, the coefficient of t^0 in $n'_0(t)$. Next, $n'_{i+1}(t) = n'_i(t) - t^i(c(m'_i))$. Thus $n'_{i+1}(t)$ is obtained by using C to cancel the leading term. We may inductively show that (i) $n'_i(t) \in N[[t]]$ and is of order at least i , (ii) m'_i is the coefficient of t^i in $n'_i(t)$, and (iii) $n'_{i+1}(t) = n(t) - \sum_{j=0}^i t^j c(m'_j)$. We also obtain the vectors $\bar{\alpha}(m'_j)$.

For $s = 1, \dots, p$, let us consider the element $\alpha_s(t) = \sum_{j=0}^{\infty} t^j \bar{\alpha}(m'_j)_s \in k$ and $n'(t) = \sum_{i=0}^{\infty} t^i c(m'_i) = \sum_{s=1}^p \alpha_s(t) c_s(t)$. It is easy to see that $n'(t)$ matches $n(t)$ in every (finite) coordinate, and thus $n'(t) = n(t)$, and is expressible as an $\mathcal{O}$ -linear combination of C . This shows that C spans $N[[t]]$.

Next, we claim that C is linearly independent over k . Indeed, if there were such a linear dependence, there would be one, say $\sum_{i=1}^p \beta_i(t) c_i(t)$, over $\mathcal{O}$ with some of the $\beta_i(0) \neq 0$. That would imply a linear dependence over $\mathbb{C}$ on the elements $\widehat{C}$, a contradiction to the choice of C .

Thus C is also a k -basis of $N(t)$ and thus $p = \dim_k(N(t)) = \dim_{\mathbb{C}}(N)$. Finally, we have $\dim_{\mathbb{C}}(\widehat{N(t)}) = |\widehat{C}| = p$. This proves the first part.

The second part follows from the fact that $n(t) \in N(t)$ for all $n \in N$. $\blacksquare$

Remark 3.5. It may well happen that $\widehat{N} \subsetneq \widehat{N(t)}$, see Example 4.4.

We record a corollary of the proof above:

Corollary 3.6. *If $m(t) \in M' = M \otimes_{\mathbb{C}} k$ is such that $m(t) \notin N(t)$, then there is an integer a and $n'(t) \in N[[t]]$ such that for $m'(t) = m(t) - t^a n'(t)$, we have $\widehat{m'(t)} \notin \widehat{N(t)}$.*

Proof. If $\widehat{m(t)} \notin \widehat{N(t)}$, then $n'(t) = 0$ and we are done.

If not, let C be the basis of $N_0[[t]]$ as above. Since $m(t) \notin N(t)$, there is a $d \geq 0$ such that $p(t) = t^d m(t) \in M \otimes \mathcal{O}$ but $p(t) \notin N[[t]]$. Starting with $i = 0$ and $p'_0(t) = p(t)$, for every i with the coefficient of t^i of $p'_i(t)$ in $\widehat{N(t)}$, let us construct the sequence $p'_{i+1}(t)$ as above. Since $p(t) \notin N[[t]]$, there is an r such that the coefficient of t^r in $p'_r(t)$ is non-zero and not an element of $\widehat{N(t)}$. Moreover, as in the proof above, $p'_r(t) = p(t) - n(t)$, for some $n(t) \in N[[t]]$. Thus, we have $(t^d m(t) - n(t)) \notin \widehat{N(t)}$. Thus, $n'(t) = t^{-d} n(t)$ is the required element and $(m(t) - n'(t)) \notin \widehat{N(t)}$ as claimed. $\blacksquare$

Proposition 3.7. *Let $\mathfrak{L}$ be a Lie subalgebra of $\mathfrak{g}$ and $N \subseteq V$ an $\mathfrak{L}$ -module. Let $\mathfrak{L}(t)$ be the k -vector space generated by the elements $\{\gamma(t) \cdot \mathbf{h} \mid \mathbf{h} \in \mathfrak{L}\}$. Similarly, let $N(t)$ be the k -vector space generated by the elements $\{\gamma(t) \cdot n \mid n \in N\}$. Then*

1. $\mathfrak{L}(t) \subseteq \mathfrak{g}'$ is a Lie subalgebra and $N(t) \subseteq V'$ an $\mathfrak{L}(t)$ -module.
2. (i) $\widehat{\mathfrak{L}(t)}$ is a Lie subalgebra of $\mathfrak{g}$, and $\dim(\widehat{\mathfrak{L}(t)}) = \dim(\mathfrak{L})$, (ii) $\widehat{N(t)} \subseteq V$ is a $\widehat{\mathfrak{L}(t)}$ -module with $\dim(\widehat{N(t)}) = \dim(N)$.
3. (i) $\widehat{\mathfrak{L}} \subseteq \widehat{\mathfrak{L}(t)}$ is a Lie subalgebra of $\mathfrak{g}$, and (ii) $\widehat{N} \subseteq \widehat{N(t)} \subseteq V$ is a $\widehat{\mathfrak{L}}$ -module.

Proof. For (1), note that typical elements $\mathbf{h}'$ and $\mathbf{g}'$ of $\mathfrak{L}(t)$ are given by $\mathbf{h}' = \sum_i \alpha_i(t) \mathbf{h}_i(t)$ and $\mathbf{g}' = \sum_j \eta_j(t) \mathbf{g}_j(t)$ where $\alpha_i(t), \eta_j(t) \in k$ and $\mathbf{h}_i, \mathbf{g}_j \in \mathfrak{L}$. Similarly, a typical $n' \in N(t)$ is given by $\sum_j \beta_j(t) n_j(t)$, with $\beta_j(t) \in k$ and $n_j \in N$. Whence $[\mathbf{h}', \mathbf{g}'] = \sum_{i,j} \alpha_i(t) \eta_j(t) [\mathbf{h}_i, \mathbf{g}_j](t)$. Since $[\mathbf{h}_i, \mathbf{g}_j] \in \mathfrak{L}$, we have $[\mathbf{h}', \mathbf{g}'] \in \mathfrak{L}(t)$. The same argument shows that $\mathbf{g}' \cdot n' \in N(t)$. This proves (1).

For (2), if $\mathbf{h}', \mathbf{g}' \in \mathfrak{L}(t)$, then by Lemma 3.1, we see that either $[\widehat{\mathbf{h}'}, \widehat{\mathbf{g}'}] = 0$ or it equals $[\widehat{\mathbf{h}'}, \widehat{\mathbf{g}'}]$. Since, $[\mathbf{h}', \mathbf{g}'] \in \mathfrak{L}(t)$, in both cases, the element $[\widehat{\mathbf{h}'}, \widehat{\mathbf{g}'}]$ is in $\widehat{\mathfrak{L}(t)}$. The same argument shows that $\widehat{N(t)}$ is a $\widehat{\mathfrak{L}(t)}$ -module. The statement about the dimensions follows from Proposition 3.4. This proves (2). Part (3) is straightforward. $\blacksquare$

Lemma 3.8. *Let v be an arbitrary element of V and $\mathfrak{g} \in \mathfrak{g}_v$. Then $\mathfrak{g}(t)v(t) = 0$ and $\hat{\mathfrak{g}} \cdot \hat{v} = 0$. Thus $\widehat{(\mathfrak{g}_v)} \subseteq \mathfrak{g}_{\hat{v}}$. Moreover, for any $\mathfrak{k}' \in \mathfrak{g}_v(t)$, we have $\mathfrak{k}'v(t) = 0$.*

Proof. If $gv = 0$, then so is $\mathfrak{g}(t)v(t)$, and consequently, its term of order $\text{ord}(\mathfrak{g}) + \text{ord}(v)$, which is $\hat{\mathfrak{g}}\hat{v}$. Coming to the last assertion, we have $\mathfrak{k}' = \sum_i \alpha_i(t)\mathfrak{k}_i(t)$, for some $\alpha_i \in k$ and $\mathfrak{k}_i \in \mathfrak{g}_v$. Whence $\mathfrak{k}'v(t) = \sum_i \alpha_i(t)\mathfrak{k}_i(t)v(t) = 0$. ■

We now come to the proof of Theorem 2.3.

Theorem 3.9. *Let y, z and γ be as in equation 1 (with $d = 0$). Then $\widehat{\mathfrak{k}(t)}$ is a Lie subalgebra of $\mathfrak{h}$ and $\dim(\widehat{\mathfrak{k}(t)}) = \dim(\mathfrak{k})$. Moreover, $\widehat{\mathfrak{k}(t)} \subseteq \mathfrak{h}_{\overline{y_e}}$, the Lie algebra stabilizer within $\mathfrak{h}$ of $\overline{y_e} \in \overline{N}$, where y_e is the tangent of approach.*

Proof. Proposition 3.7 applied to $\mathfrak{k}$ gives us that $\mathfrak{k}(t)$ is a Lie algebra. By Lemma 3.8, we have $\mathfrak{k}'y(t) = 0$ for any $\mathfrak{k}' \in \mathfrak{k}(t)$. Since $\widehat{y(t)} = z$, we have $\widehat{\mathfrak{k}'}z = 0$ and thus $\widehat{\mathfrak{k}'} \in \mathfrak{h}$. Since $\mathfrak{k}'$ was an arbitrary element of $\mathfrak{k}(t)$, we have $\widehat{\mathfrak{k}(t)} \subseteq \mathfrak{h}$. The assertion that $\dim(\widehat{\mathfrak{k}(t)}) = \dim(\mathfrak{k})$ is shown in Proposition 3.7. This proves the first assertion.

