TD 603
Water Resources

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Lecture 6: Mathematics of Groundwater flow

The basic variables and continuity

- $h(x, y, z, t)$: The head at a point in time.
- $q(x, y, z, t)$: The inflow/outflow at a point in time.
- z: the elevation of the point (x, y, z).
- velocity $v_x(x, y, z, t)$: in the x-direction.

$$v_x = K_x \partial h / \partial x$$

- saturated/water-table/unsaturated region: where $h \geq z, h = z, h < z$.

- Boundary ∂B of the terrain B.
- Boundary conditions: known h or known q.

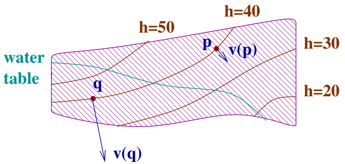
Continuity Equation

What is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}?$$

It is the rate of accumulation of water at the point (x, y, z, t)!

The condition



Thus assuming an ideal moisture condition, i.e.,

- unsaturated: moisture $= 0$.
- saturated: moisture=porosity.

we must have that water accumulation is zero if $q = 0$.

Thus we have at both p and q:

$$\frac{\partial}{\partial x}(K_x \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(K_y \frac{\partial h}{\partial y}) + \frac{\partial}{\partial z}(K_z \frac{\partial h}{\partial z}) = 0$$

Let us assume that K_x, K_y, K_z are constants. Whence, we have *for all points NOT on the water table*:

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = -q$$

E.g., $\partial^2 h/\partial x^2 > 0$ implies

$$v_x(x + dx) - v_x(x) < 0$$

Thus requiring an external outflux of water, i.e., $q > 0$.

At the water table

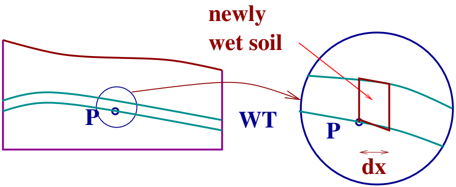
Points P on the water table may lose or gain water if h changes.

- If h increases then the water table has risen.
- More water is required for this raise.
- and the converse

For points on the water table, the outflux, i.e.,:

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2}$$

must be accounted by the rise or fall of the water-table.

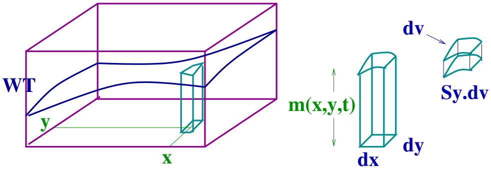


There is much argument about how this is to be modelled.

- We take the co-called Regional Groundwater Equation.
- Key Assumption: The flows in the z directions are small, i.e., h is roughly constant along z.

One Try

- $m(x, y, t)$: ht. of water table at time t.
- $S_y \frac{\partial m}{\partial t}$: Amount of water needed at x, y, z, t to move the water table.
- But is $m(x, y, t)$ computable?: Yes
- $m(x, y, t)$ is merely the z_0 so that $h(x, y, z_0, t) - z_0 = 0$
- **So not a new variable!**



Note that

$$A(t) = \int_x \int_y S_y \frac{\partial m}{\partial t} dxdy$$

is the amount of water which has come in (or gone out) from the column in the time interval $[t, t + dt]$.
Also note that:

$$B(t) = \int_{\partial T} q(x, y, z, t) ds$$

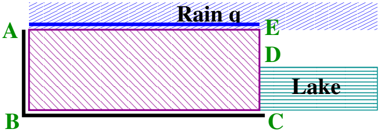
the integral on the boundary ∂T of the terrain of the net water influx.

$$A(t) = B(t)!$$

$$m \cdot K_x \frac{\partial^2 h}{\partial x^2} + m \cdot K_y \frac{\partial^2 h}{\partial y^2} = S_y \frac{\partial m}{\partial t} - q$$

- This equation is a *synthesis* of the governing equations for all regions.
- We will call this the **The Idealized Unconfined Regional Groundwater flow equation**.
- It is of limited computational value. These will focus on the saturated region.