Coming to the second assertion, let $\mathfrak{k}' \in \mathfrak{k}(t)$ be arbitrary. We already have $\mathfrak{k}'y(t) = 0$. If we write $\mathfrak{k}' = \mathfrak{k}_at^a + \mathfrak{k}_{a+1} + \dots$ and $y(t) = y_0t^0 + y_et^e + \dots$, then we have:

$$(\mathfrak{k}_at^a + \mathfrak{k}_{a+1}t^{a+1} + \dots)(y_0t^0 + y_et^e + \dots) = 0$$

with $y_0 = z$. Examining the terms of order $a, a+1, \dots, a+e-1, a+e$ in this product, we have:

$$\begin{aligned} \mathfrak{k}_{a+i}y_0 &= 0 \quad \text{for } i = 0, \dots, e-1 \\ \mathfrak{k}_ay_e + \mathfrak{k}_{a+e}y_0 &= 0 \end{aligned}$$

This tells us that $\mathfrak{k}_{a+i} \in \mathfrak{h}$ for $i = 0, \dots, e-1$. Coming to order $a+e$, since $\mathfrak{k}_a \in \mathfrak{h}$ and $\mathfrak{k}_ay_e = -\mathfrak{k}_{a+e}y_0 \in \mathfrak{g}z = T_zO(z)$, we have $\mathfrak{k}_a \cdot \overline{y_e} = \overline{0} \in \overline{N}$ and thus $\mathfrak{k}_a = \mathfrak{k}' \in \mathfrak{h}_{\overline{y_e}}$. This proves the theorem. ■

We now come to the proof of Proposition 2.5. This is done in several steps from Proposition 3.10 to Corollary 3.13.

Proposition 3.10. *For $\gamma(t) \in G(k)$ as above, let $\mathfrak{v}(t) = \dot{\gamma}(t)\gamma^{-1}(t)$, which is a matrix with entries in k . Then $\mathfrak{v}(t) \in \mathfrak{g}'$ and satisfies the path condition.*

Proof. Since $\gamma(t)$ is a k -point of G , $\gamma(t + \epsilon)$ is a $k[\epsilon]/(\epsilon^2)$ -point of G , as it satisfies the same polynomial equations over $\mathbb{C}$. Now consider

$$\gamma(t + \epsilon)\gamma^{-1}(t) = [\gamma(t) + \epsilon\dot{\gamma}(t)]\gamma^{-1}(t) = 1 + \epsilon\dot{\gamma}(t)\gamma^{-1}(t) \in GL(X)(k[\epsilon]/(\epsilon^2))$$

It follows that $\gamma(t + \epsilon)\gamma^{-1}(t)$ is also a $k[\epsilon]/(\epsilon^2)$ -point of G . Because it has the form $1 + \epsilon b(t)$, the series $b(t) = \dot{\gamma}(t)\gamma^{-1}(t)$ lies in the Lie algebra of G over k , i.e., $\mathfrak{g}'$. ■

Coming to the proof of Proposition 2.5, by Lemma 3.8, we have $\mathfrak{k}(t)y(t) = 0$. Next, for any $\mathfrak{v}(t)$ satisfying the path condition, since $\mathfrak{v}(t)y(t) = \dot{y}(t) \neq 0$, we have $\mathfrak{v}(t) \notin \mathfrak{k}(t)$. In fact, for any $\mathfrak{k}' \in \mathfrak{k}(t)$, $\mathfrak{v}'(t) = \mathfrak{v}(t) + \mathfrak{k}'$ also satisfies the path condition, viz. $\mathfrak{v}'(t)y(t) = \dot{y}(t)$. We now show that there is a $\mathfrak{k}' \in \mathfrak{k}(t)$ such that the leading term of $\mathfrak{v}(t) + \mathfrak{k}'$ does not lie in $\widehat{\mathfrak{k}(t)}$.

Proposition 3.11. *Let $\mathfrak{v}(t) \in \mathfrak{g}'$ satisfy the path condition. Then there is a $\mathfrak{g}' \in \mathfrak{k}(t)$ such that if $\mathfrak{v}'(t) = \mathfrak{v}(t) + \mathfrak{g}'$, then (i) $\mathfrak{v}'(t)$ satisfies the path condition and (ii) its leading term $\widehat{\mathfrak{v}'} \notin \widehat{\mathfrak{k}(t)}$.*

Proof. Let us apply Corollary 3.6 with $M = \mathfrak{g}$ as the G -module and $N = \mathfrak{k} \subseteq \mathfrak{g}$. Since $\mathfrak{v}(t) \notin \mathfrak{k}(t)$, we are then assured of a $\mathfrak{k}(t) \in \mathfrak{k}(t)$ and an a such that if $\mathfrak{v}'(t) = \mathfrak{v}(t) - t^a \mathfrak{k}(t)$, then $\widehat{\mathfrak{v}'(t)} \notin \widehat{\mathfrak{k}(t)}$. Then $-t^a \mathfrak{k}(t) = \mathfrak{g}' \in \mathfrak{k}(t)$ is the required element. ■

Proposition 3.12. *For general G and $\mathfrak{g} = \text{Lie}(G)$, let $\mathfrak{v}(t) \in \mathfrak{g}'$ satisfy the path condition and let $\ell = \widehat{\mathfrak{v}(t)}$ be its leading term. Suppose that $y_e \notin T_z$ - the tangent space of the G -orbit of z . Then $\ell \in \mathfrak{h} = \mathfrak{g}_z$ and $\overline{y_e}$ is an eigenvector of ℓ . Thus $[\ell, \mathfrak{h}_{\overline{y_e}}] \subseteq \mathfrak{h}_{\overline{y_e}}$.*

Proof. Let $\mathfrak{v}(t) = \sum_{i \geq a} t^i M_i$, with $M_i \in \mathfrak{g}$ and $M_a \neq 0$ satisfy $\mathfrak{v}(t)y(t) = \dot{y}(t)$. We may then write the path condition as:

$$(M_a t^a + M_{a+1} t^{a+1} + \dots)(z + y_e t^e + \dots) = e t^{e-1} y_e + \dots$$

We may rewrite the first few terms of the LHS as follows:

$$\begin{aligned} t^a M_a z + \dots + t^{a+e-1} M_{a+e-1} z &+ t^{a+e} M_{a+e} z \\ &+ t^{a+e} M_a y_e + t^{a+e+1} M_{a+1} y_e + \dots \end{aligned}$$

Since $e y_e \neq 0$, we have $a \leq e - 1$. Consider the case $0 \leq a \leq e - 1$. In this case, we have $a \leq e - 1 \leq a + e - 1$, and we have $M_{e-1} z = e y_e$ forcing $y_e \in T_z O(z)$. This contradicts the hypothesis. Thus, we have $a \leq -1$. This gives us that $M_a z = 0$ and $\ell = M_a \in \mathfrak{h}$. If $a = -1$, we have:

$$M_{e-1} z + M_a y_e = e y_e$$

Since $M_{e-1} z \in T_z O(z)$, we have $M_a \overline{y_e} = e \overline{y_e}$, and thus $\overline{y_e} \in \overline{N}$ is an eigenvector of M_a . If $a < -1$ we have:

$$M_{a+e} z + M_a y_e = 0$$

This implies $M_a \overline{y_e} = 0$ and yet again $\overline{y_e}$ is an eigenvector. Thus, in all cases, $\ell = M_a = \widehat{\mathfrak{v}(t)}$ normalizes $\mathfrak{h}_{\overline{y_e}}$ completing the proof of the proposition. ■

We have a simple corollary of the proof:

Corollary 3.13. *When $y_e \notin T_z$, as in the hypothesis, we have $a \leq -1$. When $a = -1$, we have $\ell = \widehat{\mathfrak{v}(t)} \in \mathfrak{h}$ and $\ell \overline{y_e} = e \overline{y_e}$. Since $e \neq 0$, the pair $\mathfrak{h}, \overline{y_e}$ satisfy the eigenvector condition, see Definition 5.11. When $y_e \in T_z$ and $a < e - 1$, we still have $\ell \in \mathfrak{h}$.*

We complete the proof of Proposition 2.5.

Proof. Suppose that $\mathfrak{v}(t) \in \mathfrak{g}'$ satisfies the path condition. By Proposition 3.11, we may arrive at an $\mathfrak{v}(t)$ such that $\ell = \widehat{\mathfrak{v}(t)} \notin \widehat{\mathfrak{k}(t)}$. By Proposition 3.12, we have that $\ell \in \mathfrak{h}$ and normalizes $\mathfrak{h}_{\overline{y_e}}$. This completes the proof. ■

We now consider the case when $y_e \in T_z$. This will address Proposition 2.6 and Corollary 2.7 from the introduction. We first prove:

Proposition 3.14. *Let $y \xrightarrow{\gamma} z$ and let N be a complement to T_z in V . Then there is a $\gamma'(t) \in G(k)$ such that $y'(t) = \gamma'(t) \cdot y \subseteq N \otimes \mathcal{O}$ and $y \xrightarrow{\gamma'} z$.*

We begin with a lemma:

Lemma 3.15. *Let N be a complement to T_z in V . Let $\mathfrak{s}$ be a complement to $\mathfrak{h} \subseteq \mathfrak{g}$. Let $M \subseteq G$ be a variety of dimension $\dim(\mathfrak{s})$ such that $e_G \in M$ is a non-singular point with $T_{e_G}M = \mathfrak{s}$. Then there is an ϵ -neighborhood $B_\epsilon(z) \subseteq V$ of z such that*

1. *For all $v \in B_\epsilon(z)$, the set $Mv \cap B_\epsilon(z)$ is smooth.*
2. *Moreover, there are polynomial functions $f_1, \dots, f_r$ on $V \times M \times V$, where $r = \dim(V)$, such that for $v \in B_\epsilon(z)$, the equations $f_1(v, \cdot, \cdot), \dots, f_r(v, \cdot, \cdot)$ define $\{(g, gv) \in M \times V \mid g \in M\}$ in a neighborhood of (e_G, z) .*
3. *For all $v \in B_\epsilon(z)$, there is an $n_v \in N$ such that $Mv \cap (z + N)$ is a singleton $z + n_v$ with $n_z = 0$. Moreover, the intersection at all these points is transversal.*

Proof. The set Mz is an algebraic subset of the orbit $O(z)$ and of the same dimension. Since $O(z)$ is smooth and M is smooth at e_G , so is Mz in a neighborhood of z . Since the multiplication map is smooth and algebraic, Mv is smooth in a neighborhood of z . This proves the first assertion.

For the second assertion, note that the set (v, g, gv) is merely the graph of the group action and hence an algebraic set of dimension $\dim(V) + \dim(\mathfrak{s})$. Thus, locally, at the point (z, e_G, z) , there are $r = \dim(V)$ equations $f_1, \dots, f_r$ which define this set. By specializing for $v \in B_\epsilon(z)$, we have the second assertion.

Coming to the third assertion, note that $Mz \cap (z + N) = z$ and $n_z = 0$, as asserted. Moreover, since $z + N$ is an affine space of dimension $\dim(V) - \dim(\mathfrak{s})$, there are $g_1, \dots, g_s$ affine forms on V which define it (where $s = \dim(\mathfrak{s})$). The solution $z + 0_N$ is precisely the solution to the $r + s$ equations $f_i(z, \cdot, \cdot)$ and g_j . Since $O(z)$ meets N transversally and $Mz = O(z)$ locally, we have that the $r + s$ -dimensional Jacobian $J((f_i(z, \cdot, \cdot)), (g_j))$ is non-singular at (e_G, z) . The same must hold for $J((f_i(v, \cdot, \cdot)), (g_j))$, in a small neighborhood of z . This proves that there is an isolated zero to the equations $(f_i(v, \cdot, \cdot), (g_j))$ and Mv meets N transversally. This proves the third assertion. ■

Lemma 3.16. *There is a series $m(t) \in M(\mathcal{O})$, (i.e., elements of $G(\mathcal{O})$ satisfying the defining equations of M) such that (i) $f_i(y(t), m(t), m(t)y(t)) = 0$ and (ii) $m(t)y(t) \in z + N$, and (iii) $m(0) = e_G$.*

Proof. Consider the $r + s$ polynomial equations $((f_i(y(t), \cdot, \cdot)), (g_j))$ over $M \times V$. Thus, there are $r + s$ equations in $r + s + 1$ variables, the extra parameter being t , over a locally smooth space, smooth at the point (e_G, z) . Moreover, at $t = 0$, the solution is, of course, $m(0) = e_G$ and $v(0) = z$, and the Jacobian there is non-singular. Note that $y(t) \in V \otimes \mathcal{O}$ and Newton's method applies, see [11], for example. Thus, there is a family $m(t) \in M(\mathcal{O})$ and $v(t) \in V \otimes \mathcal{O}$ which satisfies the $r + s$ algebraic equations. The governing equations imply that $v(t) = m(t)y(t)$ and $v(t) \in z + N$. This proves the lemma. ■

We are now in a position to complete the proof of Proposition 3.14.

Proof. The existence of a suitable M as in the hypothesis of Lemma 3.15 is clear. Therefore, we may construct a suitable $m(t)$ as in Lemma 3.16. Consider $\gamma'(t) = m(t)\gamma(t)$. Then we see that $\gamma'(t)y = m(t)y(t) \in z + N$. Thus $y'(t) \subseteq N$. Finally, since $m(t) \in M(\mathcal{O})$ and $m(0) = e_G$, $y'(0) = m(0)y(0) = e_G z = z$. This proves both assertions and the proposition. ■

While Proposition 3.14 constructs a 1-PF $\gamma'(t)$ with a fresh tangent of approach which is not in T_z , the algebras $\mathfrak{k}(t)$ and $\mathfrak{k}'(t)$ may be different, and hence so may $\widehat{\mathfrak{k}(t)}^\gamma$ and $\widehat{\mathfrak{k}(t)}^{\gamma'}$. However, we prove the following important corollary.

Corollary 3.17. *Let $z \in \overline{O(y)}$ be such that $\dim(O(z)) = \dim(O(y)) - 1$. Then there is a γ with a tangent of approach $y_e \notin T_z$, such that $y \xrightarrow{\gamma} z$ and $\mathfrak{h} = \widehat{\mathfrak{k}(t)} \oplus \mathbb{C}\ell$, where ℓ is obtained as $\widehat{\mathfrak{v}(t)} \in \mathfrak{g}'$ satisfying the path condition. Moreover, either (i) $\widehat{\mathfrak{k}(t)} = \mathfrak{h}_{\overline{y_e}}$ and is normalized by ℓ , or (ii) $\mathfrak{h}_{\overline{y_e}} = \mathfrak{h}$.*

Proof. That $z \in \overline{O(y)}$ assures us of a γ such that $y \xrightarrow{\gamma} z$. By Proposition 3.14, we may assume that $y_e \notin T_z$. Then, by Proposition 2.5, we have $\widehat{\mathfrak{k}(t)} \oplus \mathbb{C}\ell \subseteq \mathfrak{h}$. Since $\dim(\mathfrak{h}) = \dim(\mathfrak{k}) + 1$, we have $\widehat{\mathfrak{k}(t)} \oplus \mathbb{C}\ell = \mathfrak{h}$. This proves the first assertion. Since $\mathfrak{h}_{\overline{y_e}} \supseteq \widehat{\mathfrak{k}(t)}$, we must have either (i) $\mathfrak{h}_{\overline{y_e}} = \widehat{\mathfrak{k}(t)}$, or (ii) $\mathfrak{h}_{\overline{y_e}} = \mathfrak{h}$. In case (i), we already know that ℓ normalizes $\mathfrak{h}_{\overline{y_e}}$ and therefore, by the above equality, it normalizes $\widehat{\mathfrak{k}(t)}$. ■

Remark 3.18. An important special case is when $\gamma(t)$ is a 1-parameter subgroup (1-PS) of G . In this case, we have the weight space decompositions $V = \oplus_i V_i$ and $\mathfrak{g} = \oplus_j \mathfrak{g}_j$. The Lie algebra $\mathfrak{g}_0^+$ of Remark 3.3 is the parabolic lie algebra $\mathfrak{p}(\lambda)$ associated with λ . All proofs become simpler and $\widehat{\mathfrak{k}} = \widehat{\mathfrak{k}(t)}$. Moreover, $\widehat{\mathfrak{k}} = \sum_i \widehat{\mathfrak{k}_i}$ is a graded Lie algebra.

If $\lambda(t) = \text{diag}(t^{d_1}, \dots, t^{d_N}) \subseteq T$, a maximal torus of G , then $\mathfrak{v}(t) = t^{-1}\ell$, where $\ell = \text{diag}(d_1, \dots, d_N)$, and thus $a = -1$ (cf. Corollary 3.13). The element ℓ normalizes $\widehat{\mathfrak{k}}$ but $\ell \notin \mathfrak{h}_{\overline{y_e}} \supseteq \widehat{\mathfrak{k}}$. Moreover, if $y_e \in T_z$, but $z \notin O(y)$, there is a

$u \in U(\lambda)$ such that $uy \xrightarrow{\lambda} z$ but with the new tangent $y'_e \notin T_z$. These results appear in [1, Section 3]

Remark 3.19. In the case when G is reductive, by [8], the condition $z \in \overline{O(y)}$ assures us of a 1-PF $\gamma(t)$ of the form $\gamma(t) = h_1(t)\lambda(t)h_2(t)^{-1}$ where $h_1, h_2 \in G(\mathcal{O})$ and $\lambda(t)$ is a 1-PS of a maximal torus of G . An extension of the analysis in [2] may help define ℓ in this case, and connect it with $\lambda(t)$.