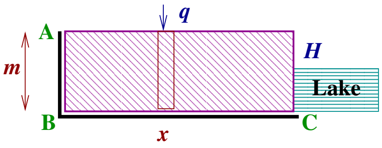
Even then, many models make simplifying assumptions.



Boundary Conditions

- AB,BC is no flow.
- AE is with water influx q mm/day.
- CD is with a knwon constant head $h = H$, the height of the water in the lake.

The Model



Assumptions

- Assume that the thickness m of the saturated layer is large compared to the variation in its height.
- Consider the saturated column situated at x. Then:

$$\frac{d}{dx}(mK_x\frac{dh}{dx}) = mK_x\frac{d^2h}{dx^2} = -q$$

Two inherent points:

- $m \sim h$, so m assumed to be the thickness.
- the head h is assumed constant in this column.

Whence:

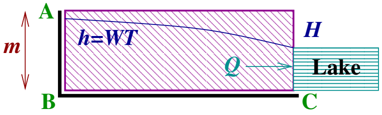
$$h(x) = \frac{-q}{2mK_x}x^2 + Ax + B$$

For some constants A, B to be determined. We have (i) $h(L) = H$ and (ii) flow at AB is zero. In other words:

$$\frac{dh}{dx}(0) = (\frac{-q}{mK_x}x + A)(0) = 0$$

This implies $A = 0$.

Several Points



Solving for (i) gives us:

$$h = H + \frac{q(L^2 - x^2)}{2mK_x}$$

- Note that this h is actually the head $h(x, y)$.
- Thus $y - h(x, y) = 0$ gives us the WT.
- The inflow into the lake is

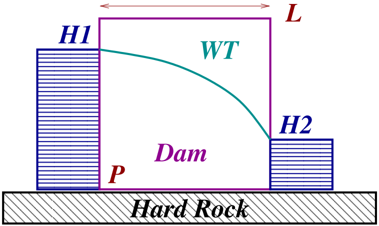
$$Q = mK_x \cdot \frac{dh}{dx}(L) = -qL$$

- Actual lake is a 3D problem, to be solved numerically.
- The asumptions imply large soil thickness and hard-rock stratum common to both lake and bank.

Problem

- What do realistic values of L, m and K give as the variation for h. Are the assumptions reasonable?
- Do the problem for $q = ax + b$, when say, the rains reduce as we move from $x = 0$ (the mountains?) to $x = L$ (the lake).

The Dupuit scheme



- A dam of width L has water stored at height H_1 on the upstream side.
- The downstream has head H_2.
- The whole dam is on impermeable rock.
- Compute the head $h(x, y)$ at all points in the dam.

The point P is $(0, 0)$.

The 1-Dupuit Assumptions

- The parameter here is x.
- The head $h(x, y)$ depends only on x. Thus the water-table at point x is at height $h(x)$.

Again, by taking a column at x, we have:

$$\frac{\partial}{\partial x}(hK_x \frac{\partial h}{\partial x}) = -q(x)$$

This implies

$$\frac{\partial^2 h^2}{\partial x^2} = \frac{-2q}{K_x}$$

The 1-D Dupuit

The basic equation:

$$\frac{\partial^2 h^2}{\partial x^2} = \frac{-2q}{K_x}$$

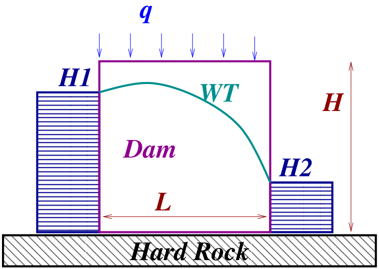
The dam problem has $q(x) = 0$. Thus

$$h^2(x, y) = \frac{-2q}{K_x} x^2 + Ax + B$$

$h(0) = H_1$ and $h(L) = H_2$ imply:

$$h^2(x) = H_1^2 - \frac{H_1^2 - H_2^2}{L} x$$

Thus, if $L = 1$ and $H_2 = 0$ and $H_2 = 1$ then $h(x) = \sqrt{1 - x}$). This explains the shape of the curve.



Problem

Now assume that the dam receives a rainfall of q. The height of the dam is H.

- Compute the water-table.
- For what rains would the top of the dam seep water?