Next, we illustrate the results of this section with some examples.

4. Examples

Example 4.1. Let us consider $G = GL_4(\mathbb{C})$ acting on $V = \mathbb{C}^4$ as column vectors. Consider the 1-parameter subgroup $\gamma(t)$ below. This allows for a weight space decomposition of V and $\mathfrak{g}$. The action of γ on a typical element $M \in \mathfrak{g}$ is given below, where each M_{ij} is a 2×2 -matrix. Note that the orders which occur are $-1, 0$ and 1 , thus we have the spaces $\mathfrak{g}_{-1}, \mathfrak{g}_0$ and $\mathfrak{g}_1$, of matrices with leading terms of order $-1, 0$ and 1 respectively. We consider $\mathfrak{k}$ as given below and compute $\widehat{\mathfrak{k}}$.

$$\gamma(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix} \quad \gamma(t)M\gamma(t)^{-1} = \begin{bmatrix} M_{11} & t^{-1}M_{12} \\ tM_{21} & M_{22} \end{bmatrix} \quad \mathfrak{k} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

For an element $\mathbf{k}$ with weight space decomposition $\sum_i \mathbf{k}_i$, the leading term $\hat{\mathbf{k}}$ is the non-zero term of smallest weight. Thus $\widehat{\mathfrak{k}}$ is a graded Lie algebra. To compute $\widehat{\mathfrak{k}}_i$, let $\mathfrak{k}_i = (\oplus_{j \geq i} \mathfrak{g}_j) \cap \mathfrak{k}$. The space $\mathfrak{k}_i / \mathfrak{k}_{i+1}$ gives us $\widehat{\mathfrak{k}}_i$.

We have $\dim(\mathfrak{k}_{-1}) = 4, \dim(\mathfrak{k}_0) = 4$ and $\dim(\mathfrak{k}_1) = 0$. Thus, we get a basis $B = B_0$ of $\widehat{\mathfrak{k}}$ of dimension 4. Moreover, since each element of B_0 is homogeneous, we have $\hat{B}_0 = B_0$ and $\widehat{\mathfrak{k}} = \mathfrak{k}$.

Next, let A be as shown below. We obtain $\mathfrak{k}' = A\mathfrak{k}A^{-1}$ as given below.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathfrak{k}' = \begin{bmatrix} a & b & c & d-a \\ c & d & 0 & -c \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

As a result, $\mathfrak{k}'(t)$ and $\widehat{\mathfrak{k}'}$ are as given below:

$$\mathfrak{k}'(t) = \begin{bmatrix} a & b & t^{-1}c & t^{-1}(d-a) \\ c & d & 0 & -t^{-1}c \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

Again, let $\mathfrak{k}'_i = \mathfrak{k}' \cap \oplus_{j \geq i} \mathfrak{g}_j$ and note that $\dim(\mathfrak{k}'_{-1}) = 4$. Now $\mathfrak{k}'_0$ consists of all elements $\mathbf{k} \in \mathfrak{k}'$ which have no term of order -1 . This forces $d = a$ and $c = 0$ and so $\dim(\mathfrak{k}'_0) = 2$. Finally $\dim(\mathfrak{k}'_1) = 0$. Thus, we have B_{-1} and B_0 , each of

cardinality two contributing to $B = B_{-1} \cup B_0$, such that $\hat{B}$ would be a basis of $\hat{\mathfrak{k}}'$. This results in $\hat{\mathfrak{k}}'$ as given, with $r, s, t, u \in \mathbb{C}$ as given below.

$$\hat{\mathfrak{k}}' = \begin{bmatrix} u & t & s & r \\ 0 & u & 0 & -s \\ 0 & 0 & u & t \\ 0 & 0 & 0 & u \end{bmatrix}$$

Note that while $\hat{\mathfrak{k}}$ is reductive, $\hat{\mathfrak{k}}'$ is a solvable Lie algebra.

Example 4.2. Consider a set of indeterminates $\{x_1, x_2, x_3\}$ and let $X = \mathbb{C} \cdot \{x_1, x_2, x_3\}$, be the space of linear combination of these indeterminates. Let $G = GL_3$ and $\mathfrak{g} = Lie(G)$. Let G act on $V = Sym^4(X)$, the space of homogeneous forms of degree 4 by substitution. In the resulting $\mathfrak{g}$ -action, an element of $\mathfrak{g}$ acts as a differential operator. More precisely, $\mathfrak{g} = (a_{ij})$ acts on a form $v \in V$ as: $\mathfrak{g}.v = \sum_{i,j} a_{ij}x_i \frac{\partial v}{\partial x_j}$. Consider $f = (x_1^2 + x_2^2 + x_3^2)^2 \in V$. The stabilizer algebra $\mathfrak{g}_f = \mathfrak{k}$ is 3-dimensional and is given below.

Consider next the 1-PS $\gamma(t) \subseteq GL_3$ given by $\gamma(x_1) = x_1, \gamma(x_2) = x_2$ and $\gamma(x_3) = tx_3$. We have $\hat{f} = g = (x_1^2 + x_2^2)^2$ as the leading term of $\gamma(t)f$ of order 0. The stabilizer $\mathfrak{g}_g = \mathfrak{h}$ is 4-dimensional and is shown below (with $a, b, c, d \in \mathbb{C}$). Note that, since x_3 is absent in g , the operators $\nabla = (bx_1 + cx_2 + dx_3) \frac{\partial}{\partial x_3}$ certainly stabilize g .

$$\mathfrak{g}_f = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \quad \gamma(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{bmatrix} \quad \mathfrak{g}_g = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ 0 & 0 & d \end{bmatrix}$$

The adjoint action of $\gamma(t)$ on $\mathfrak{k}$ and the resulting $\hat{\mathfrak{k}}$, the 3-dimensional Lie algebra of leading terms, are shown below.

$$\gamma(t)\mathfrak{k}\gamma(t)^{-1} = \begin{bmatrix} 0 & a & t^{-1}b \\ -a & 0 & t^{-1}c \\ -tb & -tc & 0 \end{bmatrix} \quad \hat{\mathfrak{k}} = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \subseteq \mathfrak{h}$$

Thus the “leading term” operation applied to $\mathfrak{k}$ inserts $\hat{\mathfrak{k}}$ into the stabilizer $\mathfrak{h}$ of the limit $g = \hat{f}$.

The tangent of approach is the form $h = 2(x_1^2 + x_2^2)x_3^2$ of order 2. Let us now compute $\mathfrak{h}_{\bar{h}}$. For this, we must compute $T_g O(g) = \mathfrak{g}g$. Note that since g is of order 0, forms in $T_g O(g)$ must be of order 0 or 1, whence we must compare (i) $(\mathfrak{h})_{-1}h$ with $(\mathfrak{g})_1g$ and (ii) compute $(\mathfrak{h})_0h$. For (i) it is easily checked that $(\mathfrak{h})_{-1}h = \{(\alpha x_1 + \beta x_2)(x_1^2 + x_2^2)x_3 | \alpha, \beta \in \mathbb{C}\} \subseteq T_g O(g)$. For instance, $x_3 \frac{\partial g}{\partial x_2} = 4x_2(x_1^2 + x_2^2)x_3$ is in $T_g O(g)$. On the other hand, for (ii) we see that $(\mathfrak{h})_0h = 0$. From this, it easily follows that $\hat{\mathfrak{k}} = \mathfrak{h}_{\bar{h}} \subset \mathfrak{h}$.

Example 4.3. Let $G = GL_2(\mathbb{C})$ acting on $V = \mathbb{C}^2$ (treated as column vectors). Let $\gamma(t)$ be as below and let us compute $\hat{V}$ as well as $\widehat{V \otimes k}$.

$$\gamma(t) = \begin{bmatrix} 1 & t-1 \\ 0 & t^2 \end{bmatrix} \quad \gamma(t) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha - \beta \\ 0 \end{bmatrix} + t \begin{bmatrix} \beta \\ 0 \end{bmatrix} + t^2 \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

For a general $v \in V$, there are two cases: $\alpha \neq \beta$ or $\alpha = \beta$. In either case we see that $\widehat{v} = \begin{bmatrix} * \\ 0 \end{bmatrix}$. Thus $\dim(\widehat{V}) = 1$. On the other hand, we see that

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad tv_2(t) - (1+t)v_1(t) = t^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus $\widehat{V} \neq \widehat{V(t)}$.

Example 4.4. Let $G, \gamma(t), y$ and z be as given below:

$$G = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \quad (a \neq 0) \quad \gamma(t) = \begin{bmatrix} t & 1-t \\ 0 & t \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We have

$$\mathfrak{g}_y = \bar{0} \quad \text{and} \quad \mathfrak{g}_z = \mathfrak{h} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

We have $\gamma(t), y(t)$ and $\mathfrak{v}(t) = \dot{\gamma}(t)\gamma^{-1}(t)$ are given below:

$$y(t) = \begin{bmatrix} 1 \\ t \end{bmatrix} \quad \mathfrak{v}(t) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t^{-1} & t^{-2}(t-1) \\ 0 & t^{-1} \end{bmatrix} = \begin{bmatrix} t^{-1} & -t^{-2} \\ 0 & t^{-1} \end{bmatrix}$$

We see that:

$$\mathfrak{v}(t) = -t^{-2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + t^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathfrak{g} \otimes \mathbb{C}((t))$$

We see that $y_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $a = -2$, as in the proof of Proposition 2.5. We also see that $\widehat{\mathfrak{v}(t)} \in \mathfrak{h}$ and stabilizes $\overline{y_e}$, indeed $\widehat{\mathfrak{v}(t)}y_e = z \in T_z$.

Example 4.5. This is Example 7 from [2]. Let $G = SL_2(\mathbb{C})$ and $V = \text{Sym}^3(\mathbb{C}^*)$. Let $p = x^2y$ and $q = x^3$. Note that G_p is trivial and G_q is the unipotent group of dimension 1. Let $\gamma(t)$ and $\gamma(t)^{-1}$ be as given below:

$$\gamma(t) = \begin{bmatrix} t^{-1} & 0 \\ -t^{-2} & t \end{bmatrix} \quad \gamma(t)^{-1} = \begin{bmatrix} t & 0 \\ t^{-2} & t^{-1} \end{bmatrix}$$

We have:

$$p([x, y]^T)(t) = p(\gamma(t)^{-1}[x, y]^T) = (tx)^2(t^{-2}x + t^{-1}y) = x^3 + tx^2y$$

Thus we have $p \xrightarrow{\gamma} q$ with $e = 1$ and x^2y is the tangent vector, which is in $T_q O(q)$. It is shown in [2] that while $q \in \overline{O(p)}$, it cannot be obtained as the limit of a 1-PS in G .

Analysing the above limit, we see that $G_p = K$ is trivial and $G_q = H$ is the lower unipotent group of dimension 1. The provisional construction $\mathfrak{v}(t)$ is given below:

$$\mathfrak{v}(t) = \dot{\gamma}(t)\gamma^{-1}(t) = \begin{bmatrix} -t^{-2} & 0 \\ 2t^{-3} & 1 \end{bmatrix} \begin{bmatrix} t & 0 \\ t^{-2} & t^{-1} \end{bmatrix} = \begin{bmatrix} -t^{-1} & 0 \\ 3t^{-2} & t^{-1} \end{bmatrix}$$

We thus have:

$$\mathbf{v}(t) = t^{-2}\mathbf{g}_{-2} + t^{-1}\mathbf{g}_{-1} = t^{-2} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} + t^{-1} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

with $\ell = \widehat{\mathbf{v}(t)} = \mathbf{g}_{-2}$. Note that $\mathbf{v}(t) \in \mathfrak{g}' = \mathfrak{sl}_2 \otimes k$ and $a = -2$. We check that the path condition indeed holds:

$$(t^{-2} \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} + t^{-1} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix})(x^3 + tx^2y) = x^2y$$

Since $a < -1$, Corollary 3.13 applies, and we have $\ell \in \mathfrak{h}$.

Example 4.6. Let us continue with Example 4.5, with $V = \text{Sym}^3(\mathbb{C}^*)$ and $p = x^2y$ and $q = x^3$, but with the group $G' = GL_2(\mathbb{C})$. Note that G'_p is of dimension 1 while G'_q is of dimension 2. Let $\gamma'(t)$ and $\gamma'(t)^{-1}$ be as given below:

$$\gamma'(t) = \begin{bmatrix} 1 & 0 \\ -t^{-1} & t^{-1} \end{bmatrix} \quad \gamma'(t)^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & t \end{bmatrix}$$

We have:

$$p([x, y]^T)(t) = p(\gamma'(t)^{-1}[x, y]^T) = x^2(x + ty) = x^3 + tx^2y$$

achieving the same limit and tangent of approach as $\gamma(t)$ above. Moreover, we have:

$$\mathbf{v}'(t) = \dot{\gamma}'(t)\gamma'^{-1}(t) = \begin{bmatrix} 0 & 0 \\ t^{-2} & -t^{-2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & t^{-1} \end{bmatrix}$$

This gives us $\ell' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, the semi-simple element in $\mathfrak{g}'_q$. Note that $a = -1$.

While q is not accessible as a limit of p via a 1-PS in $SL_2(\mathbb{C})$, it is indeed so in $GL_2(\mathbb{C})$.

Example 4.7. This example illustrates the construct of $\gamma'(t)$ as in Proposition 3.14. Let $X = \mathbb{C}x + \mathbb{C}y$ and $V = \text{Sym}^2(X)$. Let $p = x^2 + y^2$ and $\gamma(t)$ and its action be as given below:

$$\gamma(t) = \begin{bmatrix} t & 1 \\ 0 & 1 \end{bmatrix} \quad \gamma(t) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} tx + y \\ y \end{bmatrix}$$

We then have $\gamma(t) \cdot p = p(t) = t^2x^2 + 2txy + 2y^2$, and thus $q = 2y^2$ is the limit and $2xy$ is the tangent of approach. $T_q = T_q(O(q))$ is given by $\mathbb{C}y^2 + \mathbb{C}xy$ and thus the tangent lies in T_q . Let $M(a, b)$ be the space of matrices given below and the complement $N = \mathbb{C}x^2$ to T_q in V .

$$M(a, b) = \begin{bmatrix} 1 & 0 \\ b & 1 + a \end{bmatrix}$$

We may check that $T_{e_G}M \cdot q = T_q$. Upon substituting $M(a, b)$ in $p(t)(x, y)$, we get:

$$p'(t) = p(t)(M(a, b)(x, y)) = t^2x^2 + 2tx(bx + (1 + a)y) + 2(bx + (1 + a)y)^2$$

The condition that $p'(t) \in 2y^2 + \mathbb{C}x^2$ gives us $a = 0$ and $b = -t/2$. This gives $m(t)$ and $\gamma'(t)$ as shown below.

$$m(t) = \begin{bmatrix} 1 & 0 \\ -t/2 & 1 \end{bmatrix} \quad \gamma'(t) = \gamma(t) \cdot m(t) = \begin{bmatrix} t/2 & 1 \\ -t/2 & 1 \end{bmatrix}$$

One may check that

$$\gamma'(t)p = (t^2/2)x^2 + 2y^2$$

5. The normal cone and the tangent ideal

We now come to Theorem 2.10 on the construction of the G -invariant ideal $\bar{J} \subseteq R$. We compute its dimension and also show that the variety of $\bar{J}$ extends $O(z)$ in the direction y_e .

Let us recall the ideals $I_z, I_y \subseteq \mathbb{C}[V]$ and the associated graded ring $R = \bigoplus_{i \geq 0} R_i$ where $R_i = I_z^i / I_z^{i+1}$ (with $I_z^0 = \mathbb{C}[V]$), as in Definition 2.8. For any ideal $I \subseteq I_z$, let $I_i = I_z^i \cap I$ and $\bar{I}_i = (I_z^{i+1} + I_i) / I_z^{i+1}$. Then (I_i) is a I_z -stable filtration and $\bar{I} = \bigoplus_{i \geq 1} \bar{I}_i$ is an ideal of R . Also recall that $I_y \subseteq I_z$ and that, when G is reductive, the ideal $\bar{I}_y$ is a graded G -module isomorphic to I_y .

We construct the ideal $\bar{J} \subseteq R$ and the open set $O \subseteq \text{Spec}(R)$ from Theorem 2.10 in several steps.

Let $f \in \mathbb{C}[V]$ and ϵ be an indeterminate. Let $w \in O(z)$ and $v \in V \cong T_w V$. We may evaluate f at the point $w + \epsilon v$ to obtain a polynomial $p_f = \sum_{i \geq 0} p_i \epsilon^i$.

Definition 5.1. For $w \in O(z)$ and $v \in T_w V$ as above, let $D_{w,v}^i(f)$ be the coefficient of ϵ^i in the expansion of $f(w + \epsilon v)$.