Transmissivity

Note that the terms

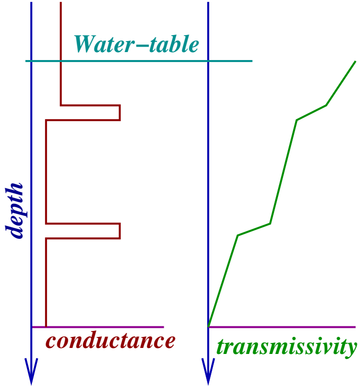
- mK_x in the lake problem.
- hK_x in the dam problem.

is **transmissivity (T) of the aquifer** .
In other words, it measures the power of the aquifer to respond to a differential in the head.

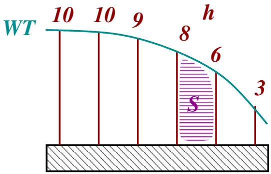
- It depends on the location of the water-table.
- In principle, the transmissivity depends on the thickness of the saturated part of the aquifer, and changes with time.

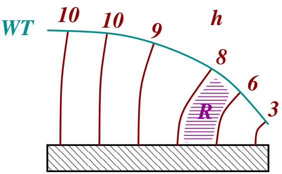
The general definition is

$$T = \int K(l)dl$$



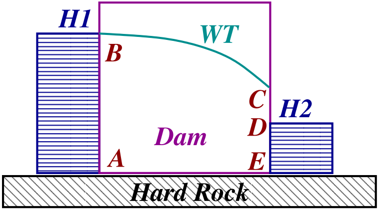
But is Dupuit correct?



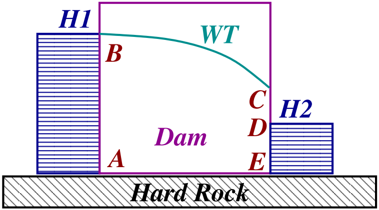


- The assumptions of h being constant vertically is wrong and is true only when the slope of WT is small.
- We will see soon that iso-h lines intersect the WT at 90 degrees.
- The *Dupuit* approximation actually approximates region R by regions S.

The actual water-table for the simple dam is shown below.
Water seeps out of the face CD!



The Dam Again



AB : constant head H_1.

AE : no flow ($\partial h/\partial z = 0$).

DE : constant head H_2.

CD : seepage face, var. head z.

Water seeps out at $CD \Rightarrow$ head there equals atmospheric pressure, i.e., z.

The problem is that point C and curve BC are unknown. Whence, the curve BC will need two defining conditions.

- $h = z$: BC is the water-table.
- $\partial h/\partial n = 0$: No flow across BC.

where n is the normal to the water-table.

In other words, velocity vectors at points on the WT are along it.

The second condition $\Rightarrow$

$$\nabla h \cdot \nabla(h - z) = 0$$

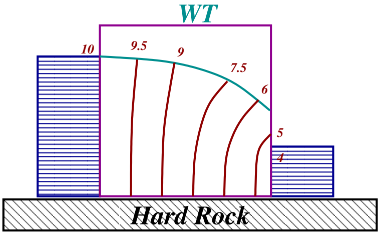
(where ∇ is the *gradient*)

Finally

This simplifies to:

$$(\frac{\partial h}{\partial x})^2 + (\frac{\partial h}{\partial z})^2 = \frac{\partial h}{\partial z}$$

- The conditions $h - z = 0$ and $\partial h / \partial n = 0$ are standard for locating the missing water-table in general situations.
- They say that the water-table hits all equi-potential surfaces at right angles.



- There is no analytic solution and only a computational one. The heads are shown in the figure above.
- Note that the equi-p lines meet the WT and leave the rock-bottom at 90-degrees.
- Also see that the $h = 5$ lines meets the seepage face at an angle. This gives the seepage at the point.

Discussion

1. Are there situations when K_x varies with x?
2. Are there any other properties of water/soil which seem to be ignored?
3. In the lake problem, how is it that we assume m as fixed and then h as varying.
4. What do you think is the general seepage condition?
5. Look at the "exact" dam solution. What would you do if the model had to include constant rain q?
6. Is there any additional condition for locating point C?