For $f \in I_z$, $D_{w,v}^1$ is, of course, the familiar derivation associated with the tangent vector $v \in T_w V$. We then have:

Proposition 5.2. *With the above notation, for any $w \in O(z), v \in T_w V$, we have:*

1. *For $f \in I_z^r$ and $r \geq 0$, we have $D_{w,v}^i(f) = 0$ for $i < r$. Moreover, if $f' \in I_z^{r+1}$, then $D_{w,v}^r(f + f') = D_{w,v}^r(f)$.*
2. *For $f, f' \in I_z^r$ and $f'' \in I_z^s$, we have:*

$$\begin{aligned} D_{w,v}^r(f + f') &= D_{w,v}^r(f) + D_{w,v}^r(f') \\ D_{w,v}^{r+s}(ff'') &= D_{w,v}^r(f) D_{w,v}^s(f'') \end{aligned}$$

3. *For $w \in O(z)$ and $v \in T_w V$ let $v' \in T_w O(z)$, the tangent space of the orbit $O(z)$ at w . Then for an $f \in I_z^r$ with $r > 0$, we have $D_{w,v+v'}^r(f) = D_{w,v}^r(f)$.*
4. *For any $g \in G$, and $f \in I_z^r$, let f^g denote the function $f^g(x) = f(g^{-1}x)$. Then, $f^g \in I_z^r$ as well. Moreover, for any $g' \in G$, we have:*

$$D_{gw,gv}^r f^{g'} = D_{(g')^{-1}gw, (g')^{-1}gv}^r(f)$$

Proof. The proofs of (1), (2) and (3) are straightforward. For (4), note that $f^{g'}(gw + \epsilon gv) = f((g')^{-1}(gw + \epsilon gv)) = f((g')^{-1}g(w + \epsilon v))$. This proves the proposition. ■

Recall $y(t) = \gamma(t)y$ as in Eq.(1) reproduced below:

$$y(t) = \gamma(t)y = z + t^e y_e + \dots + t^i y_i + \dots$$

For the translate gy , we have:

$$g\gamma(t)g^{-1}(gy) = g\gamma(t)y = gy(t) = gz + t^e gy_e + \dots + t^i gy_i + \dots$$

Lemma 5.3. *Given any polynomial $f \in \mathbb{C}[V]$ and for any $g \in G$, we make the substitution as below:*

$$f(gy(t)) = f(gz + t^e gy_e + \dots) = f_0 + f_1 t^1 + \dots$$

Then $f_0 = f(gz)$, $f_1 = \dots = f_{e-1} = 0$ and $f_e = D_{gz, gy_e}^1(f)$.

Proof. It suffices to prove this for $g = e_G$, the identity. Let us create a sequence $(\epsilon_i)_{i \geq e}$ of independent indeterminates and consider the substitution:

$$f(z + \sum_{i \geq e} \epsilon_i y_i) = f(z) + \sum_{i \geq e} \epsilon_i p_i + \sum_{i \leq j} \epsilon_i \epsilon_j p_{ij} + \dots$$

where p_i, p_{ij} etc. are suitable constants. By substituting $\epsilon_i = \epsilon$ and $\epsilon_j = 0$ for $i \neq j$, we get $p_i = D_{z, y_i}^1(f)$. By substituting $\epsilon_i = t^i$ and noting that e is the minimum of all t -degrees gives us the result. ■

This gives us the following two corollaries:

Corollary 5.4. *Using the notation as above:*

1. If $f \in I_z^k$ and $g \in G$, then $f(gy(t))$ is a power series in t of the form:

$$f(gy(t)) = \alpha t^{ke} + \text{higher terms in } t$$

where $\alpha = D_{gz, gy_e}^k(f)$.

2. If $f \in I_y \cap I_z^k$ and $g \in G$, then $D_{gz, gy_e}^k(f) = 0$.

We now come to the definition of the ideal $\bar{J}$.

Definition 5.5. For the above data, and for $i \geq 1$, let $\bar{J}_i \subseteq R_i$ be defined as follows:

$$\bar{J}_i = \{\bar{f} \in I_z^i / I_z^{i+1} \mid D_{gz, gy_e}^i(\bar{f}) = 0 \text{ for all } g \in G\}$$

Proposition 5.6. *With the above Definition of $\bar{J}_i$, we have:*

1. $\bar{J}_r R_s \subseteq \bar{J}_{r+s}$ and

$$2. (\bar{I}_y)_i \subseteq \bar{J}_i.$$

3. Let $\bar{J}_{z,y_e}$ or simply $\bar{J} = \bigoplus_{i \geq 1} \bar{J}_i$. Then $\bar{J}$ is a G -invariant ideal of R and $\bar{I}_y \subseteq \bar{J} \subseteq \bar{I}_z = \bigoplus_{i \geq 1} R_i$.

Henceforth, we call $\bar{J} = \bar{J}_{z,y_e}$ as the **tangent ideal** for the data (z, y_e) .

Proof. For (1), we note that for any representatives $f \in \bar{J}_r$ and $f' \in I_z^s$, have

$$D_{gz,gy_e}^{r+s}(ff') = D_{gz,gy_e}^r(f)D_{gz,gy_e}^s(f') = 0$$

This proves (1). Assertion (2) follows from Corollary 5.4 above. For (3) above, (1) ensures that $(\bar{J}_i)$ is I_z -stable. Its G -invariance follows from the fact that if for some $f \in I_z^k$, $D_{gz,gy_e}^k(f)$ vanishes for all g , then, by Proposition 5.2 (2), so do $D_{gz,gy_e}^k(f^{g'})$, for any g' -translate of f . This proves (3) and completes the proof of the proposition. $\blacksquare$

Proposition 5.7. *The ideal $\bar{J}$ above is a radical ideal of R . Moreover, for every $i \geq 1$ we have $R_i \neq \bar{J}_i$.*

Proof. As $\bar{J}$ is a graded ideal of R , it suffices to show that if power of a homogeneous element $\bar{f}_r$ of R is in $\bar{J}$, then $\bar{f}_r$ itself is in $\bar{J}$. Note that, by Proposition 5.2,

$$D_{w,v}^{rp}(\bar{f}_r^p) = (D_{w,v}^r(\bar{f}_r))^p$$

Using the definition of $\bar{J}$, we see that $\bar{f}_r^p \in \bar{J}_{rp}$ implies that $\bar{f}_r \in \bar{J}_r$. Coming to the second assertion, by the non-singularity of the point $z \in \overline{O(z)} \subseteq V$, there is an $f \in I_z$ such that $D_{z,y_e}^1(f) \neq 0$. Then $f^i \in R_i$ and $D_{z,y_e}^i(f^i) \neq 0$. Thus $\bar{f}^i \notin \bar{J}_i$. $\blacksquare$

We now compute the dimension of the tangent ideal $\bar{J}$. We start with an easy consequence of the ring homomorphism $R_0 \rightarrow R$:

Proposition 5.8. *There is a morphism $\text{Spec}(R) \rightarrow \overline{O(z)}$. Thus, $O' = \pi^{-1}(O(z))$ is an open set of $\text{Spec}(R)$. Moreover, for any open set $U \subseteq O(z)$ we have the map $\text{Spec}(R_U) \rightarrow U$.*

Let C be the affine variety $\overline{O(z)} \times V$ and C^o be the open subset $O(z) \times V$ which is also the trivial bundle over $O(z)$. Let $TO(z) = \{(gz, gv) | g \in G, v \in T_z\} \subset C^o$ be the tangent bundle and $B^o = \{(gz, v + gT_z) | g \in G, v \in V\}$, the quotient bundle. Note that $B^o \cong G \times^H \bar{N}$, the associated bundle over $O(z)$ with fiber $\bar{N}$.

We denote a typical element of B^o by $(w, \bar{v})$, where $w = gz \in O(z)$ and $\bar{v} = v + gT_z$. Let π be the surjection $\pi : C^o \rightarrow B^o$ with fiber gT_z . For any open set $U \subseteq O(z)$, let $B_U^o = \{(w, \bar{v}) \in B^o | w \in U\}$. We prove,

Proposition 5.9. *For any $w \in O(z)$ we have an open set $U \subseteq O(z)$, such that (i) $\text{Spec}(R_U) \cong B_U^o$ (ii) for any $f \in (R_U)_i$, its evaluation on any $(w, \bar{v})$ is given by $D_{w,v}^i f$.*

Proof. Part (i) is classical, see for example [5, Appendix B.7]. For part (ii), note that the isomorphism $\text{Spec}(R_U)$ with B_U^o arises from the evaluation map $R_i \times C^o \rightarrow \mathbb{C}$ given by $(\bar{f}_i, (w, v)) \rightarrow D_{w,v}^i \bar{f}_i$. This gives us the ring homomorphism $R \rightarrow \mathbb{C}[C^o]$. Properties of $D_{w,v}^i$ as given in Proposition 5.2, imply that this factorizes as $R \xrightarrow{i} \mathbb{C}[B^o] \xrightarrow{\pi^*} \mathbb{C}[C^o]$. Part (ii) follows. ■

For any $v \in V$, let $[v]$ denote the line $\mathbb{C}^*v$. We define:

$$B_J^o = \{(gz, \alpha g \bar{y}_e) | g \in G, \alpha \in \mathbb{C}^*\} \subseteq B^o$$

We have the following proposition:

Proposition 5.10. *For the above data, we have:*

1. B_J^o is a fiber bundle over $O(z)$ with fiber $O_H([\bar{y}_e]) \subseteq \bar{N}$.
2. For any U as in Proposition 5.9, $(B_J^o)_U$ is a fiber bundle over $O(z)$ with fiber $\overline{O_H([\bar{y}_e])} \subseteq \bar{N}$ and is the variety of $\bar{J}_U$.

Proof. Part (1) is clear - The G -orbit of $(z, \bar{y}_e) \in B^o$ with the fiber over gz given by $\{(ghz, (ghg^{-1})g\bar{y}_e) | h \in H\}$.

For (2), let us look at $C_J^o = \pi^{-1}B_J^o = \{gz, \alpha y_e + gv | g \in G, \alpha \in \mathbb{C}^*, v \in T_z\}$. Clearly C_J^o is an algebraic set - it is the image of $G \times \mathbb{C}^*$ acting on $(z, y_e + T_z)$. Let $\overline{C_J^o}$ be its closure. Then:

Claim: If $(gz, v) \in \overline{C_J^o}$, then $v \in g(\overline{O_H(y_e)} + T_z)$.

Proof: It suffices to prove this for $g = e_G$, the identity, i.e., for an element $(z, v) \in \overline{C_J^o}$. But the set $(z, V) \cap C_J^o$ is $(z, O_H([y_e]) + T_z)$, and so $v \in \overline{O_H([y_e]) + T_z}$. This implies that $\bar{v} \in \overline{O_H([\bar{y}_e])}$.

This proves the first assertion in (2). For the second assertion, note that via the local ring isomorphism $\mathbf{i} : R_U \rightarrow \mathbb{C}[B_U^o]$, $\bar{J}_U$ is the vanishing ideal of the algebraic set $(B_J^o)_U$. By the first assertion, the fiber over z is precisely $\overline{O_H([\bar{y}_e])}$ as stated. This proves the proposition. ■

In general $O_H([\bar{y}_e])$ is either an orbit or a 1-parameter collection of orbits. This is captured in the definition below:

Definition 5.11. We say that $\mathfrak{h}, y_e$ satisfy the **eigenvector condition** if there is an $\mathfrak{h} \in \mathfrak{h}$ such that $\mathfrak{h}\bar{y}_e = \alpha \bar{y}_e$ with $\alpha \neq 0$. We then set $\kappa = 0$. Otherwise, we set $\kappa = 1$.

Proposition 5.12. *The dimension of $V(\bar{J})$ in O is $\dim(G) - \dim(H_{\bar{y}_e}) + \kappa$.*

Proof. The dimension of $V(\bar{J}) \subseteq \text{Spec}(R)$ equals that of a generic localization $V(\bar{J}_U) \subseteq \text{Spec}(R_U)$. By Proposition 5.10 (2), it is $\dim(O(z)) + \dim(O([\bar{y}_e]))$. But this is precisely $\dim(G) - \dim(H) + \dim(O([\bar{y}_e])) = \dim(G) - \dim(H_{\bar{y}_e}) + \kappa$ as claimed. ■

We now come to the proof of Theorem 2.10.

Theorem 5.13. *Suppose that $y \xrightarrow{\gamma} z$ with y_e as the tangent of approach. Moreover, suppose that $\hat{\mathfrak{k}} \subsetneq \mathfrak{h}_{\overline{y_e}}$. Then there is a G -invariant ideal $\overline{J}_{z,y_e}$ or simply $\overline{J} = \bigoplus_{i \geq 1} \overline{J}_i$ of R such that:*

1. *For any $\overline{f} \in \overline{J}_r$ and any representative $f \in I_z^r$, we have $D_{z,y_e}^r(f) = 0$.*
2. *$\overline{I}_y \subseteq \overline{J} \subsetneq \overline{I}_z$.*
3. *Let $V(\overline{J}) \subseteq \text{Spec}(R)$ be the variety of $\overline{J}$. Then there is an open subset $O \subseteq \text{Spec}(R)$ such that the dimension of $V(\overline{J}) \cap O$ is $(\dim(G) - \dim(H_{\overline{y_e}})) + \kappa$.*

Proof. (1) follows from the definition of $\overline{J}$, as in Definition 5.5. The first containment in (2) follows from Proposition 5.6. That $\overline{J} \neq \overline{I}_z$ is shown in Proposition 5.7. This proves (2). The assertion on the dimension is proved above in Proposition 5.12. $\blacksquare$

We know in general that $\hat{\mathfrak{k}} \subseteq \mathfrak{h}_{\overline{y_e}}$. By the above theorem, if $\dim(\mathfrak{k}) < \dim(\mathfrak{h}_{\overline{y_e}}) - \kappa$, then, in the normal cone $\text{Spec}(R)$, there is indeed an intermediate variety between $\text{Spec}(R/\overline{I}_z) = \text{Spec}(\mathbb{C}[V]/I_z) = \overline{O(z)}$ and $\text{Spec}(R/\overline{I}_y)$. Note that, when G is reductive $\mathbb{C}[V] \cong R$ and $\mathbb{C}[V]/I_y \cong R/\overline{I}_y$ as G -modules and have the same dimensions.

Conjecture 5.14. Assume that G is reductive. If $\dim(\mathfrak{k}) < \dim(\mathfrak{h}_{\overline{y_e}}) - \kappa$, then there is a strictly intermediate variety $\overline{O(z)} \subsetneq W \subsetneq \overline{O(y)}$ of dimension $\dim(G) - \dim(\mathfrak{h}_{\overline{y_e}}) + \kappa$.

Remark 5.15. The above conjecture can be proved easily in the following special case. We assume that G is a reductive group with a non-trivial center which acts uniformly and non-trivially on V . Further, we assume that y is in the null-cone and $\lambda(t)$ is a 1-PS of G which drives y to $z = \bar{0}$. Such a 1-PS is guaranteed to exist due to the Hilbert-Mumford theorem [13, 14].

So, we have $y = y_e + \dots$ and $\lambda(t)y = y_e t^e + \dots$ with $e > 0$. Note that, we have $H = G_z = G$. With $\ell \in \mathfrak{g}$ as the Lie algebra element corresponding to $\lambda(t)$, $\ell \cdot y_e = e y_e$. As a result the eigenvector condition holds and $\kappa = 0$.

Note that, the additional hypothesis ensures that both orbits $O(y)$ and $O(y_e)$ are cones and are of dimensions $\dim(G) - \dim(K)$ and $\dim(G) - \dim(G_{y_e})$ respectively. Further, the line over y_e belongs to $\overline{O(y)}$. So, if $\dim(K) < \dim(G_{y_e})$, then $\overline{O(y_e)} \subsetneq \overline{O(y)}$.

6. Examples

We present three illustrative examples of the construction of the ideal $\overline{J}$.

Example 6.1. Let $V = \mathbb{C}^2$, and $z = [0, 0]^T$ and $y = [1, 1]^T$. Let G be the parabolic group given below. V consists of 3 orbits, viz., $O(z)$ of the sole point z , $O(y)$ of $\mathbb{C}^2$ minus the X -axis, and $O([1, 0]^T)$ of the X -axis minus z . We also have $H = G_z = G$.

Consider the two families γ_1 and γ_2 below and the paths $\gamma_1(t) \cdot y$ and $\gamma_2(t) \cdot y$ which take y to z but with tangents $y_1 = [1, 1]^T$ and $y_2 = [1, 0]^T$.

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid ac \neq 0 \right\} \quad \gamma_1(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \quad \gamma_2(t) = \begin{bmatrix} t & 0 \\ 0 & t^2 \end{bmatrix}$$

We see that:

$$\mathfrak{h} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \mathfrak{h}_{\overline{y_1}} = \mathfrak{k} = \widehat{\mathfrak{k}}^{\gamma_1} = \begin{bmatrix} a & 1-a \\ 0 & 0 \end{bmatrix} \quad a, b, c \in \mathbb{C}$$

On the other hand,

$$\mathfrak{h}_{\overline{y_2}} = \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix}, \widehat{\mathfrak{k}}^{\gamma_2} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \quad b, c \in \mathbb{C}$$

Let $\overline{J}_1$ and $\overline{J}_2$ be the tangent ideals for the data (z, y_1) and (z, y_2) respectively. Then $\overline{J}_1$ consists of all forms f of degree k whose k -th derivatives vanish in the directions $[1, 1]^T$ and its H -orbit. Since this is the set $O(y)$, we have $(\overline{J}_1)_i = (0)$ for all i . The ideal $J_1 = (0) \subseteq \mathbb{C}[V]$, is such that $\overline{(J_1)} = \overline{J}_1$. The variety of J_1 is $\overline{O(y)}$. Note that since $\mathfrak{h}_{\overline{y_1}} = \widehat{\mathfrak{k}}^{\gamma_1}$, no such tangent-based intermediate variety is expected.

On the other hand, since $\mathfrak{h}_{\overline{y_2}} \neq \widehat{\mathfrak{k}}^{\gamma_2}$, an intermediate variety is indeed expected. Now, the H -orbit of $y_2 = [1, 0]$ is its own multiples, $(\overline{J}_2)_i$ is the ideal generated by the forms $\{y^j x^{i-j} \mid j > 0\}$. Again, there is the ideal $J_2 = (y) \subseteq \mathbb{C}[V]$ such that $\overline{(J_2)} = \overline{J}_2$. The variety $W_2 = \overline{O([1, 0]^T)}$ is a strictly intermediate variety - $\overline{O(z)} \subsetneq W_2 \subsetneq \overline{O(y)}$.

Example 6.2. We consider an example from $\mathbb{R}^3$. Let us consider the group $G = S^1 \times \mathbb{R}^*$ acting on $\mathbb{R}^3$ given by the matrix $g(\theta, r)$ as:

$$g(\theta, r) = \begin{bmatrix} r \cos \theta & r \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \\ 0 & 0 & r \end{bmatrix}$$

Consider the point $y = [1, 0, 1]^T$ and its orbit $O(y) = \{(r \cos \theta, r \sin \theta, r) \mid \theta \in [0, 2\pi), r \neq 0\}$. The point $z = [0, 0, 0]^T$ is in $\overline{O(y)}$ via $\gamma(t) = \text{diag}(t, t, t)$, with $y_e = [1, 0, 1]$. Here again, we have $\mathfrak{h}_{\overline{y_e}} = \widehat{\mathfrak{k}}^\gamma$. Let us compute the tangent ideal $\overline{J}$ for this data and its variety.

Note that $\overline{O(z)} = O(z) = \{z\}$, $H = G_z = G$ and $I_z = (x, y, z)$. Let $f_1 = ax + by + cz \in I_z$, for some real a, b, c , $f_2 = x^2 \in I_z^2$ and $f_3 = x^2 + y^2 - z^2 \in I_z^2$. We have:

$$\begin{aligned} D_{z, y_e}^1(f_1) &= a + c \\ D_{z, y_e}^2(f_2) &= 1 \\ D_{z, y_e}^2(f_3) &= 0 \end{aligned}$$

Applying a general group element $g(\theta, r)$ and evaluating $D_{gz, gy_e}^k(f_i)$ gives us that $f_1 \notin \overline{J}_1$ for any non-zero tuple (a, b, c) . Indeed $\overline{J}_1 = (0)$. On the other hand, $f_2 \notin \overline{J}_2$ but $f_3 \in \overline{J}_2$. In fact, for the ideal $J' = (x^2 + y^2 - z^2)$, we see that $\overline{J'} = \overline{J}$. This gives $\overline{O(y)}$ as W' the variety of the tangent ideal. Note that W' has a singularity at z .

The final example of $\bar{J}$ is from the classical representation of matrices under the adjoint action.

Example 6.3. Let us consider $V = \mathbb{C}^{3 \times 3}$ with the coordinate functions $X = (X_{ij})$ in $(\mathbb{C}^{3 \times 3})^*$. Consider the action of $\mathbb{C}^* \times GL_3(\mathbb{C})$ on $\mathbb{C}^{3 \times 3}$ by $(t, g)(A) = t * gAg^{-1}$. It is easy to check that this is a group action. Consider the matrix C and the family $\gamma(t)$ below:

$$C = \begin{bmatrix} 1 & 0 & 1 \\ c_{21} & 1 & 0 \\ c_{31} & c_{32} & 1 \end{bmatrix} \quad \gamma(t) = t^2 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{bmatrix}$$

The entries $c_{ij} \in \mathbb{C}$ are chosen such that the matrix C has distinct eigenvalues. Let N_1 , N_2 and I be the matrices below:

$$N_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that:

$$\gamma(t) \cdot C = N_1 + t^2 I + \text{terms with higher degree in } t$$

Thus, in the notation of this section, we have $y = C$, $z = N_1$ and $y_e = I$. Moreover, $\overline{O(C)}$ is given by $\overline{O(C) \cup O(N_2) \cup O(N_1) \cup \mathbf{0}}$, where $\mathbf{0}$ is the zero matrix. The ideal of the orbit closure $\overline{O(N_1)}$ is given by the equation $X^2 = 0$.

Since $X^2(N_1) = X^3(N_1) = 0$ these are elements of I_z , we may evaluate $D_{z, y_e}^1(X^2)$ and $D_{z, y_e}^1(X^3)$. and we have:

$$\begin{aligned} D_{z, y_e}(X^2) &= 2N_1 \neq 0 \\ D_{z, y_e}(X^3) &= N_1^2 I + N_1 I N_1 + I N_1^2 = 0 \end{aligned}$$

Thus $X^2 \notin \bar{J}_1$ while $X^3 \in \bar{J}_1$. Note that $X^3(N_2) = 0$ and this points to the variety $W = \overline{O(N_2)}$ as the intermediate variety. Note that $\overline{O(z)} \subsetneq W \subsetneq \overline{O(y)}$.

7. Remarks

The motivation for this study comes from computational complexity theory in theoretical computer science. Let $X_n = \mathbb{C}^{n \times n}$ be the space of $n \times n$ -matrices, and let $\det_n(X_n) \in \text{Sym}^n(X_n^*)$ be the usual determinant. An important result of Valiant [16] is that the form $\det_n$ is universal in the following sense. For any fixed space X , and $f \in \text{Sym}^m(X^*)$, there is a large enough $n \geq m$, so that f is the determinant of an $n \times n$ matrix with entries as affine linear forms on X .

A celebrated problem in this connection is the permanent versus determinant problem, where $X = X_m$, the space of $m \times m$ -matrices and $f = \text{perm}_m(X_m) \in \text{Sym}^m(X_m^*)$, the usual permanent of the matrix $x \in X_m$. A long-standing conjecture (of Valiant) is that there is no determinant $\det_n$ implementing the permanent perm_m , where n is bounded by a polynomial in m .

By using a homogenizing variable x_{nn} , Mulmuley and Sohoni [12] show in their *geometric complexity* paper that Valiant's construction of f as a determinant

implies that the form $f' = x_{nn}^{n-m}f$ lies in the orbit closure of $\det_n$ under the action of $GL(X_n)$. Thus, there is indeed a one parameter family $\gamma(t)$ such that $\det_n \xrightarrow{\gamma} f'$. When f is perm_m , the permanent polynomial of an m by m matrix they also propose the use of geometric invariant theory of the stabilizers K_n , of $\det_n$ and $H_{m,n}$, of the padded permanent, $\text{perm}_{m,n} = x_{nn}^{n-m}\text{perm}_m$, to settle Valiant's conjecture.

It is $\det_n$ which serves as y and the padded permanent $\text{perm}_{m,n}$ which serves as z , with both $z, y \in V = \text{Sym}^n(X_n^*)$. Theorem 2.3 applies and inserts $\widehat{\text{Lie}(K_n)}(t)$ into $\text{Lie}(H_{m,n})$. Theorem 2.10 suggests at the existence of intermediate forms between $\det_n$ and $x_{nn}^{n-m}\text{perm}_m$ and intermediate stabilizers. This, to our knowledge, is the first result which connects the two stabilizers, and allows for a deeper analysis of the problem.

A second remark concerns the triple $K = G_y, H = G_z$ and γ . Let $G(t) = \gamma(t)G\gamma(t)^{-1} \subseteq G(k)$ (elements of G with entries from k). Then $G(t)$ is "bounded", in that there is a uniform upper bound on the order of pole of any element $g(t) \in G(t)$. We also have the bounded subgroup $K(t) = \gamma(t)K\gamma^{-1}(t)$ associated with the group K obtained as the $\gamma(t)$ -conjugate of the stabilizer of y in G .

The study of such bounded subgroups and the structure of maximal subgroups within them, is subsumed in the Bruhat-Tits theory of buildings. This was initiated by Iwahori and Matsumoto [8], and developed in the monumental work of Bruhat and Tits [3, 4] and others - see the survey [9]. An important data for any bounded group $L(t) \subseteq G(k)$ associated with the reductive subgroup $L \subseteq G$, is obtained by fixing a root system $R \subset \text{Lie}(L)$ and constructing a function $f : R \rightarrow \mathbb{Z}$ recording the order of poles of unipotent subgroups U_α attached to every $\alpha \in R$. The function f is required to be concave and is used to construct a bounded group L^f which is intimately connected to the original group $L(t)$.

We have seen in Proposition 3.2, the function $\text{ord}_\gamma : \mathfrak{k} \rightarrow \mathbb{Z}$, which assigns to $\mathbf{k} \in \mathfrak{k}$, the order of the leading term $\widehat{\mathbf{k}}(t)$. This is of course, the order of zero of the element $\mathbf{k}(t)$. We have seen that for any $\mathbf{k}, \mathbf{k}' \in \mathfrak{k}$ we have **convexity**, i.e., $\text{ord}_\gamma([\mathbf{k}, \mathbf{k}']) \geq \text{ord}_\gamma(\mathbf{k}) + \text{ord}_\gamma(\mathbf{k}')$. Thus, if we are to fix a maximal torus $T_K \subseteq K$ and a root system $R \subseteq \mathfrak{k}$, then the function $f_\gamma = -\text{ord}_\gamma : R \rightarrow \mathbb{Z}$ is concave. It is likely that our Lie subalgebra $\widehat{\mathfrak{k}}^\gamma$ is connected to the bounded group K^{f_γ} .

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